COM 202 - Signal Processing

2025 Final Exam Solutions

Multiple Choice Questions: each multiple choice question is worth 4 points.

Exercise 1. Signal Processing in Python

What is the output obtained by running the following code?

```
import numpy as np

def myfun(x: np.ndarray, h: np.ndarray) -> np.ndarray:
    out = []
    for i in range(len(x) - len(h) + 1):
        v = np.sum(x[i:i+len(h)] * h[::-1])
        out.append(v)
    return out

print( myfun(np.array([0,1,-2,3,0,2]), np.array([-1,1,-1])) )
```

a) `[3 -6 5 -5]`

b) `[0 -1 3 -6 5 -5 2 -2]`

c) `[-5 5 -6 3]`

d) `[3 -6 5 -5 2]`

Solution: *The code computes the convolution of the input arrays at all indexes for which the arrays fully overlap; the correct answer is* `[3, -6, 5, -5]`

Exercise 2. DTFT in Python

Consider a real-valued, finite support sequence $\mathbf{x}$, where $x[n]$ is zero for $n \notin [0, N-1]$. Given a frequency value $\omega_0 = 2\pi(A/B)$, where A, B are nonzero integers, you want to compute numerically the value $|X(\omega_0)|$, i.e. the magnitude of the DTFT of the sequence at $\omega = 2\pi(A/B)$.

Which of the following Python functions does **not** return the value $|X(2\pi(A/B))|$?

a)
```
def dtft_point(A, B, x):
    return np.abs(np.sum(x * np.exp(-2j * np.pi * A / B)))
```

b)
```
def dtft_point(A, B, x):
    N = len(x)
    return np.abs(np.fft.fft(x, N * B)[N * A])
```

c)
```
def dtft_point(A, B, x):
    w = 2 * np.pi * np.arange(len(x)) * A / B
    return np.abs(np.sum(x * np.exp(-1j * w)))
```

d)
```
def dtft_point(A, B, x):
    t = 2 * np.pi * np.arange(len(x)) * A / B
    return np.sqrt(np.sum(x * np.cos(t)) ** 2 + np.sum(x * np.sin(t)) ** 2)
```

Solution:

The incorrect function is:

```
def dtft_point(A, B, x):
    return np.abs(np.sum(x * np.exp(-2j * np.pi * A / B)))
```

since the exponential is missing the index variable and so it returns the value $\left|\sum_{n=0}^{N} x[n]e^{-j2\pi(A/B)}\right| = \left|\sum_{n=0}^{N} x[n]\right| = |X(0)|$.

Exercise 3. DFT

Consider two sequences $\mathbf{x}, \mathbf{y} \in \mathbb{C}^4$ whose length-4 DFTs are, respectively,

$$\mathbf{X} = \begin{bmatrix} 1+j & 2 & 1-2j & 3j \end{bmatrix}^T$$
$$\mathbf{Y} = \begin{bmatrix} 1+j & 2 & 0 & 3j \end{bmatrix}^T$$

Indicate the value of the squared distance between $\mathbf{x}$ and $\mathbf{y}$, namely, the value of

$$||\mathbf{x}-\mathbf{y}||^2 = \sum_{n=0}^{3} |x[n]-y[n]|^2.$$

a) $\frac{5}{4}$

b) $\frac{\sqrt{5}}{4}$

c) 5

d) $\sqrt{5}$

Solution: *The correct answer is* $\frac{5}{4}$.

Let $\mathbf{z} = \mathbf{x} - \mathbf{y}$. *Since the DFT is linear,* $\mathbf{Z} = \mathbf{X} - \mathbf{Y}$*; by the energy conservation property of the DFT (an orthogonal change of basis) we have*

$$\sum_{n=0}^{3} |x[n]-y[n]|^2 = \sum_{n=0}^{3} |z[n]|^2 \qquad = \frac{1}{4}\sum_{k=0}^{3} |Z[k]|^2 \qquad = \frac{1}{4}|1-2j|^2 = \frac{5}{4}$$

Exercise 4. DFT

Compute the 4 DFT coefficients of the length-4 signal $\mathbf{x} = \begin{bmatrix} 2 & 1 & 1 & 1 \end{bmatrix}^T$.

a) $\mathbf{X} = \begin{bmatrix} 5 & 1 & 1 & 1 \end{bmatrix}^T$

b) $\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 5 \end{bmatrix}^T$

c) $\mathbf{X} = \begin{bmatrix} 5 & 0 & 0 & 0 \end{bmatrix}^T$

d) $\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 2 \end{bmatrix}^T$

Solution: *The correct answer is* $\mathbf{X} = \begin{bmatrix} 5 & 1 & 1 & 1 \end{bmatrix}^T$.

