ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Advanced Information, Computation, Communication II

June 27, 2017

08h15 – 11h15

Important Notes

- No document or electronic device is allowed.
- For each question, there is exactly one correct answer. We assign negative points to the wrong answers, in such a way that, in average, random choices give zero points. (Same as not answering.)
- Mark your answer with a thick 'X' in the corresponding box. If you want to change your answer, color the box completely and mark the new answer with a 'X'.
- Each page has a code on top of it (see the top of this page). Do not write on it.
- For technical reasons, pencils are not allowed.

The prime numbers smaller than 100 are:
$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89$ et 97.

Except otherwise specified, all the numbers are in base 10.

Room:
Seat:

Student Name / Sciper no.:
Blank Exam

Problem 1 [8 points]
Consider the 4 binary codes below:

	Codes			
Source Symbol	$C1$	$C2$	$C3$	$C4$
a	11	01	1100	01
b	10	001	00	111
c	101	000	0100	011
d	100	10	1	00
e	01	110	0101	010
f	001	111	1101	0110

Answer the following True/False questions [1 point each]:
Warning: the order of the questions is randomized, hence not necessarily according to the above table.

- ☐ True ☐ False — $C4$ is uniquely decodable
- ☐ True ☐ False — $C1$ is uniquely decodable
- ☐ True ☐ False — $C3$ is uniquely decodable
- ☐ True ☐ False — $C2$ is uniquely decodable

- ☐ True ☐ False — There exists an instantaneous code that has the same codeword lengths as $C2$
- ☐ True ☐ False — There exists an instantaneous code that has the same codeword lengths as $C1$
- ☐ True ☐ False — There exists an instantaneous code that has the same codeword lengths as $C3$
- ☐ True ☐ False — There exists an instantaneous code that has the same codeword lengths as $C4$

Problem 2 [7 points]
Consider a code $\mathcal{C}$ used for source coding. Answer the following True/False questions [1 point each]:

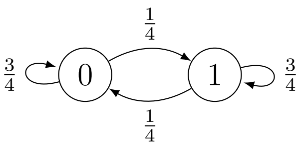
- ☐ True ☐ False — $\mathcal{C}$ does not satisfy Kraft's inequality $\Rightarrow$ $\mathcal{C}$ is with prefix
- ☐ True ☐ False — We can always replace a uniquely decodable code by a prefix-free code that has the same codeword lengths
- ☐ True ☐ False — $\mathcal{C}$ satisfies Kraft's inequality $\Rightarrow$ $\mathcal{C}$ is prefix-free
- ☐ True ☐ False — The codeword lengths do not satisfy Kraft's inequality $\Rightarrow$ $\mathcal{C}$ is not uniquely decodable
- ☐ True ☐ False — $\mathcal{C}$ is with prefix $\Rightarrow$ $\mathcal{C}$ is not uniquely decodable
- ☐ True ☐ False — $\mathcal{C}$ satisfies Kraft's inequality $\Rightarrow$ $\mathcal{C}$ is uniquely decodable
- ☐ True ☐ False — $\mathcal{C}$ is with prefix $\Rightarrow$ $\mathcal{C}$ does not satisfy Kraft's inequality

Problem 3 [9 points]
Consider a biased coin with $p(H) = \frac{1}{3}$ and $p(T) = 1 - p(H) = \frac{2}{3}$. We flip the coin indefinitely. The sequence of flips can be modeled as the output of the source $\mathcal{S} = S_1, S_2, \ldots$

Answer the following True/False questions [1 point each]:

