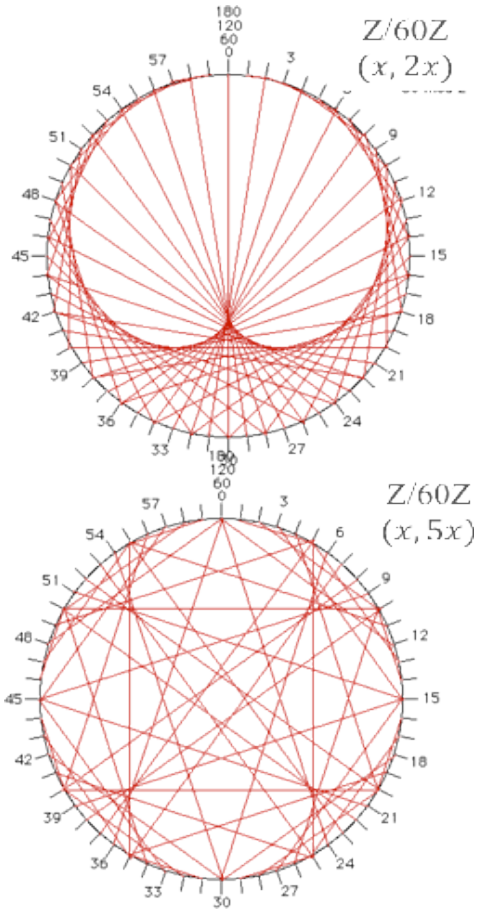
Week 7: Modular Arithmetic (Textbook Chapter 8)



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Slides by Prof. M. Gastpar and Prof. em. B. Rimoldi

EPFL

Spring Semester 2025

Outline

Introduction and Organization

Entropy and Data Compression

Cryptography

One-Time Pad, Perfect Secrecy, Public-Key (Diffie-Hellman)

Rudiments of Number Theory

Modular Arithmetic

Commutative Groups

Public-Key Cryptography

Summary of Chapter 2

Channel Coding

LAST WEEK

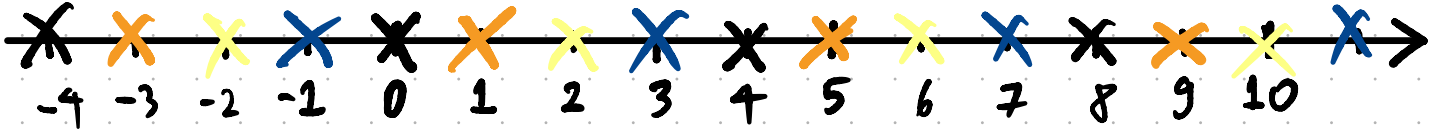
- MODULO RULES

$$(a + b) \bmod m = \left((a \bmod m) + (b \bmod m)\right) \bmod m$$

$$(ab) \bmod m = \left((a \bmod m)(b \bmod m)\right) \bmod m$$

CONGRUENCE BY COLOR

Let $m = 4$ for example.



$-4 \equiv 0 \equiv 4 \equiv 8 \ldots \pmod 4$

$-3 \equiv 1 \equiv 5 \equiv 9 \ldots \pmod 4$

$-2 \equiv 2 \equiv 6 \equiv 10 \ldots \pmod 4$

$-1 \equiv 3 \equiv 7 \equiv 11 \ldots \pmod 4$

THIS WEEK

HOW TO FIND THE MULTIPLICATIVE INVERSE?

$$5x \mod 13 = 1$$

$$5x \equiv 1 \pmod{13}$$

Why Modular Arithmetic

Modular arithmetic is a foundation of number theory.

We need number theory for cryptography and for channel coding.

Instead of considering integers and congruences $\pmod m$, and write "equations" like

$$a + b \equiv c \pmod m$$
$$a \cdot b \equiv d \pmod m,$$

we would like to write the "usual" kind of equations like

$$a + b = c$$
$$a \cdot b = d,$$

even when the operations are $\bmod m$.

This can be done, if we give new meaning to a, b, c and d, namely we make them the congruence classes $[a]_m$, $[b]_m$, $[c]_m$ and $[d]_m$.

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Definition (Congruence Classes)

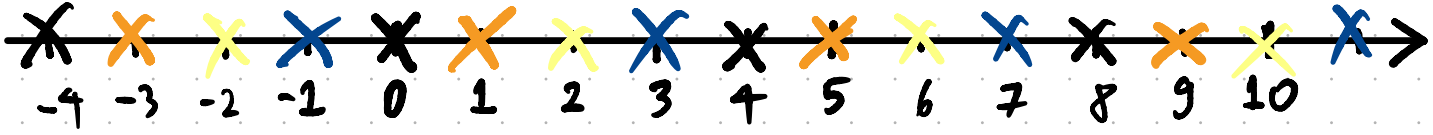
Let $m > 1$ be an integer, called the modulus.

The set of all integers congruent to $a \pmod{m}$ is called the **congruence class of** a **modulo** m.

It is denoted by $[a]_m$.

CONGRUENCE BY COLOR

Let $m = 4$ for example.



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Example

- $[24]_2$ is the set of even integers. Same as $[0]_2$, $[2]_2$, etc.
- $[23]_2$ is the set of odd integers. Same as $[1]_2$, $[3]_2$, etc.
- $[a]_m = [b]_m$ iff $a \equiv b \pmod{m}$.

Definition ($\mathbb{Z}/m\mathbb{Z}$)

The set of all congruence classes modulo m is denoted by $\mathbb{Z}/m\mathbb{Z}$ (which is read "$\mathbb{Z}$ mod m") .

Note: Some authors use the notation $\mathbb{Z}_m$.

Example

- $\mathbb{Z}/2\mathbb{Z} = \{[0]_2, [1]_2\}$.
- $\mathbb{Z}/3\mathbb{Z} = \{[0]_3, [1]_3, [2]_3\}$.
- etc.

NB: An element of $\mathbb{Z}/m\mathbb{Z}$ can be written in many ways

$$[a]_m = [a+m]_m = [a+2m]_m = \cdots$$

In particular:

- if $a = mq + r$, with $0 \leq r \leq m-1$, then

  $$[a]_m = [r]_m.$$

  We say that $[r]_m$ is in reduced form.