One way to solve the problem is to write

$$x[n] = \delta[n] + 1$$

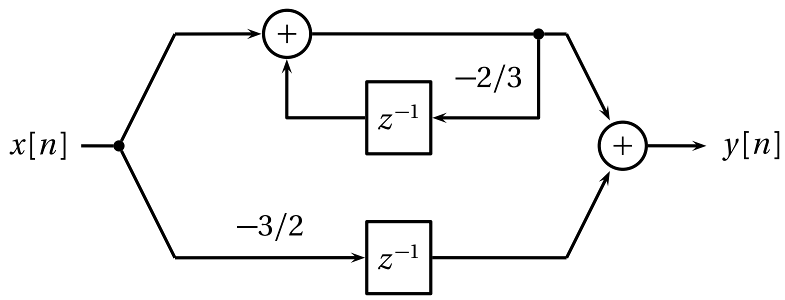
The DFT coefficients of these two signals are standard (see Appendix B) and we obtain

$$X[k] = 1 + 4\delta[k]$$

and thus $\mathbf{X} = \begin{bmatrix} 5 & 1 & 1 & 1 \end{bmatrix}^T$.

Exercise 5. Transfer function

What is the transfer function of the system represented by the following block diagram?



a) $H(z) = \dfrac{1 - (3/2)z^{-1} - z^{-2}}{1 + (2/3)z^{-1}}$

b) $H(z) = \dfrac{1 - (3/2)z^{-1} - z^{-2}}{1 - (2/3)z^{-1}}$

c) $H(z) = \dfrac{1-(3/2)z^{-1}}{1+(2/3)z^{-1}}$

d) $H(z) = 1$

Solution: *The transfer function of the top branch is that of a first-order feedback loop*

$$H_t(z) = \frac{1}{1+(2/3)z^{-1}}$$

while the bottom branch is a simple delay with a gain term:

$$H_b(z) = (-3/2)z^{-1}$$

The two branches are connected in parallel and so the global transfer function is

$$H(z) = H_t(z) + H_b(z) = \frac{1-(3/2)z^{-1}-z^{-2}}{1+(2/3)z^{-1}}.$$

Exercise 6. DTFT

Given a real-valued, discrete-time signal $\mathbf{x}$ with DTFT $X(\omega)$, indicate the correct expression for the DTFT of the signal

$$y[n] = \begin{cases} 2x[n] & n \text{ even}, \\ 2x[-n] & n \text{ odd}. \end{cases}$$

[Hint: rewrite the definition of $y[n]$ using the fact that the sequence $1+e^{-j\pi n}$ is zero for n odd while the sequence $1-e^{-j\pi n}$ is zero for n even.]

a) $Y(\omega) = X(\omega) + X(\omega - \pi) + X(-\omega) - X(-\omega - \pi)$

b) $Y(\omega) = X(\omega) + X^*(\omega) + X(-\omega) + X^*(-\omega)$

c) $Y(\omega) = (1/2)X(\omega/2) + (1/2)X(-\omega/2)$

d) $Y(\omega) = 2X(\omega) + 2X^*(-\omega)$

Solution:

The signal $\mathbf{y}$ can be expressed as

$$y[n] = (1+e^{-j\pi n})\,x[n] + (1-e^{-j\pi n})\,x[-n] = x[n] + x[-n] + e^{-j\pi n}(x[n] - x[-n]) = x[n] + x[-n] + e^{j\pi n}(x[n] - x[$$

since $e^{-j\pi n} = e^{-j\pi n}$. *Using the DTFT properties of linearity and phase shift, we have*