- ☐ True ☐ False $H(S_1|S_2) < H(S_1)$
- ☐ True ☐ False $H(S_1, S_2, S_3) = (3 \log_2 3 - 2)$ bits
- ☐ True ☐ False $H(S_1) = H(S_2)$
- ☐ True ☐ False $H(S_1|S_2) = H(S_1, S_2)$
- ☐ True ☐ False $H(S_1, S_2, S_3) = (2 \log_2 3 - 3)$ bits
- ☐ True ☐ False $H(S_1, S_2, \ldots, S_n) = n \cdot H(S_n|S_1, S_2, \ldots, S_{n-1})$
- ☐ True ☐ False $H(S_2, S_3) = 2 \cdot H(S_2|S_3)$
- ☐ True ☐ False $H(S_1, S_2) < H(S_1) + H(S_2)$
- ☐ True ☐ False $H(S_2, S_3|S_1) = H(S_1, S_2, S_3) - H(S_2, S_3)$

Problem 4 [6 points]
Let $S_1, S_2, S_3, \ldots$ be an infinite sequence produced by source $\mathcal{S}$. All S_i take values in $\{0, 1\}$ and the probability $p_{S_{n+1}|S_n}(\cdot|\cdot)$ is schematically represented in the following graph:



For instance, the directed edge from 0 to 1 means that $p_{S_{n+1}|S_n}(1|0) = \frac{1}{4}$.
We assume that $p_{S_1}(0) = \frac{1}{4}$ and $p_{S_1}(1) = \frac{3}{4}$.
For the source $\mathcal{S}$ defined above, let $H(\mathcal{S})$ be the entropy of a symbol and $H^*(\mathcal{S})$ be the entropy rate.

Answer the following True/False questions [1 point each]:

- ☐ True ☐ False $p_{S_n}(0) = \frac{2^n - 1}{2^{n+1}}$ is correct and $p_{S_n}(0) = \frac{2^n - 1}{2^n + 2}$ is incorrect
- ☐ True ☐ False $H(\mathcal{S}) = 1$ bit
- ☐ True ☐ False $H^*(\mathcal{S}) = (2 - \log_2 3)$ bits
- ☐ True ☐ False $H(S_1, \ldots, S_n) = (n-1)H(S_n|S_{n-1}) + H(S_1)$
- ☐ True ☐ False The source $\mathcal{S}$ is regular
- ☐ True ☐ False $H(S_n|S_1, S_2, \ldots, S_{n-1}) = H(S_{n-1}|S_1, S_2, \ldots, S_{n-2})$

Problem 5 [7 points]
Consider a biased dice with $p(1) = p(2) = \frac{1}{4}$ and $p(3) = p(4) = p(5) = p(6) = \frac{1}{8}$. We throw this dice n times, for some fixed integer $n \geq 2$. We denote by $S = (S_1, S_2, \ldots, S_n)$ the source formed by the outcomes of the n throws. For S, we construct a binary Shannon-Fano code Γ_{SF} with an average codeword length $L(S, \Gamma_{SF})$ and a binary Huffman code Γ_H with an average codeword length $L(S, \Gamma_H)$. Let $H(S)$ be the entropy of S.

Answer the following True/False questions [1 point each]:

- ☐ True ☐ False $H(S_1, S_2, \ldots, S_n) = n \cdot H(S_1)$
- ☐ True ☐ False $H(S_n | S_1, S_2, \ldots, S_{n-1}) = n \cdot L(S, \Gamma_{SF})$
- ☐ True ☐ False $H(S) = 2n \log_2 3$ bits
- ☐ True ☐ False $L(S, \Gamma_{SF}) < L(S, \Gamma_H)$
- ☐ True ☐ False $H(S) = \frac{5n}{2}$ bits
- ☐ True ☐ False $L(S, \Gamma_H) = H(S)$
- ☐ True ☐ False $H(S_1) = \frac{L(S, \Gamma_{SF})}{n}$

Problem 6 [2 points]
Answer the following True/False questions [1 point each]:

- ☐ True ☐ False $(\mathbb{Z}/4\mathbb{Z} \setminus \{[1]_4, [3]_4\}, +)$ is a commutative group
- ☐ True ☐ False $(\mathbb{Z}/31\mathbb{Z} \setminus \{[0]_{31}\}, \cdot)$ is a commutative group