- every element of $\mathbb{Z}/m\mathbb{Z}$ has a unique representation in reduced form;

- $[b]_m$ is in reduced form iff $0 \leq b \leq m-1$.

Example

Which statements are correct?

1. $[-13]_9 = [5]_9$
2. $[13]_9 = [-5]_9$
3. $[13]_9 = [5]_9$
4. $[-13]_9 = [-5]_9$

Solution

1. $[-13]_9 = [5]_9$ is correct: $9 \mid -18$
2. $[13]_9 = [-5]_9$ is correct: $9 \mid 18$
3. $[13]_9 = [5]_9$ is incorrect: 9 does not divide 8
4. $[-13]_9 = [-5]_9$ is incorrect: 9 does not divide -8

In $\mathbb{Z}/m\mathbb{Z}$ we define the sum and the product as follows:

- $[a]_m + [b]_m = [a+b]_m$
- $[a]_m[b]_m = [ab]_m$

The sum set

$\{x : x = x_1 + x_2,$ where $x_1 \in [a]_m,$ $x_2 \in [b]_m\}$

The result is the same regardless the choice of representatives. In fact:

- If we choose $[a+km]_m$ instead of $[a]_m$
- and $[b+lm]$ instead of $[b]_m$
- then we obtain $[a+km]_m + [b+lm]_m = [a+km+b+lm]_m$ which is the same as $[a+b]_m$.

Idem for multiplication.

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Idem for multiplication.

Example (addition and multiplication in $\mathbb{Z}/3\mathbb{Z}$)

If the value of m is implicit, e.g. $m = 3$, then we may write a instead of $[a]_3$.

The addition and multiplication tables are:

$\mathbb{Z}/3\mathbb{Z}$	$+$	$[0]_3$	$[1]_3$	$[2]_3$
	$[0]_3$	$[0]_3$	$[1]_3$	$[2]_3$
	$[1]_3$	$[1]_3$	$[2]_3$	$[0]_3$
	$[2]_3$	$[2]_3$	$[0]_3$	$[1]_3$

$\mathbb{Z}/3\mathbb{Z}$	$\times$	$[0]_3$	$[1]_3$	$[2]_3$
	$[0]_3$	$[0]_3$	$[0]_3$	$[0]_3$
	$[1]_3$	$[0]_3$	$[1]_3$	$[2]_3$
	$[2]_3$	$[0]_3$	$[2]_3$	$[1]_3$

Example (addition and multiplication in $\mathbb{Z}/4\mathbb{Z}$)

$\mathbb{Z}/4\mathbb{Z}$	$+$	0	1	2	3
	0	0	1	2	3
	1	1	2	3	0
	2	2	3	0	1
	3	3	0	1	2

$\mathbb{Z}/4\mathbb{Z}$	$\times$	0	1	2	3
	0	0	0	0	0
	1	0	1	2	3
	2	0	2	0	2
	3	0	3	2	1

Properties of $+$ in $\mathbb{Z}/m\mathbb{Z}$

The sum has the following properties:

- associativity:

$$[a]_m + ([b]_m + [c]_m) = ([a]_m + [b]_m) + [c]_m;$$

- there exists an additive identity, namely $[0]_m$:

$$[a]_m + [0]_m = [0]_m + [a]_m = [a]_m;$$

- there exists an inverse with respect to addition: every $[a]_m$ has an inverse, denoted $(-[a]_m)$, such that

$$[a]_m + (-[a]_m) = (-[a]_m) + [a]_m = [0]_m;$$

  the inverse of $[a]_m$ is $[-a]_m$;
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$$[a]_m + [b]_m = [b]_m + [a]_m;$$

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The multiplication has the following properties:

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$$[a]_m([b]_m[c]_m) = ([a]_m[b]_m)[c]_m;$$
- multiplicative identity, namely $[1]_m$:
$$[a]_m[1]_m = [1]_m[a]_m = [a]_m;$$
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The two operations have the following property:

- distributivity:

$$[a]_m([b]_m + [c]_m) = [a]_m[b]_m + [a]_m[c]_m;$$

THE NOTATION $k[a]_m$ IN $\mathbb{Z}/m\mathbb{Z}$

For an arbitrary positive integer k, $k[a]_m$ is a short hand for

$$\underbrace{[a]_m + [a]_m + \cdots + [a]_m}_{k \text{ times}}.$$

We can easily verify that

$$k[a]_m = [ka]_m = [k]_m[a]_m.$$

Some elements of $\mathbb{Z}/m\mathbb{Z}$ have the **multiplicative inverse**.

The multiplicative inverse of $[a]_m$, if it exists, is an element $[b]_m$ such that

$$[a]_m[b]_m = [b]_m[a]_m = [1]_m.$$

The multiplicative inverse, if it exists it is unique, and it is denoted by $([a]_m)^{-1}$.

Furthermore $\left(([a]_m)^{-1}\right)^{-1} = [a]_m$.

Proof that the inverse, if it exists, is unique:

- Suppose $ab = 1$ and $ac = 1$.
- Then $ab = ac$. Multiplying both sides by b yields
- $bab = bac$. But $ba = ab = 1$. Hence $b = c$. □

Proof that if b is the inverse of a, then the inverse of b is a.

If b is the inverse of a, then $ab = ba = 1$, which implies that a is the inverse of b. □

Exercise ($\mathbb{Z}/4\mathbb{Z}$)

Which elements of $\mathbb{Z}/4\mathbb{Z}$ have the multiplicative inverse? What is it?

$\mathbb{Z}/4\mathbb{Z}$	$\times$	0	1	2	3
	0	0	0	0	0
	1	0	1	2	3
	2	0	2	0	2
	3	0	3	2	1

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Solution

We see that

- $[1]_4$ and $[3]_4$ have the inverse ($[1]_4$ and $[3]_4$, respectively).
- $[2]_4$ has no inverse.
- $[0]_m$ has no inverse, regardless of m.

