EPFL

Prof. M. Gastpar

Quiz 1 (Homeworks 1, 2 & 3)

Due on Moodle

on Monday, March 10, 2025, at 23:59.

Quiz 1

SCIPER : **111111**

- This quiz is to be solved individually.
- Try not to use any of the course materials other than the formula collection on a first attempt.
- Once you are done, enter your answers into Moodle. Moodle will give you feedback. You can update your answers as many times as you want before the deadline.
- For each question there is **exactly one** correct answer. We assign **negative points** to the **wrong answers** in such a way that a person who chooses a wrong answer loses **25 %** of the points given for that question.

Question 1

[2 Points] In a binary Huffman code, symbols with equal probability always have codewords of equal length.

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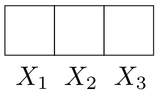
Solution: This is false. Consider a source with 3 symbols with probabilities $(1/3, 1/3, 1/3)$. While all symbols have the same probabilities, two of them will be assigned length 2, while one will be assigned a length of one.

Question 2 [5 points] *Note: This is an* **open** *question. In the real exam, we will grade your arguments. Here for the quiz, we do not have the capacity to do this. Therefore, you will merely enter your final answer into a multiple choice grid on Moodle. However, do make sure to carefully look at the solution and compare to your answer. How many points would you have given yourself?*

Consider the following three boxes.



We fill the three boxes with bits with the following procedure:

(a) We select one box uniformly at random and we fill it with 1;

(b) For each of the remaining two boxes, we fill it with either 0 or 1 independently and uniformly at random.

We denote the value in the i-th box with the random variable X_i, as in the figure. What is $H(X_1, X_2, X_3)$?

Solution, Version 1: First, we notice that X_i's are not independent. We can observe this with an example: $P(X_i = 0) > 0$, $\forall i \in \{1, 2, 3\}$, however $P(X_1 = X_2 = X_3 = 0) = 0$. So, we start by calculating the joint probability. Notice the events that have the same number of 1's will be equally likely, for instance $P_{X_1X_2X_3}(0, 0, 1) = P_{X_1X_2X_3}(0, 1, 0)$. Thus, we can calculate the probability of 4 different evetns and deduce other event probabilities from them. We know $P_{X_1X_2X_3}(0, 0, 0) = 0$ because of the part (a) of the procedure.

$$P_{X_1X_2X_3}(0,0,1) = P[X_3 \text{ is chosen in (a)}]P[X_1 = 0]P[X_2 = 0] \tag{1}$$
$$= \frac{1}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{12} \tag{2}$$

$$P_{X_1X_2X_3}(0,1,1) = P[X_2 \text{ is chosen in (a)}]P[X_1 = 0]P[X_3 = 1] \tag{3}$$
$$+ P[X_3 \text{ is chosen in (a)}]P[X_1 = 0]P[X_2 = 1] \tag{4}$$
$$= \frac{1}{3}\frac{1}{2}\frac{1}{2} + \frac{1}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{6} \tag{5}$$

$$P_{X_1X_2X_3}(1,1,1) = P[X_1 \text{ is chosen in (a)}]P[X_2=1]P[X_3=1] \quad (6)$$
$$+ P[X_2 \text{ is chosen in (a)}]P[X_1=1]P[X_3=1] \quad (7)$$
$$+ P[X_3 \text{ is chosen in (a)}]P[X_1=1]P[X_2=1] \quad (8)$$
$$= \frac{1}{3}\frac{1}{2}\frac{1}{2} + \frac{1}{3}\frac{1}{2}\frac{1}{2} + \frac{1}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{4} \quad (9)$$

X_1	X_2	X_3	$P_{X_1X_2X_3}(x_1,x_2,x_3)$
0	0	0	0
1	0	0	$\frac{1}{12}$
0	1	0	$\frac{1}{12}$
0	0	1	$\frac{1}{12}$
1	1	0	$\frac{1}{6}$
1	0	1	$\frac{1}{6}$
0	1	1	$\frac{1}{6}$
1	1	1	$\frac{1}{4}$

Then, we can calculate the entropy easily since we know the joint probability $P_{X_1X_2X_3}(x_1,x_2,x_3)$:

$$H(X_1,X_2,X_3) = -3\frac{1}{12}\log\frac{1}{12} - 3\frac{1}{6}\log\frac{1}{6} - \frac{1}{4}\log\frac{1}{4} \quad (10)$$
$$= -\frac{1}{4}(-2-\log 3) - \frac{1}{2}(-1-\log 3) - \frac{1}{4}(-2) \quad (11)$$
$$= \frac{1}{4}(2+\log 3) + \frac{1}{2}(1+\log 3) + \frac{1}{2} \quad (12)$$
$$= \frac{3}{2} + \frac{3}{4}\log 3. \quad (13)$$

Solution, Version 2: Here is an alternative solution, leveraging the chain rule. Let (X_1,X_2,X_3) be as in the problem statement, the contents of the three boxes. Now, let N_1 =Number of ones in (X_1,X_2,X_3). It is immediate to see that $N_1 \in \{1,2,3\}$ and that $p(N_1=1)=1/4, p(N_1=2)=1/2, p(N_1=3)=1/4$. Then, we can observe, using the chain rule in two different ways (much like what we've done in class):

$$H(X_1,X_2,X_3,N_1) = H(X_1,X_2,X_3) + H(N_1|X_1,X_2,X_3) \quad (14)$$
$$= H(N_1) + H(X_1,X_2,X_3|N_1). \quad (15)$$