$$Y(\omega) = X(\omega) + X(\omega - \pi) + X(-\omega) - X(-\omega - \pi).$$

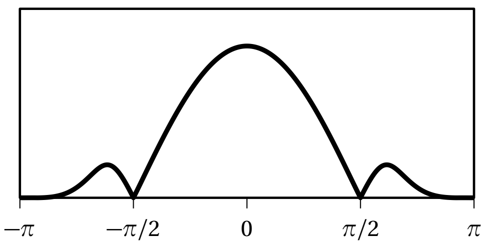
Exercise 7. Frequency response

Consider a stable, causal LTI system with transfer function

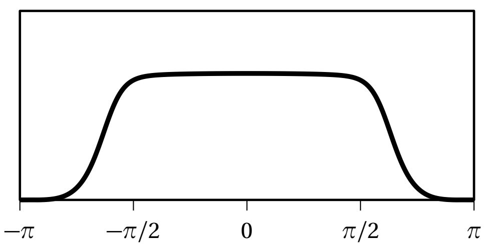
$$H(z) = \frac{(z^{-2}+1)\,(0.17 + 0.67z^{-1} + z^{-2} + 0.67z^{-3} + 0.17z^{-4})}{1 - 0.78z^{-1} - 0.68z^{-2} - 0.18z^{-3} - 0.03z^{-4}}$$

The magnitude of the system's frequency response $|H(\omega)|$ is shown in one of the following plots. Indicate which one.

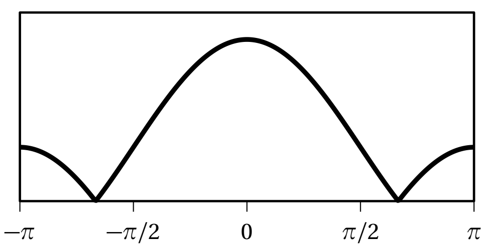
a)



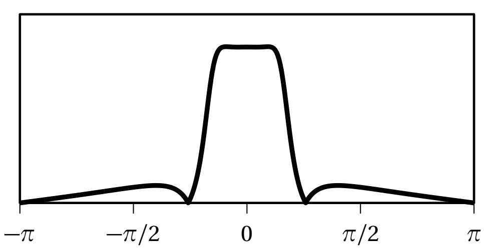
b)



c)



d)



Solution:

The initial term $(1+z^{-2})$ *in the transfer function corresponds to a pair of complex-conjugate zeros on the unit circle at* $z=\pm j=e^{\pm j\pi/2}$. *The magnitude response will therefore be exactly zero at* $\omega=\pm\pi/2$ *and only one plot fulfills this condition.*

Exercise 8. System Properties

Consider the system defined by the input-output relationship

$$y[n]=\frac{1}{2L+1}\sum_{k=n-L}^{n+L}x[k] \tag{1}$$

for L a positive integer. Which of the following statements is *false*?

a) This system is causal

b) This system is BIBO stable

c) This system is a low-pass filter

d) This system is LTI

Solution:

The system is ***not causal*** *since its the output at time n depends on L future samples (as well as L past samples). Indeed, the system is a zero-centered Moving Average of length* $2L+1$.

Exercise 9. Interpolation

A five-sample discrete-time signal $x[n]$ is interpolated into a function $x_c(t)$ using a first-order local interpolator with interpolation interval $T_i=1$:

$$x_c(t)=\sum_{n=0}^{4}x[n]\,i_1(t-n), \qquad t\in\mathbb{R}, 0\le t\le 4$$

where

$$i_1(t)=\begin{cases}1-|t| & |t|\le 1\\ 0 & \text{otherwise}\end{cases}$$

This results in an interpolated function $x_c(t)$ where data points are connected with straight lines. The function is then resampled with period $T_s=0.5$ to produce the sequence $y[n]=x_c(nT_s)$.

Which of the following statements is *true*?

a) $y[1] = (x[0] + x[1])/2$

b) $y[1] = x[1]$

c) $y[1] = 0$

d) $y[1] = (x[0] + x[2])/2$

Solution: *The correct answer is $y[1] = (x[0] + x[1])/2$. The interpolator connects the dots between each sample in $x[n]$. Then, the sampling picks out the original values of $x[n]$ as well as the interpolated values in between each sample.*