For any positive integer k,

- $([a]_m)^k$ is a short hand for $\underbrace{[a]_m[a]_m \cdots [a]_m}_{k \text{ times}}$;
- $([a]_m)^0 = [1]_m$ (empty product).
- Note that we do not consider negative exponents $([a]_m)^{-k}$ because it is problematic in general, with the exception of $([a]_m)^{-1}$, if course, which is simply the multiplicative inverse of $[a]_m$ whenever it exists.

Example

$$([3]_7)^{12} = \left(([3]_7)^2\right)^6 = ([2]_7)^6 = \left(([2]_7)^3\right)^2 = ([1]_7)^2 = [1]_7.$$

EXERCISE:

Suppose $[a]_m \in \mathbb{Z}/m\mathbb{Z}$ has multiplicative inverse.

Question: Does there exist k such that

$$(*) \qquad \left([a]_m\right)^k = [0]_m \quad ?$$

Let b be the multiplicative inverse of a.

$$b^k a^k = b^k 0 = 0$$

$$b^{k-1} \underbrace{b a}_{=1} a^{k-1} = 0 \Rightarrow 1 = 0$$

$\times$

EXERCISE:

Suppose $[a]_m \in \mathbb{Z}/m\mathbb{Z}$ has multiplicative inverse.

Question: Does there exist k such that

(*) $\left([a]_m\right)^k = [0]_m$ (?)

ANSWER: NO

DENOTE $[b]_m \stackrel{\text{def}}{=} \left([a]_m\right)^{-1}$.

(*) IMPLIES:

$$\underbrace{\left([b]_m\right)^k}_{\text{=}} \left([a]_m\right)^k = [0]_m$$

BUT $([b]_m)^{k-1}\underbrace{[b]_m}_{k-1}[a]_m([a]_m)^{k-1}$

$= ([bm])^{k-1}([a]_m)^{k-1}$

$= \ldots = [1]_m \neq [0]_m$

HENCE (*) MUST BE WRONG.

An equation of the form

$$[a]_m x = [b]_m$$

has a **unique** solution iff $[a]_m$ has the inverse. In this case,

$$x = ([a]_m)^{-1}[b]_m.$$

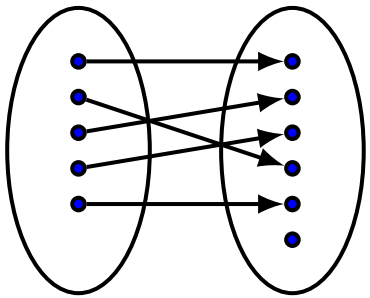
We prove a more general statement.

First a brief terminology review.

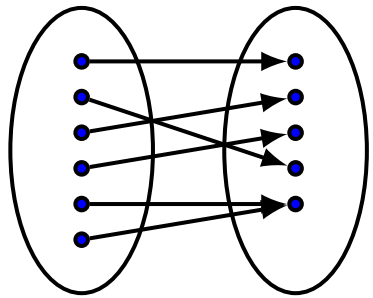
Recall that for a function $f : \mathcal{E} \to \mathcal{F}$

- $\mathcal{E}$ is the domain
- $\mathcal{F}$ is the codomain
- $f(\mathcal{E})$ is the image
- (the word *range* is sometimes used for the codomain, and sometimes for the image)

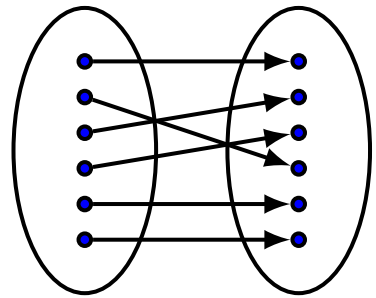
PIGEONHOLE PRINCIPLE



injective
(one-to-one)



surjective
(onto)



bijective
(one-to-one and onto)

Let $f : \mathcal{E} \to \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are finite sets.

- f injective $\Rightarrow |\mathcal{E}| \leq |\mathcal{F}|$
- f surjective $\Rightarrow |\mathcal{E}| \geq |\mathcal{F}|$
- f bijective $\Rightarrow |\mathcal{E}| = |\mathcal{F}|$

THEOREM

In $\mathbb{Z}/m\mathbb{Z}$, the following statements are equivalent: for a certain $[a]_m$:

(1) $[a]_m$ has the inverse;

(2) For all $[b]_m$, $[a]_m x = [b]_m$ has a unique solution;

(3) There exists a $[b]_m$, such that $[a]_m x = [b]_m$ has a unique solution.

Proof:

(1) $\Rightarrow$ (2): We multiply both sides of $[a]_m x = [b]_m$ by $[a]_m^{-1}$ and obtain the equivalent equation $x = [a]_m^{-1}[b]_m$, showing that there is a solution and the solution is unique.

(2) $\Rightarrow$ (1): For $[b]_m = [1]_m$ we obtain $[a]_m x = [1]_m$, which has a solution by assumption. The solution is the inverse of a.

(2) $\Rightarrow$ (3): True since (3) is a weaker statement than (2).

PROOF OF (3) $\Rightarrow$ (2).

CLAIM: $\exists\, b$ s.t. unique sol. $\Rightarrow \forall b$ unique sol.

PROOF OF (3) $\Rightarrow$ (2).

CLAIM: $\exists$ b s.t. unique sol. $\Rightarrow \forall b$ unique sol.

CLAIM': $\exists$ b s.t. either no sol or multiple sol

$\Rightarrow \nexists$ b s.t. unique sol.

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CLAIM', FIRST HALF: $\exists$ b s.t. multiple sol

$\Rightarrow \nexists$ b s.t. unique sol.

PROOF: Let $[a]_m[x_1]_m = [b^*]_m$

$[a]_m[x_2]_m = [b^*]_m$ $([x_1]_m \neq [x_2]_m)$

Now define $[x_3]_m = [x_1]_n - [x_2]_m$

With this: $[a]_m [x_3]_m = [0]_m$ $\neq [0]_m$

Now select any $[\bar{b}]_m$.

Suppose $[a)_n [x_4]_m = [\bar{b}]_m$

But then

$$[a]_m \left([x_4]_n + [x_3]_m \right) = [\bar{b}]_m$$

So we have multiple solutions.

PROOF OF (3) $\Rightarrow$ (2).

CLAIM: $\exists$ b s.t. unique sol. $\Rightarrow$ $\forall b$ unique sol.