Evidently, N_1 is a function of (X_1,X_2,X_3). Therefore, $H(N_1|X_1,X_2,X_3)=0$. Hence, we find

$$H(X_1,X_2,X_3) = H(N_1) + H(X_1,X_2,X_3|N_1). \quad (16)$$

The tricky part is now $H(X_1,X_2,X_3|N_1)$. Let us tackle the various cases for N_1 in turn. When $N_1=1$, we have exactly three options, namely $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and by symmetry, they are all equally likely, hence,

$$H(X_1, X_2, X_3|N_1 = 1) = \log_2(3). \tag{17}$$

By the same token,

$$H(X_1, X_2, X_3|N_1 = 2) = \log_2(3). \tag{18}$$

The last case is the easiest. It is direct to see that whenever $N_1 = 3$, all $X_i = 1$, and thus

$$H(X_1, X_2, X_3|N_1 = 3) = 0. \tag{19}$$

Combining, we find

$$H(X_1, X_2, X_3|N_1) = \frac{1}{4}\log_2(3) + \frac{1}{2}\log_2(3) + \frac{1}{4}\cdot 0 = \frac{3}{4}\log_2(3). \tag{20}$$

Moreover,

$$H(N_1) = -\frac{1}{4}\log_2\frac{1}{4} - \frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} = \frac{3}{2}. \tag{21}$$

Hence, we conclude that

$$H(X_1, X_2, X_3) = H(N_1) + H(X_1, X_2, X_3|N_1) = \frac{3}{2} + \frac{3}{4}\log_2(3). \tag{22}$$

Grading Notes: For Solution, Version 1: 3 points for correctly calculating all the probabilities. 2 points for correctly evaluating and simplifying the entropy expression.

For Solution, Version 2: 2 points for the idea of introducing an additional random variable N_1 (or similar - there are other choices of additional random variables that lead to workable proofs). 1 point for calculating the probabilities correctly. 2 points for correctly evaluating and simplifying the entropy expressions.

Question 3:

[6 Points] About a binary source code for a source whose alphabet has 5 letters, we only know the codeword lengths: $\ell_1 = 1, \ell_2 = 2, \ell_3 = 4, \ell_4 = 5, \ell_5 = 6$.

Answer the following true/false questions.

This source code could be a Huffman code.

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Solution: Draw the tree. Code can obviously be improved.

It is possible that this source code is uniquely decodable.

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Solution: Kraft inequality.

The average codeword length of this source code could be equal to the entropy of the source.

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Solution: This could only happen if the symbol probabilities were $p_i = 2^{-\ell_i}$. But for the proposed codeword lengths, this does not sum to one. So, there is no way of assigning probabilities to the symbols such that the average codeword length for this code attains exactly the entropy.

Question 4 [3 points] Let E and F be two events. Suppose that they satisfy $p(E|F) = p(E) > 0$. Then we must have $p(F|E) = p(F)$.

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Solution: By definition,

$$p(E|F) = \frac{p(E \cap F)}{p(F)}. \tag{23}$$

In this problem, we know that $p(E|F) = p(E)$. Combining this with the definition, we thus have $p(E \cap F) = p(E)p(F)$. Now, again by the definition,

$$p(F|E) = \frac{p(F \cap E)}{p(E)} = \frac{p(F)p(E)}{p(E)} = p(F). \tag{24}$$

So the claim is correct. (In words, $p(E|F) = p(E)$ means that E and F are independent events. But then, we must also have $p(F|E) = p(F)$.)

Question 5 [2+3 points] Let S be a random variable taking values in $\{a, b, c, d, e\}$ with the following probabilities.

	a	b	c	d	e
$p_S(\cdot)$	1/3	1/3	1/9	1/9	1/9

Let Γ_D be the D-ary Huffman code for S. Let $L(S, \Gamma_D)$ be the average codeword-length of Γ_D, and let $H_D(S)$ be the D-ary entropy of S. Answer the following true/false questions.

(a) If $D = 3$, then $L(S, \Gamma_D) = H_D(S)$.

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(b) $L(S, \Gamma_D) > H_D(S)$ for every $D > 3$.

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Solution:

(a) It is easy to check they are equal:

$$L(S, \Gamma_D) = 2\frac{1}{3} \times 1 + 3\frac{1}{9} \times 2 = \frac{4}{3} \tag{25}$$

$$H_D(S) = -2\frac{1}{3} \log \frac{1}{3} - 3\frac{1}{9} \log \frac{1}{9} = \frac{4}{3} \tag{26}$$

(b) For $D = 4$, it is easy to verify that $L(S, \Gamma_D) = \frac{11}{9} > \frac{4}{3} \log_4 3 = H_D(S)$. For $D > 4$, $H_D(S) < 1$, however $L(S, \Gamma_D) \geq 1$.