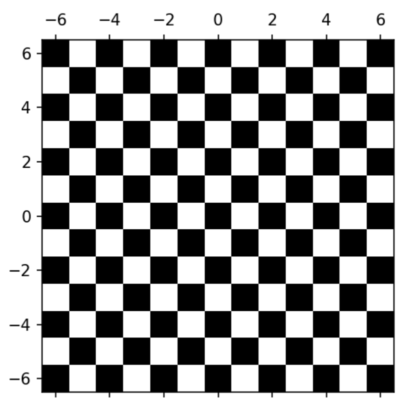
Exercise 10. Image Processing

Consider an digital image model in which pixels are encoded over 101 distinct grayscale levels, from black (zero) all the way to white (100). In this model, the 2D signal defined as

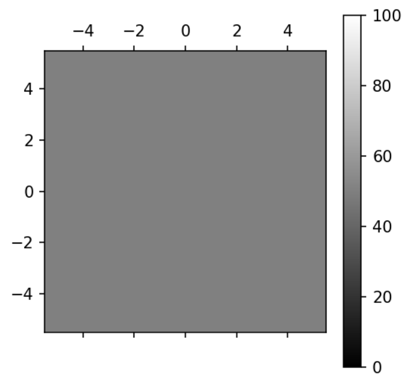
$$c[n_1, n_2] = 50(1 - (-1)^{n_1+n_2}).$$

forms the the checkerboard pattern shown in figure (a), with neighboring pixels alternating between black and white.

The signal $c[n_1, n_2]$ is filtered with a zero-centered, 3×3 FIR and the output is the uniformly gray image shown in figure (b) (assume the input image is so large that border effects can be neglected).



(a)



(b)

Which of the four FIRs listed below turns the checkerboard into a uniformly gray image?

a) $h[n_1, n_2] = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

b) $h[n_1, n_2] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

c) $h[n_1, n_2] = \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

d) $h[n_1, n_2] = \frac{1}{5}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Solution:

The correct filter is $h[n_1, n_2] = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ *since, no matter where it's placed on the checkerboard, its 8 samples with value one will overlap with 4 white pixels.*

Exercise 11. Digital storage

A 10-second piece of analog audio is sampled at $F_s = 40000$ Hz and quantized with 4 bits per sample. How many bits would it take to store the digital data uncompressed?

(a) $16 \cdot 10^5$ bits

(b) 10^6 bits

(c) $8 \cdot 10^4$ bits

(d) more bits than atoms in the universe

Solution: *The correct answer is* $16 \cdot 10^5$ *bits. At* 40000 *Hz we have* 4×10^5 *samples over ten seconds. Taking four bits for each sample gives the correct asnwer.*

Exercise 12. Constant-Coefficient Difference Equations *(16 points)*

A causal discrete-time filter is defined by the difference equation

$$y[n] = x[n] - \frac{1}{6}x[n-1] - \alpha x[n-2] + \frac{2}{3}y[n-1]$$

where α is a real number.

(a) *[4 pts]* Compute the expression for the transfer function $H(z)$.

(b) *[2 pts]* Is the system stable?

(c) *[4 pts]* For what values of α does the system become an FIR filter? Call this value α_0.

(d) *[6 pts]* Using $\alpha = \alpha_0$, find a signal $x[n]$ such that, when $x[n]$ is the input to the system, the output is $y[n] = \delta[n]$.

Solution:

(a) the transfer function is

$$H(z) = \frac{1 - \frac{1}{6}z^{-1} - az^{-2}}{1 - \frac{2}{3}z^{-1}}$$

(b) The system is stable since its single pole in $z = \frac{2}{3}$ is less than one in magnitude.

(c) The system is an FIR filter only if there are no poles and, for this to happen, $H(z)$ should have a zero in $2/3$ that cancels the existing pole. From

$$1 - \frac{1}{6}z^{-1} - az^{-2}\Big|_{z=2/3} = \frac{3}{4} - a\frac{9}{4} = 0$$

we obtain $\alpha_0 = \frac{1}{3}$.