CLAIM': $\exists$ b s.t. either no sol or multiple sol

$\Rightarrow \nexists$ b s.t. unique sol.

CLAIM', FIRST HALF: $\exists$ b s.t. multiple sol

$\Rightarrow \nexists$ b s.t. unique sol.

CLAIM', SECOND HALF: $\exists$ b s.t. no sol

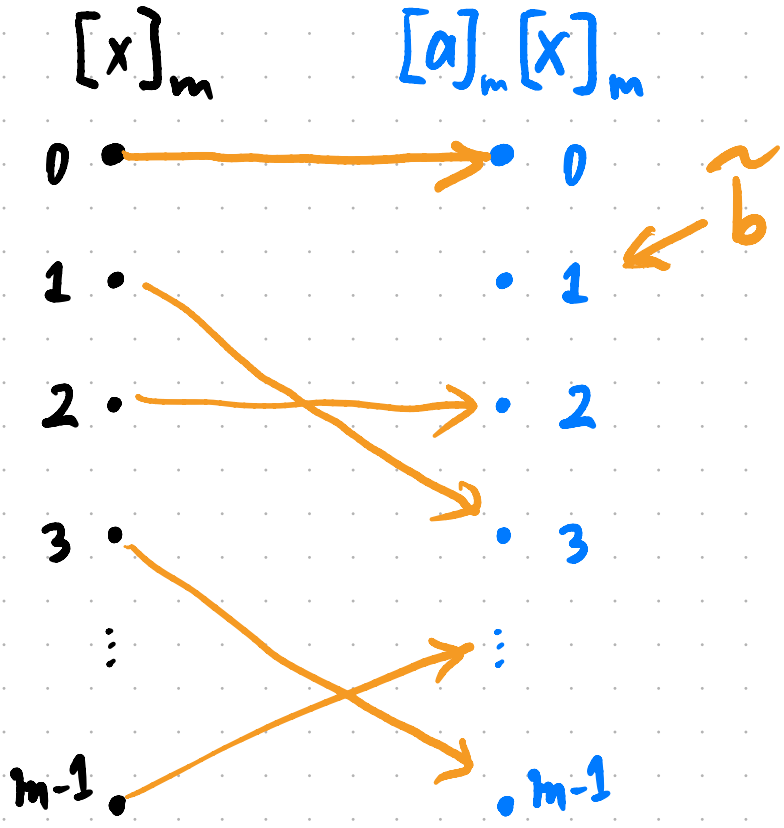
$\Rightarrow \nexists$ b s.t. unique sol.

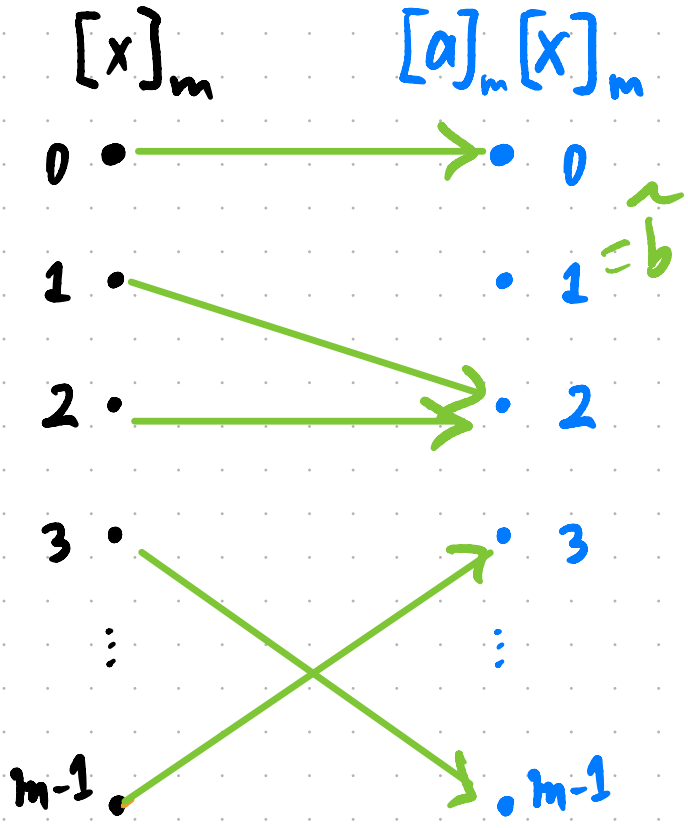
PROOF:

LET $[\tilde{b}]_m$ BE S.T. $[a]_m[x]_m \neq [\tilde{b}]_m$

FOR ALL

$[x]_m \in \mathbb{Z}/m\mathbb{Z}$


$[x]_m$
$[a]_m[x]_m$
0
1
2
3
m-1
0
1
2
3
m-1
$\tilde{b}$



Then, by pigeon hole, there must exist b^* s.t.

$$ax = b^*$$

has multiple solutions.

HENCE:

BY CLAIM', FIRST HALF

WE CAN COMPLETE THE PROOF.

□

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Exercise ($\mathbb{Z}/9\mathbb{Z}$)

If it exists, find the solution of $[4]_9 x = [3]_9$

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Solution

x	0	1	2	3	4	5	6	7	8
$[4]_9 x$	0	4	8	3	7	2	6	1	5

Pedestrian solution: From the above table we see that the solution is $[3]_9$. This approach requires having the above $[4]_9 x$ table.

Preferable solution (when possible): If it exists, we find the inverse of $[4]_9$. For now, we use the table to find $([4]_9)^{-1} = [7]_9$. Hence

$$x = [7]_9[3]_9 = [3]_9.$$

We will see how to find the inverse without constructing the table.