Problem 7 [4 points]
Answer the following True/False questions [1 point each]:

- ☐ True ☐ False $(\mathbb{Z}/4\mathbb{Z}, +)$ and $((\mathbb{Z}/2\mathbb{Z})^2, +)$ are isomorphic
- ☐ True ☐ False $(\mathbb{Z}/5\mathbb{Z}^*, \cdot)$ and $(\mathbb{Z}/8\mathbb{Z}^*, \cdot)$ are isomorphic
- ☐ True ☐ False $(\mathbb{Z}/8\mathbb{Z}^*, \cdot)$ and $(\mathbb{Z}/4\mathbb{Z}, +)$ are isomorphic
- ☐ True ☐ False $((\mathbb{Z}/2\mathbb{Z})^2, +)$ and $(\mathbb{Z}/8\mathbb{Z}^*, \cdot)$ are isomorphic

Problem 8 [3 points]
Alice and Bob have public trapdoor one-way functions, respectively f_A and f_B. Alice wants to send a signed message to Bob. The original message is m. She sends (check one):

- ☐ $f_B^{-1}(m)$
- ☐ $f_A(m)$
- ☐ $f_A^{-1}(m)$
- ☐ $f_B(m)$

Problem 9 [4 points]

Consider the RSA public key $(m, e) = (55, 3)$. What is the decryption of the RSA ciphertext $c = 2$? Check one:

- ☐ 18
- ☐ 2
- ☐ 7
- ☐ 3
- ☐ 8

Problem 10 [4 points]

Let m and n be coprime positive integers. How many elements are there in $\mathbb{Z}/mn\mathbb{Z}^*$? ($\phi(\cdot)$ represents Euler's totient function.) Check one:

- ☐ $mn - \phi(m) - \phi(n) + \phi(mn)$
- ☐ $mn - \phi(m)\phi(n)$
- ☐ $\phi(m + n)$
- ☐ $\phi(m) + \phi(n)$
- ☐ $\phi(m)\phi(n)$

Problem 11 [4 points]

Let E be an encryption scheme, with message T and key K. $E_K(T)$ represents the encryption of T with the key K.

Answer the following True/False questions [1 point each]:

☐ True	☐ False	If K and T are uniformly distributed over their respective alphabets, then E provides perfect secrecy
☐ True	☐ False	If K, T and $E_K(T)$ all have same length, then E provides perfect secrecy
☐ True	☐ False	If E provides perfect secrecy, then $H(T) = H(K)$
☐ True	☐ False	If $H(E_K(T)) = H(K)$, then E provides perfect secrecy

Problem 12 [4 points]
Let H be the parity-check matrix of a linear code of parameters (n, k, d_{min}).

Answer the following True/False questions [1 point each]:

- ☐ True ☐ False If the code is not MDS, there exists a collection of $n-k$ columns of H that are linearly dependent
- ☐ True ☐ False d_{min} is the smallest number of linearly dependent columns of H
- ☐ True ☐ False All collections of $n-k+1$ columns of H are linearly dependent
- ☐ True ☐ False The rank of H is $n-k$

Problem 13 [5 points]
Suppose that the input to a binary channel is a codeword of the binary code described by the following parity-check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Suppose that the channel output is $y = (1, 0, 0, 1, 0)$. You know that the channel introduces at most one error (at most one flipped position). Which is the correct answer? Check one:

- ☐ The error is in position 5
- ☐ The error is in position 2
- ☐ The error is in position 1
- ☐ The error is in position 4
- ☐ The error is in position 3
- ☐ There is no error

Problem 14 [4 points]
Suppose that the input to a binary channel is a codeword of the binary code described by the following parity-check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Suppose that the channel output is $y = (1, ?, ?, 1, 0)$. Which is the correct answer? Check one:

- ☐ The transmitted codeword is $(1, 1, 1, 1, 0)$
- ☐ The transmitted codeword is $(1, 0, 1, 1, 0)$
- ☐ The transmitted codeword is $(1, 0, 0, 1, 0)$
- ☐ The transmitted codeword is $(1, 1, 0, 1, 0)$
- ☐ There is more than one way to fill in

Problem 15 [3 points]
Answer the following True/False questions [1 point each]:

☐ True ☐ False A linear code that has minimum distance 5 can have at most one codeword of the form $(1, 0, 0, 0, c_5, c_6, c_7, c_8)$

☐ True ☐ False Every (n, k) block code, whether linear or not, has a generator matrix

☐ True ☐ False Every $(8, 4)$ linear code has at least one codeword of the form $(1, 0, 0, 0, c_5, c_6, c_7, c_8)$

Problem 16 [4 points]
Fix a finite field $\mathbb{F}$, let $p(x) = a + bx + cx^2 + dx^3$ be a polynomial over $\mathbb{F}$ of degree 3, and suppose that a_1, a_2, a_3, a_4, a_5 are distinct field elements. Suppose you are given k points of the form $(a_i, y_i) \in \mathbb{F}^2$.

Answer the following True/False questions [1 point each]:

☐ True ☐ False For $k = 3$ a polynomial of the form $p(x)$ such that $y_i = p(a_i)$, $i = 1, \ldots, k$, always exists and is unique.

☐ True ☐ False For $k = 3$ a polynomial of the form $p(x)$ such that $y_i = p(a_i)$, $i = 1, \ldots, k$, always exists but it is not necessarily unique.

☐ True ☐ False For $k = 5$ a polynomial of the form $p(x)$ such that $y_i = p(a_i)$, $i = 1, \ldots, k$, may or may not exist. If it exists, it is unique.

☐ True ☐ False For $k = 5$ a polynomial of the form $p(x)$ such that $y_i = p(a_i)$, $i = 1, \ldots, k$, may or may not exist and may or may not be unique.

Problem 17 [2 points]
Answer the following True/False questions [1 point each].

There exists a linear code over a finite field $\mathbb{F}$ that has the following parameters:

☐ True ☐ False $n = 10^3$, $|\mathbb{F}| = 1024$, the code has 2^{50} codewords, $d_{min} = 996$

☐ True ☐ False $n = 100$, the code has 63 codewords, $d_{min} = 6$

Problem 18 [4 points]
Consider a Reed-Solomon code over $\mathbb{F}_5 = (\mathbb{Z}/5\mathbb{Z}, +, \cdot)$ with parameters $k = 1$, $n = 4$, defined by $a_1 = 3$, $a_2 = 0$, $a_3 = 4$, $a_4 = 2$. Answer the following True/False questions [2 points each].

☐ True ☐ False $(3, 0, 4, 2)$ is a valid codeword

☐ True ☐ False $(1, 1, 1, 1)$ is a valid codeword

Problem 19 [4 points]
Answer the following True/False questions [2 points each].

The following generator matrices G and $\tilde{G}$ generate the same code over $\mathbb{F}_5 = (\mathbb{Z}/5\mathbb{Z}, +, \cdot)$:

☐ True ☐ False $G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ $\tilde{G} = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

☐ True ☐ False $G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ $\tilde{G} = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

Problem 20 [2 points]
Which of the following is a parity-check matrix for the generator matrix $G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ of a code over $\mathbb{F}_5 = (\mathbb{Z}/5\mathbb{Z}, +, \cdot)$? Check one:

☐ $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}$

☐ $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}$

Problem 21 [5 points]
Consider a linear code $\mathcal{C}$ of parameters (n, k), constructed on $\mathbb{F}_{11}$. How many distinct generator matrices are there for this code? Check one:

☐ 11^k

☐ $11^n - 11^k$

☐ $\frac{n!}{k!(n-k!)}$

☐ 11^{n-k}

☐ $\prod_{i=0}^{k-1}(11^k - 11^i)$

☐ $\sum_{i=0}^{k} \frac{n!}{i!(n-i)!}$