(d) Since $Y(z) = H(z)X(z)$, if $y[n] = \delta[n]$ then $Y(z) = 1$ and so the z-transform of the desired input signal will be $X(z) = 1/H(z)$. When $\alpha = \alpha_0 = \frac{1}{3}$, the transfer function becomes

$$H(z) = 1 + \frac{1}{2}z^{-1}$$

and therefore

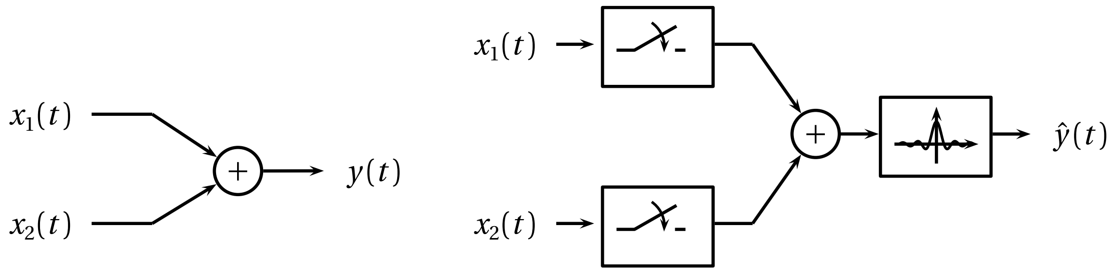
$$X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}.$$

This corresponds to an exponentially decaying signal (you can remember the impulse response of a leaky integrator, for instance) and we will have

$$x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

Exercise 13. Sampling *(12 points)*

Consider the continuous-time system shown in figure (a) below, whose output is the sum of its two input signals. In order to implement this system, you build the device shown in figure (b), using two samplers and an ideal sinc interpolator, all of which work at the same rate F_s.



System-(a) System-(b)

In the following questions, the continous-time signals $s_1(t)$ and $s_2(t)$ are known to be real-valued and bandlimited, and their maximum positive frequency is $F_N = 4000$ Hz.

(a) *[4 pts]* Determine the minimum value for the rate F_s so that when the inputs are $x_1(t) = s_1(t)$ and $x_2(t) = s_2(t)$, System-(b) produces exactly the same output as System-(a). What is the corresponding sampling interval T_s?

(b) *[4 pts]* Suppose now that the two input signals are $x_1(t) = s_1(t-t_0)$ and $x_2(t) = s_2(t+t_0)$ for some $t_0 > 0$. For the two systems to produce the same result, should you change the value of the rate F_s with respect to the previous case?

(c) *[4 pts]* Finally, suppose that the inputs are $x_1(t) = s_1(\alpha t)$ and $x_2(t) = s_2(\alpha t)$ with $0 < \alpha < 1$. For the two systems to still produce the same result, should you change the value of the rate F_s with respect to the first case? Does T_s become bigger, smaller, or stay the same?

Solution:

(a) Since the signals are real, the magnitude of the CTFT is symmetric, and so the smallest negative frequency is negative 4000 *Hz. According to the sampling theorem,* $F_s = 8000$ *Hz and* $T_s = \frac{1}{8000}$ *seconds.*

(b) Since a shift in time corresponds to scaling by a complex exponential in the frequency domain, this does not impact the maximum positive frequency. Thus, F_s *and* T_s *are the same as in part a)*

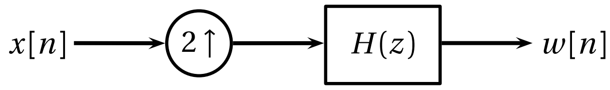
(c) The CTFT of the new signals will be $\frac{1}{\alpha}X_1\left(\frac{\omega}{\alpha}\right)$ *and* $\frac{1}{\alpha}X_2\left(\frac{\omega}{\alpha}\right)$*. Thus, the new maximum positive frequency will be* $\alpha F_N < F_N$*. Since the bandwidth of both signals becomes smaller, you have two alternative options:*

- *you can use the same sampling rate as before (and T_s remains unchanged as well); the system will work as intended although internally the signals will be oversampled by a factor $1/\alpha$;*
- *you can reduce the sampling rate to $F_s = 8000 \cdot \alpha$ Hz and $T_s = \frac{1}{8000 \cdot \alpha}$ seconds, which will minimize the number of samples (and thus the number of operations) per second.*

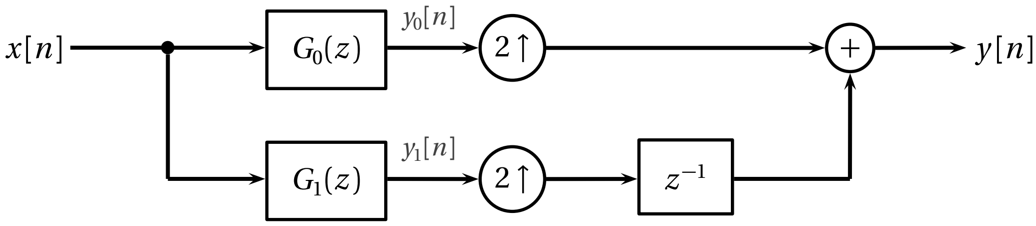
Exercise 14. Multirate interpolation *(12 points)*

Consider two possible implementations of a multirate system that increases the rate of the input by a factor of two. In both implementations, all filters are FIR.