Example

If it exists, find the solution of $[2]_7 x + [3]_7 = [1]_7$

1. $\Leftrightarrow [2]_7 x = [1]_7 + (-[3]_7)$ (adding on both sides the negative of $[3]_7$ — always exists)
2. $\Leftrightarrow [2]_7 x = [-2]_7$
3. $\Leftrightarrow x = ([2]_7)^{-1}[5]_7$ (multiplying both sides by the inverse of $[2]_7$, which exists)
4. $\Leftrightarrow x = [4]_7[5]_7$ $(([2]_7)^{-1} = [4]_7)$
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If it exists, find the solution of $[3]_9 x + [2]_9 = [5]_9$

1. $\Leftrightarrow [3]_9 x = [5]_9 + [-2]_9$
2. $\Leftrightarrow [3]_9 x = [3]_9$

$([3]_9)^{-1}$ does not exist (see table below).

x	0	1	2	3	4	5	6	7	8
$[3]_9 x$	0	3	6	0	3	6	0	3	6

Yet, from the above table, we see that there are three solutions, namely

$$x = [1]_9,\ x = [4]_9,\ x = [7]_9.$$

THEOREM

Let $m > 1$ be integer.

The element $[a]_m \in \mathbb{Z}/m\mathbb{Z}$ has a multiplicative inverse iff $\gcd(a, m) = 1$.

The proof is postponed (see Bézout's identity).

Example (Multiplicative inverses in $\mathbb{Z}/4\mathbb{Z}$)

$\mathbb{Z}/4\mathbb{Z}$	$\times$	0	1	2	3
	0	0	0	0	0
	1	0	1	2	3
	2	0	2	0	2
	3	0	3	2	1

$\gcd(a, 4) = 1$ for $a = 1, 3$.

Theorem ($\mathbb{Z}/p\mathbb{Z}$ with p prime)

If p is prime, all elements of $\mathbb{Z}/p\mathbb{Z}$ except $[0]_p$ have a multiplicative inverse.

Proof:

$$\gcd(a, p) = 1 \text{ for } a = 1, 2, \ldots, p-1$$

$$\gcd(0, p) = p.$$

□

Recall the Mod 97 – 10 procedure

1. Append 00 (i.e., multiply the number by 100)
2. Let r be the remainder after division by 97
3. The check digits are $c = 98 - r$ (written as a 2-digit number)
4. Replace 00 with c
5. Check: the resulting number mod 97 equals 1

Recall the example

1. $n = 212351234$
2. $n \longmapsto 21235123400 + 98 - 91 = 21235123407$
3. Check: $21235123407 \mod 97 = 1$. Check passed
4. If we transpose: 21253123407
5. Check: $21253123407 \mod 97 = 2$. Check **not** passed

WHY IT DETECTS TRANSPOSITIONS

Let us use the new notation to remind ourselves why an unmodified number passes the check:

Recall that $n \to 100n + 98 - (100n \mod 97)$.

The test is passed if [number with check digits]$_{97} = [1]_{97}$

This is the case:

$$[100n+98-(100n \mod 97)]_{97} = [[100n]_{97}+98-[100n]_{97}]_{97} = [98]_{97} = [1]_{97}.$$

NOW SUPPOSE WE SWAP TWO CONSECUTIVE DIGITS:

$$m = 212ba123407$$

$$\downarrow$$

$$\tilde{m} = 212ab123407$$

NOW SUPPOSE WE SWAP TWO CONSECUTIVE DIGITS:

$$n = 212\,ba\,123407$$

$$\tilde{n} = 212\,ab\,123407$$

$$= n - 10^6(a+10b) + 10^6(b+10a)$$

NOW SUPPOSE WE SWAP TWO CONSECUTIVE DIGITS:

$$m = 212\,ba\,123407$$

$$\downarrow$$

$$\tilde{m} = 212\,ab\,123407$$

$$= m - 10^6(a+10b) + 10^6(b+10a)$$

WE ARE CHECKING

$$\tilde{m} \mod 97 = 1 \quad (?)$$

$$\tilde{m} \mod 97$$

$$= \left[m - 10^6 (a + 10b) + 10^6 (b + 10a) \right] \mod 97$$

$\tilde{m} \mod 97$

$= \left[m - 10^6 (a + 10b) + 10^6 (b + 10a) \right] \mod 97$

$= \underbrace{m \mod 97}_{=1} + \left(- 10^6 (a + 10b) + 10^6 (b + 10a) \right) \mod 97$

FOR THIS TO BE EQUAL TO 1, WE MUST HAVE

$\left(- 10^6 (a + 10b) + 10^6 (b + 10a) \right) \mod 97 = 0$

HENCE: CHECK IS PASSED IFF

$$\left[-10^6(a+10b)+10^6(b+10a)\right]_{97}=[0]_{97}$$

HENCE : CHECK IS PASSED IFF

$$\left[-10^6(a+10b)+10^6(b+10a)\right]_{97}=[0]_{97}$$

$$\left[10^6\left[-a-10b+b+10a\right]\right]_{97}$$

$$=\left[10^6\cdot 9\cdot(a-b)\right]_{97}=[0]_{97}$$

FOR THIS TO HAPPEN, WE NEED:

$[10^6]_{97} = [0]_{97}$ ← CANNOT HAPPEN $[10^k]_{97} \neq [0]_{97}$

OR

$[9]_{97} = [0]_{97}$ ← DOES NOT HOLD

OR

$[a-b]_{97} = [0]_{97}$ ← THIS HAPPENS IF AND ONLY IF $a = b$

Two consecutive digits ba of a decimal number are worth $10^k(a+10b)$ for some nonnegative integer k.

After we transpose them they are worth $10^k(b+10a)$.

The check detects the transposition, unless

$$[10^k(b+10a) - 10^k(a+10b)]_{97} = [0]_{97}$$

$$\Leftrightarrow\ [10^k(9a-9b)]_{97} = [0]_{97}$$

$$\Leftrightarrow\ [10^k 9(a-b)]_{97} = [0]_{97}$$

$$\Leftrightarrow\ ([10]_{97})^k [9]_{97} [a-b]_{97} = [0]_{97}$$

$$\Leftrightarrow\ [a-b]_{97} = [0]_{97}$$ (all non-zero elements of $\mathbb{Z}/97\mathbb{Z}$ have an inverse)

We conclude that the transposition is not detected iff $a = b$, i.e., if there is no transposition.

