Problem 1.1.

Consider two random variables X and Y, where $X \in \{0,1\}$ and $Y \in \{A,B,C\}$. Let their joint distribution be given by the following table:

	$Y=A$	$Y=B$	$Y=C$
$X=0$	$1/16$	$1/8$	$1/16$
$X=1$	$1/8$	$1/8$	$1/2$

1. Find the marginal distributions of X and Y, respectively.
2. Find the conditional distributions of X given $Y=B$ and given $Y=C$, respectively.
3. Find the conditional distribution of Y given $X=1$.

Problem 1.2.

Roll a fair dice twice and define the sample space $\Omega = \{\omega = (\omega_1, \omega_2) \in \{1,\dots,6\}^2\}$.

1. Compute the following probabilities:
   (a) $p(\{\omega \in \Omega : \omega_1 = \omega_2\})$. Note: in the future, when there is no risk of confusion, we will use the short-hand notation $p(\omega_1 = \omega_2)$ to denote $p(\{\omega \in \Omega : \omega_1 = \omega_2\})$.
   (b) $p(\omega_1 < \omega_2)$. *Hint:* Draw Ω (in two dimensions) and try to identify the events $\{\omega \in \Omega : \omega_1 < \omega_2\}$, $\{\omega \in \Omega : \omega_1 > \omega_2\}$ and $\{\omega \in \Omega : \omega_1 = \omega_2\}$.
   (c) $p(\omega_1 \leq \omega_2)$.
2. For this question we need to introduce random variables. Let $X_1 : \Omega \to \mathbb{R}$ be the random variable defined by $X_1((\omega_1,\omega_2)) = \omega_1$ and $X_2 : \Omega \to \mathbb{R}$ be defined by $X_2((\omega_1,\omega_2)) = \omega_2$. Furthermore, let $Y_1 = X_1 \pmod 6$, $Y_2 = X_2 \pmod 6$ and $Y = (Y_1 + Y_2) \pmod 6$. Consider the three events $E_1 = \{Y_1 = Y_2\}$ and $E_2 = \{Y_1 = Y\}$. (Once again, we are using a short-hand notation: $E_1 = \{Y_1 = Y_2\}$ really means $E_1 = \{\omega \in \Omega : Y_1(\omega) = Y_2(\omega)\}$, etc.)
   Are E_1 and E_2 independent?

Problem 1.3.

Lisa has 2 coins in her pocket: a fair coin A which produces with equal probability heads and tails (H,T) and a fake coin B which produces only tails (T). She picks at random one of the coins and flips it twice. The sample space is $\Omega = \{\omega = (\omega_0, \omega_1, \omega_2) \in \{A,B\} \times \{H,T\}^2\}$. Consider the random variable $X = (X_0, X_1, X_2)$ where $X_i(\omega) = \omega_i$ for $i = 0,1,2$.

1. List the values of X that have non-zero probability.
2. We know that Lisa chose the coin A. For all possible values of s_1 and s_2, compute $p(X_1 = s_1 | X_0 = A)$, $p(X_2 = s_2 | X_0 = A)$ and $p(\{X_1 = s_1\} \cap \{X_2 = s_2\} | X_0 = A)$. (In order to avoid heavy notations, we use curly brackets only when necessary. That is, $X_0 = A$ refers to the event $\{X_0 = A\} = \{\omega \in \Omega : X_0(\omega) = A\}$, etc.)
3. Now we assume that we do not know which coin was chosen. For all possible values of s_1 and s_2, compute $p(X_1 = s_1)$, $p(X_2 = s_2)$ and $p(\{X_1 = s_1\} \cap \{X_2 = s_2\})$. Are the

events $E_1 = \{X_1 = H\}$ and $E_2 = \{X_2 = H\}$ independent?

Problem 1.4.

Flip a fair coin 4 times, and denote the outcomes $X = (X_0, X_1, X_2, X_3)$. Let $Y = (X_0, X_1)$. Considering that heads is encoded as the digit 0 and tails as the digit 1, the outcome can be seen as a binary string which encodes an integer $X' = X_3 \cdot 8 + X_2 \cdot 4 + X_1 \cdot 2 + X_0$. Let $Z = X'(\mathrm{mod} 5)$.
Compute $H(X)$, $H(Y)$ and $H(Z)$.

Problem 1.5.

In class we have seen that two events A and B are independent if $p(A|B) = p(A)$, or equivalently, if $P(A \cap B) = P(A)P(B)$. This second definition is symmetric in A and B. It can be extended to three or more events. Indeed, we say that three events A, B, and C are independent (also sometimes called mutually independent) if

a) $P(A \cap B) = P(A)P(B)$ and $P(A \cap C) = P(A)P(C)$ and $P(B \cap C) = P(B)P(C)$.

b) $P(A \cap B \cap C) = P(A)P(B)P(C)$.

In this homework problem, we show by counter-example that both conditions must be imposed: Neither does a) imply b), nor does b) imply a). For terminology, when A, B and C only satisfy a), they are called *pairwise independent* events.

1. Show by counter-example that a) does **not** imply b).
   (Construct a concrete setting: extraction of balls, throw of dice, flips of coins with three events A, B, C such that a) holds but b) does not);
2. Show by counter-example that b) does **not** imply a).
   (Construct a concrete setting with three events A, B, C such that b) holds but a) does not);