System A, shown in the following diagram, uses a lowpass filter with cutoff frequency $\omega_c = \pi/2$ after the upsampler; the impulse response of the filter $h[n]$ has length $M = 2L$.



System B, shown in the following diagram, uses two filters and two upsamplers in parallel:



(a) *[2 pts]* Determine C_A, the number of multiplications per *input* sample required by System A.

(b) *[4 pts]* In system B, call $y_0[n]$ and $y_1[n]$ the outputs of the two FIR filters $G_0(z)$ and $G_1(z)$. Express $y[n]$ in terms of $y_0[n]$ and $y_1[n]$.

(c) *[4 pts]* Given $h[n]$, the length-$2L$ impulse reponses of the filter $H(z)$ in System A, determine the the impulse responses of the filters $G_0(z)$ and $G_1(z)$ in System B so that the two systems produce the same output.

(d) *[2 pts]* What is C_B, the number of multiplications per *input* sample required by System B?

Solution:

(a) *In System A every output sample $w[n]$ is generated by an FIR filter of length M, which will require M multiplications. Since the input signal $x[n]$ is upsampled by two before it reaches the filter, the FIR must compute two ouput samples for each input sample and so, in total, the number of multiplications per input sample is*

$$C_A = 2M$$

(b) *in System B, call $p_i[n]$ the signals exiting the two upsamplers; we have*

$$\begin{cases} p_i[2n] = y_i[n] \\ p_i[2n+1] = 0 \end{cases}$$

After the delay on the bottom branch, $y[n] = p_0[n] + p_1[n-1]$ and so

$$\begin{cases} y[2n] = p_0[2n] + p_1[2n-1] = p_0[2n] + p_1[2(n-1)+1] = y_0[n] \\ y[2n+1] = p_0[2n+1] + p_1[2n] = y_1[n] \end{cases}$$

The sequence of output samples $y[n]$ is thus the interleaving of the the sequences $y_0[n]$ and $y_1[n]$; starting at $n = 0$, for instance, the output is

$$y[0] = y_0[0],\ \ y[1] = y_1[0],\ \ y[2] = y_0[1],\ \ y[3] = y_1[1],\ \ y[4] = y_0[2],\ \ y[5] = y_1[2], \ldots$$

(c) *there are two ways of solving this question:*

- *Working in the time domain, the signals produced by the two FIR filters in system B are*

$$y_i[n] = (g_i * x)[n] = \sum_{k=0}^{M_i-1} g_i[k]x[n-k], \quad i = 0,1$$

where $M_{0,1}$ are the lengths of the two impulse responses $g_{0,1}[n]$. Since we want $y[n] = w[n]$, let's look at the even- and odd-indexed values of $w[n]$. In System A, call $x_2[n]$ the signal exiting the upsampler:

$$x_2[n] = \begin{cases} x[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

With this, the even-indexed output samples are computed as

$$\begin{aligned} w[2n] &= (h * x_2)[2n] \\ &= \sum_{k=0}^{M-1} h[k]x_2[2n-k] \\ &= \sum_{i=0}^{M/2-1} h[2i]x_2[2n-2i] \end{aligned}$$

(because, since $x_2[n] = 0$ for n odd, all terms for k odd are equal to zero)

$$= \sum_{i=0}^{L-1} h[2i]x[n-i].$$

In order to have $w[2n] = y_0[n]$ it must be

$$\sum_{i=0}^{L-1} h[2i]x[n-i] = \sum_{k=0}^{M_0} g_0[k]x[n-k]$$

meaning that $G_0(z)$ is a length-L FIR whose impulse response contains the even-indexed samples of $h[n]$. Similarly, for the odd-indexed output samples, we obtain

$$w[2n+1] = \sum_{i=0}^{L-1} h[2i+1]x[n-i] = \sum_{k=0}^{M_1} g_1[k]x[n-k]$$

so that, in the end, System A and System B are equivalent if

$$g_i[k] = \begin{cases} h[2k+i] & 0 \le k < L = M/2 \\ 0 & \textit{otherwise} \end{cases} \qquad i = 0, 1.$$