NOW SUPPOSE WE SWAP TWO CONSECUTIVE DIGITS:

$$m = 212\,ba\,123407$$

$$\downarrow$$

$$\tilde{m} = 212\,ab\,123407$$

$$= m - 10^6(a+10b) + 10^6(b+10a)$$

WE ARE CHECKING

$$\tilde{m} \mod 97 = 1 \quad (?)$$

NOW SUPPOSE WE SWAP TWO NON-CONSECUTIVE DIGITS:

$$m = 212b5123a07$$

$$\downarrow$$

$$\tilde{m} = 212a5123b07$$

NOW SUPPOSE WE SWAP TWO NON-CONSECUTIVE DIGITS:

$$m = 212b5123a07$$

$$\downarrow$$

$$\tilde{m} = 212a5123b07$$

$$= m - 10^2(a + 10^5 b) + 10^2(b + 10^5 a)$$

WE ARE CHECKING

$$\tilde{m} \mod 97 = 1 \quad (?)$$

$$\left(10^{2} \left(10^{5} - 1 \right) (a-b) \right) \bmod 97 = 0$$

IN GENERAL:

$$\left(10^{k} \left(10^{n} - 1 \right) (a-b) \right) \bmod 97 = 0$$

$$\left(10^{2} \left(10^{5} - 1 \right) (a-b) \right) \bmod 97 = 0$$

IN GENERAL:

$$\left(10^{k} \left(10^{n} - 1 \right) (a-b) \right) \bmod 97 = 0$$

QUESTION: DOES THERE EXIST n

$$\left[10^{n} - 1 \right]_{97} = [0]_{97}$$

YESTERDAY

- COMPUTATIONS WITH EQUIVALENCE CLASSES.

$[a]_m$

<u>Ex</u>: $m = 3$

$[2]_3 = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$

$[2]_3 + [2]_3 = \{\ldots, -2, 1, 4, 7, 10, 13, \ldots\}$

$= [1]_3$

$\mathbb{Z}/n\mathbb{Z}$

$+$	$[0]_n$	$\cdots$	$[n-1]_n$
$[0]_n$	$[0]_n$		
$[1]_n$			$[0]_n$
$\vdots$			
$[n-1]_n$			

$\cdot$	

- MULTIPLICATIVE INVERSE

$$ax \equiv 1 \pmod{m}$$

$$[a]_m x = [1]_m$$

EXISTS IF AND ONLY IF

$$\gcd(a, m) = 1.$$

PROOF : TODAY.

EX: IF $\gcd(a,m) = 1$

THEN YOU CANNOT FIND

$k \in \mathbb{Z}_+$ s.t. $[a]_m^k = [0]_m$

EX: IF $\gcd(a,n) = 1$ & $\gcd(b,n) = 1$

THEN: $[a]_n [b]_n = [0]_n$

IF AND ONLY IF

$[a]_n = [0]_n$ OR $[b]_n = [0]_n$

What for?

- Recall that $[a]_m$ has an inverse (in $\mathbb{Z}/m\mathbb{Z}$) iff $\gcd(a, m) = 1$.
- The Euclidean algorithm is a technique for quickly finding the gcd of two integers. (Much faster than via the prime factor decomposition, which is hard to do for large numbers.)
- When $\gcd(a, m) = 1$, Bézout's identity gives us the inverse of $[a]_m$.

$a \mid b$

a divides b

if $\exists\ u \in \mathbb{Z}$ s.t. $a \cdot u = b$

a divides b

if $\exists\ u \in \mathbb{Z}$ s.t. $a \cdot u = b$

Does a divide 0 ?

YES!

PROPERTIES OF THE GCD

0) $\gcd(a, 0) = a$

1) $\gcd(a, b) = \gcd(b, a)$

Hence: W.l.o.g assume $a \geq b$.

2) $\gcd(a, b) = \gcd(-a, b)$

3) $\gcd(a, b) = \gcd(a - kb, b)$

PROOF:

IF d DIVIDES a AND b,
THEN d ALSO DIVIDES $a - kb$ AND b.

IF d DIVIDES $a - kb$ AND b
THEN d ALSO DIVIDES a AND b
BECAUSE IF d DIV. $a - kb$ AND b
THEN d DIVIDES $a - kb + lb$ AND b.

3) $\gcd(a, b) = \gcd(a - kb, b)$

Proof: For any $d \in \mathbb{Z}$:

d divides <u>both</u> a and b

if and only if

d divides <u>both</u> $a - kb$ and b

3) $\gcd(a, b) = \gcd(a - kb, b)$

4) Let $a = bq + r$. with $r \in \{0, 1, \ldots, b-1\}$

$\gcd(a, b) = \gcd(b, r)$

4) Let $a = bq_1 + r_1$.

$$\gcd(a, b) = \gcd(b, r_1)$$

→ **EUCLIDEAN ALGORITHM**

Let $b = r_1 q_2 + r_2$

$$\Rightarrow \gcd(a, b) = \gcd(b, r_1) = \gcd(r_1, r_2)$$

EXAMPLE:

$$\gcd(150, 27)$$
$$= \gcd(27, 15)$$
$$= \gcd(15, 12)$$
$$= \gcd(12, 3)$$
$$= \gcd(3, 0)$$

$150 = 27 \times 5 + 15$

$27 = 15 \times 1 + 12$

$15 = 12 \cdot 1 + 3$

$12 = 3 \times 4 + 0$

$\Rightarrow \gcd(150, 27) = 3$

EXAMPLE:

$\gcd(150, 27)$

$= \gcd(27, 15)$

$= \gcd(15, 12)$

$= \gcd(12, 3)$

$= \gcd(3, 0) = 3$

$150 = 5 \times 27 + 15$

$27 = 1 \times 15 + 12$

EXAMPLE:

$$\gcd(12345678906, 12345678901)$$

$$= \gcd(12345678901, 5)$$

$$= \gcd(5, 1)$$

$$= \gcd(1, 0)$$

EXAMPLE:

$$\gcd(12345678906, 12345678901)$$

$$= \gcd(12345678901, 5)$$

$$= \gcd(5, 1)$$

$$= \gcd(1, 0) = 1$$

THEOREM (EUCLID, TEXTBOOK THM 8.3)

Let a and b be integers, not both zero. Then, for any integer k

$$\gcd(a, b) = \gcd(b, a - kb)$$

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Let a and b be integers, not both zero. Then, for any integer k

$$\gcd(a, b) = \gcd(b, a - kb)$$

Proof:

If d divides a and b, then it divides b and $a - kb$.

Similarly, if d divides both b and $a - kb$, then it divides b and $a - kb + kb = a$.

Since the set of divisors of a and b is the same as the set of divisors of b and $a - kb$, the greatest divisor is the same in both cases. □

Basic ingredients to compute the gcd

- $\gcd(a, b) = \gcd(\pm a, \pm b) = \gcd(b, a)$.
- Hence we can focus on the computation of $\gcd(a, b)$ with $0 \leq b \leq a$.
- If $a = qb + r$ is the Euclidean division, then

$$\gcd(a, b) = \gcd(b, a - qb) = \gcd(b, r),$$

  with $0 \leq r < b$. This is progress.

- Hence $\gcd(a, b) = \gcd(b, r_1) = \gcd(r_1, r_2) = \cdots = \gcd(r_n, 0) = r_n$, where

$$\begin{aligned} a &= bq_1 + r_1, \quad 0 \leq r_1 < b \\ b &= r_1 q_2 + r_2, \quad 0 \leq r_2 < r_1 \\ r_i &= r_{i+1} q_{i+2} + r_{i+2}, \quad 0 \leq r_{i+2} < r_{i+1}. \end{aligned}$$

Example

$\gcd(a,b)$	$a = bq + r$
$= \gcd(b,r)$	
$= \gcd(122,22)$	$122 = 22 \times 5 + 12$
$= \gcd(22,12)$	$22 = 12 \times 1 + 10$
$= \gcd(12,10)$	$12 = 10 \times 1 + 2$
$= \gcd(10,2)$	$10 = 2 \times 5 + 0$
$= \gcd(2,0)$	
$= 2$	

EUCLIDEAN ALGORITHM (RECURSIVE)

Algorithm 1 gcd(a, b : positive integers)

1: **if** $a < b$ **then**
   return gcd(b, a)
2: **else if** $b = 0$ **then**
   return a
3: **else**
   return gcd($b, a \% b$)
4: **end if**

EXERCISE

Compute $\gcd(12345678906, 12345678901)$

EXERCISE

Compute $\gcd(12345678906, 12345678901)$

SOLUTION

$$\begin{aligned}\gcd(12345678906, 12345678901) &= \gcd(12345678901, 5)\\ &\overset{(*)}{=} \gcd(5, 1)\\ &= \gcd(1, 0)\\ &= 1,\end{aligned}$$

where in $(*)$ we use the fact that a number $xxxxxxx0$ is divisible by 5.

THEOREM (BÉZOUT'S IDENTITY (TEXTBOOK THEOREM 8.4))

Let a and b be integers, not both zero.

There exist integers u and v, such that

$$\gcd(a, b) = au + bv$$

$$\gcd(4, 6) = 2 = 4 \cdot u + 6 \cdot v$$

$$u = -1, \quad v = 1.$$

UNIQUENESS ?

$$\gcd(a,b) = au + bv$$

$$= a(u + \ell b) + b(v - \ell a)$$

$$= au' + bv'$$

$\Rightarrow$ u and v are <u>not</u> unique

(In fact, there are infinitely many choices)

We prove Bézout's identity by means of the extended Euclidean algorithm, which finds solutions to Bézout's identity

$$\gcd(a, b) = au + bv,$$

where a and b are given, and u, v and $\gcd(a, b)$ are returned by the algorithm.

Note: if $\gcd(a, b) = au + bv$, then $\gcd(-a, b) = au + bv$ and $\gcd(a, -b) = au + bv$.

Hence it suffices that we consider nonnegative numbers a, b, not both zero.

EXTENDED EUCLID ALGORITHM

OBS: $\gcd(a,b) = \gcd(b,r)$, $a = bq + r$

$\downarrow$

suppose we have $\tilde{u}, \tilde{v}$ such that

$$\gcd(b,r) = b\tilde{u} + r\tilde{v}$$

$\Downarrow$

$$\gcd(b,r) = b\tilde{u} + (a - bq)\tilde{v}$$

$$= a\tilde{v} + b(\tilde{u} - q\tilde{v})$$

$\Lleftarrow$

$$\gcd(a,b) = a\tilde{v} + b(\tilde{u} - q\tilde{v})$$

USE THIS RECURSIVELY!

$\gcd(a,b)$	$a = bq + r$	q	$\tilde{u}$	$\tilde{v}$	$u = \tilde{v}$	$v = (\tilde{u} - q\tilde{v})$
$(150, 33)$	$150 = 33 \times 4 + 18$	4			2	-9
$(33, 18)$	$33 = 18 \times 1 + 15$	1	-1	2	-1	2
$(18, 15)$	$18 = 15 \times 1 + 3$	1	1	-1	1	-1
$(15, 3)$	$15 = 3 \times 5 + 0$	5	0	1	0	1
$(3, 0)$		0	1	0		

$\gcd(150, 33) = 3 = 2 \times 150 - 9 \times 33$

$\gcd(a,b)$	$a = bq + r$	q	$\tilde{u}$	$\tilde{v}$	$u = \tilde{v}$	$v = (\tilde{u} - q\tilde{v})$

Proof:

- Iteration step: Suppose $a \geq b$.
  - $\gcd(a, b) = \gcd(b, r)$, where $a = bq + r$;
  - suppose we have found $\tilde{u}$ and $\tilde{v}$ such that $\gcd(b, r) = b\tilde{u} + r\tilde{v}$;
  - use $r = (a - bq)$ to rewrite

    $$\gcd(a, b) = \gcd(b, r) = b\tilde{u} + r\tilde{v} = b\tilde{u} + (a - bq)\tilde{v} = a\tilde{v} + b(\tilde{u} - q\tilde{v}) \stackrel{!}{=} au + bv;$$

  - comparing terms: $u = \tilde{v}$ and $v = (\tilde{u} - q\tilde{v})$.

- Final step: $\gcd(a, 0) = a \Rightarrow \tilde{u} = 1$, $\tilde{v} = 0$.
  Note: in this last step, $\tilde{v}$ is not unique.