- *Working in the z-transform domain, we have*

$$W(z) = H(z)X(z^2)$$
$$Y(z) = G_0(z^2)X(z^2) + z^{-1}G_1(z^2)X(z^2) = (G_0(z^2) + z^{-1}G_1(z^2))X(z^2)$$

so that, in order to have $Y(z) = W(z)$ it must be

$$H(z) = G_0(z^2) + z^{-1}G_1(z^2).$$

which means

$$\sum_{n=0}^{M-1} h[n]z^{-n} = \sum_{n=0}^{M_0-1} g_0[n]z^{-2n} + \sum_{n=0}^{M_1-1} g_1[n]z^{-(2n+1)}$$

This equation relates two polynomials in z and by comparing the coefficients of each power of z we can establish that

$$g_0[n] = h[2n], \quad n = 0, 1, \ldots, L$$
$$g_1[n] = h[2n+1], \quad n = 0, 1, \ldots, L$$

(d) *In system B, every input sample is processed in parallel by two FIR filters of length L, so the total number of multiplications per input sample is*

$$C_B = L + L = M = C_A/2$$

Exercise 15. Optimal Denoising *(12 points)*

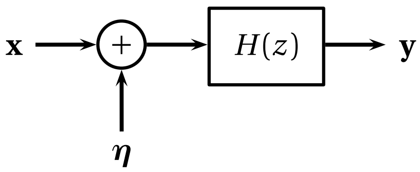
In the system shown in figure (a) below:

- the input **x** is a random process whose Power Spectral Density $P_x(\omega)$ is shown figure (b) for $\omega \in [-\pi, \pi]$; the PSD is described analytically by the expression

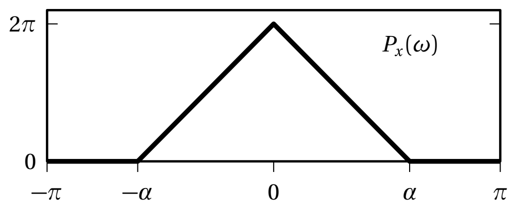
$$P_x(\omega) = \begin{cases} 2\pi\left(1 - \dfrac{|\omega|}{\alpha}\right) & |\omega| \leq \alpha \\ 0 & \alpha < |\omega| \leq \pi \end{cases}$$

- $\boldsymbol{\eta}$ is a zero-mean, white noise process with PSD $P_\eta(\omega) = \sigma^2$, independent of the input process **x**
- the denoising filter $H(z)$ is an ideal lowpass with cutoff frequency λ

$$H(\omega) = \text{rect}\left(\frac{\omega}{2\lambda}\right)$$



(a)



(b)

A performance metric for this system is the Signal to Noise-and Distortion ratio (SNDR); this is defined as the the power of the "clean" input process **x** divided by the power of the error signal $\mathbf{e} = \mathbf{y} - \mathbf{x}$, which will depend on the chosen cutoff frequency λ. Remember that the total power of a process can be obtained as the integral of its PSD over $[-\pi, \pi]$ so that:

$$\text{SNDR}_\lambda = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(\omega)d\omega}{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_e(\omega)d\omega}$$

(a) *[3 pts]* compute the SNDR when $\lambda = \pi$ (which is equivalent to removing the lowpass filter)

(b) *[3 pts]* compute the SNDR when $\lambda = \alpha$

(c) *[6 pts]* what is the optimal cutoff frequency λ_{opt} that maximizes the SNDR?