- Via successive applications of the above iteration, eventually we reach the form $\gcd(a, b) = au + bv$. □

Example

$\gcd(a,b)$	$a = bq + r$	$u = \tilde{v}$	$v = (\tilde{u} - q\tilde{v})$
$\gcd(122, 22)$	$122 = 22 \times 5 + 12$		
$\gcd(22, 12)$	$22 = 12 \times 1 + 10$		
$\gcd(12, 10)$	$12 = 10 \times 1 + 2$		
$\gcd(10, 2)$	$10 = 2 \times 5 + 0$		
$\gcd(2, 0) = 2$			

Example (cont.)

$\gcd(a, b)$	$a = bq + r$	$u = \tilde{v}$	$v = (\tilde{u} - q\tilde{v})$
$\gcd(122, 22)$	$122 = 22 \times 5 + 12$		
$\gcd(22, 12)$	$22 = 12 \times 1 + 10$		
$\gcd(12, 10)$	$12 = 10 \times 1 + 2$		
$\gcd(10, 2)$	$10 = 2 \times 5 + 0$	0	1
$\gcd(2, 0) = 2$		1	0

Example (cont.)

$\gcd(a, b)$	$a = bq + r$	$u = \tilde{v}$	$v = (\tilde{u} - q\tilde{v})$
$\gcd(122, 22)$	$122 = 22 \times 5 + 12$		
$\gcd(22, 12)$	$22 = 12 \times 1 + 10$		
$\gcd(12, 10)$	$12 = 10 \times 1 + 2$	1	$(0 - 1 \times 1) = -1$
$\gcd(10, 2)$	$10 = 2 \times 5 + 0$	0	1
$\gcd(2, 0) = 2$		1	0

Example (cont.)

$\gcd(a, b)$	$a = bq + r$	$u = \tilde{v}$	$v = (\tilde{u} - q\tilde{v})$
$\gcd(122, 22)$	$122 = 22 \times 5 + 12$		
$\gcd(22, 12)$	$22 = 12 \times 1 + 10$	-1	$(1 - 1(-1)) = 2$
$\gcd(12, 10)$	$12 = 10 \times 1 + 2$	1	-1
$\gcd(10, 2)$	$10 = 2 \times 5 + 0$	0	1
$\gcd(2, 0) = 2$		1	0

Example (cont.)

$\gcd(a, b)$	$a = bq + r$	$u = \tilde{v}$	$v = (\tilde{u} - q\tilde{v})$	sporadic checks
$\gcd(122, 22)$	$122 = 22 \times 5 + 12$	2	$-1 - 5 \times 2 = -11$	
$\gcd(22, 12)$	$22 = 12 \times 1 + 10$	-1	2	$-22 + 12 \cdot 2 \overset{\checkmark}{=} 2$
$\gcd(12, 10)$	$12 = 10 \times 1 + 2$	1	-1	$12 - 10 \overset{\checkmark}{=} 2$
$\gcd(10, 2)$	$10 = 2 \times 5 + 0$	0	1	
$\gcd(2, 0) = 2$		1	0	

$$\gcd(122, 22) = 122 \times 2 + 22 \times (-11)$$

Extended Euclidean algorithm (recursive)

Algorithm 2 Euclid(a, b : nonnegative integers, not both zero)

1: **if** $a < b$ **then**
   $(u, v, d) = \text{Euclid}(b, a)$
   return (v, u, d)
2: **else if** $b = 0$ **then**
   return $(1, 0, a)$
3: **else**
   $(q, r) \leftarrow$ quotient & remainder
   $(u, v, d) = \text{Euclid}(b, r)$
   return $(v, u - v * q, d)$
4: **end if**

Now we are in the position to prove the following result (stated earlier without proof).

THEOREM

Let $m > 1$ be integer.

The element $[a]_m \in \mathbb{Z}/m\mathbb{Z}$ has a multiplicative inverse iff $\gcd(a, m) = 1$.

PROOF:

Claim (1): $\gcd(a, m) = 1 \Rightarrow \left([a]_m\right)^{-1}$ exists.

$\gcd(a, m) = 1$

$\Rightarrow \exists\ u, v$ s.t. $1 = au + mv$

$\Rightarrow [1]_m = [au]_m + [mv]_m$

$\Rightarrow [1]_m = [a]_m [u]_m$

PROOF:

Claim (2): $([a]_m)^{-1}$ exists $\Rightarrow \gcd(a, m) = 1$.

Let us define $[u]_m \triangleq ([a]_m)^{-1}$.

$\Rightarrow [a]_m [u]_m = [1]_m$

or $[au]_m = [1]_m$

which means $au + mv = 1$ for <u>some</u> $v \in \mathbb{Z}$

$\Downarrow$

$\frac{a}{d} u + \frac{m}{d} v = \frac{1}{d}$ for <u>some</u> $v \in \mathbb{Z}$

But suppose that d divides <u>both</u> a and m.

Proof: $\gcd(a, m) = 1$ implies the existence of integers u and v such that $1 = au + mv$ (Bézout). Hence

$$[1]_m = [au + mv]_m = [au]_m = [a]_m[u]_m,$$

proving that $[u]_m$ is the inverse of $[a]_m$ in $\mathbb{Z}/m\mathbb{Z}$.

For the other direction, if $[u]_m$ is the inverse of $[a]_m$ in $\mathbb{Z}/m\mathbb{Z}$, then $[a]_m[u]_m = [1]_m$ or, equivalently, $[au]_m = [1]_m$. This implies that

$$au + mv = 1$$

for some integer v. If d is a divisor of both a and m, then we can write

$$\frac{a}{d}u - \frac{m}{d}v = \frac{1}{d}.$$

The left hand side is an integer, whereas the right hand side is an integer iff $d = \pm 1$. Hence 1 is the greatest integer that divides a and m. □

Corollary

$\gcd(a, m) = 1$ iff there exist integers u and v such that $1 = au + mv$.

Proof:

If $\gcd(a, m) = 1$, by Bézout, there exist integers u and v such that $1 = au + mv$.

For the other direction, suppose that $1 = au + mv$, where u and v are integers. Then $[1]_m = [au + mv]_m = [au]_m = [a]_m[u]_m$, showing that $[u]_m$ is the inverse of $[a]_m$ in $\mathbb{Z}/m\mathbb{Z}$.

By the theorem that we just proved, $\gcd(a, m) = 1$. □