Solution:

The total power of the input is the integral of the input's PSD, namely, the area of the triangle of base 2α and height 2π:

$$W_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(\omega)d\omega = \alpha$$

The error signal is

$$\mathbf{e} = \mathbf{y} - \mathbf{x} = \mathbf{h} * (\mathbf{x} + \boldsymbol{\eta}) - \mathbf{x} = \mathbf{h} * \boldsymbol{\eta} + \mathbf{h} * \mathbf{x} - \mathbf{x}$$

The filter is an ideal lowpass, so $H(\omega) = 1$ over $[-\lambda, \lambda]$ and zero otherwise; therefore

$$\mathbf{h} * \mathbf{x} - \mathbf{x} = \mathbf{h}_c * \mathbf{x}$$

where $\mathbf{h}_c$ is the impulse response of the complementary highpass filter

$$H_c(\omega) = H(\omega) - 1$$

Since $\boldsymbol{\eta}$ and $\mathbf{x}$ are independent, the PSD of the error signal is

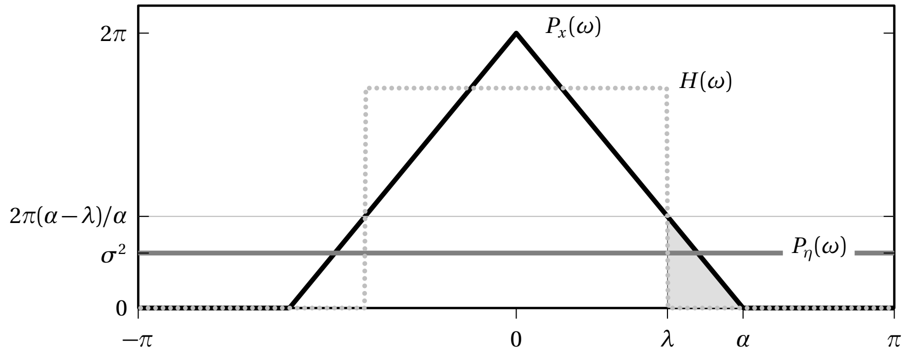
$$P_e(\omega) = |H(\omega)|^2\sigma^2 + |H_c(\omega)|^2 P_x(\omega)$$

The total power of the error is the integral of the error's PSD, and it will depend on the cutoff frequency λ; since the PSD is symmetric, we have:

$$\begin{aligned} W_e(\lambda) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2\sigma^2 d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_c(\omega)|^2 P_x(\omega)d\omega \\ &= \frac{1}{\pi}\int_{0}^{\lambda} \sigma^2 d\omega + \frac{1}{\pi}\int_{\lambda}^{\pi} P_x(\omega)d\omega \\ &= \frac{\lambda}{\pi}\sigma^2 + D(\lambda) \end{aligned}$$

If we draw a picture of the PSDs and the filter as in the figure below, we can see that $D(\lambda)$ can be evaluated geometrically as the area of the shaded region, a right triangle with sides $(\alpha - \lambda)$ and $2\pi(1 - \lambda/\alpha)$:

$$D(\lambda) = \frac{1}{\pi}\int_{\lambda}^{\pi} P_x(\omega)d\omega = \begin{cases} \frac{(\alpha-\lambda)^2}{\alpha} & 0 < \lambda < \alpha \\ 0 & \alpha \le \lambda \le \pi \end{cases}$$



(a) *If* $\lambda = \pi$, $W_e(\pi) = \sigma^2$ *so the SNDR is*

$$SNDR_\pi = \frac{\alpha}{\sigma^2}$$

(b) *If* $\lambda \geq \alpha$, *the PSD of the input is not affected by the lowpass operation whereas the PSD of the noise is set to zero for* $|\omega| > \lambda$; *the power of the error is* $W_e(\lambda) = (\sigma^2/\pi)\lambda$ *and the SNDR for* $\lambda = \alpha$ *is thus*

$$SNDR_\alpha = \frac{\pi}{\sigma^2} \geq SNDR_\pi$$

(c) *Since the power of the signal is a constant, the maximum SNDR is achieved by the cutoff frequency that minimizes the power of the error. For* $\lambda \geq \alpha$, $W_e(\lambda)$ *decreases linearly with* λ *whereas for* $0 \leq \lambda < \alpha$ *we have*

$$W_e(\lambda) = \frac{\sigma^2}{\pi}\lambda + \frac{(\alpha - \lambda)^2}{\alpha};$$

this is a quadratic function of the cutoff frequency whose global minimum can be found by solving

$$\frac{\partial W_e}{\partial \lambda} = \frac{\sigma^2}{\pi} - 2\frac{(\alpha - \lambda)}{\alpha} = 0.$$

This gives the optimal cutoff frequency

$$\lambda_{opt} = \alpha\left(1 - \frac{\sigma^2}{2\pi}\right)$$
