Problem 12.1.

Consider a $(6,3)$ binary linear code where the message bits x_1, x_2, x_3 are encoded into the codeword bits $c_1, c_2, \dots, c_6$ as follows:

$$
\begin{aligned}
c_1 &= x_1 \\
c_2 &= x_2 \\
c_3 &= x_3 \\
c_4 &= x_1 + x_2 \\
c_5 &= x_2 + x_3 \\
c_6 &= x_1 + x_2 + x_3
\end{aligned}
$$

1. Find a generator matrix of the code.
2. Find a parity-check matrix of the code.
3. What is the minimum distance of the code?
4. Upon transmission over the erasure channel, we have received the word $[1, 0, ?, ?, 1, 0]$. Can we find the transmitted codeword? If yes, compute it.
5. Assume that we receive the words $y_1 = [1, 0, 1, 0, 0, 1]$ and $y_2 = [1, 0, 1, 1, 1, 1]$ and they both have 1 erroneous bit. Determine the corresponding transmitted codewords c_1 and c_2, and also the corresponding message bits x_1 and x_2 corresponding to the two codewords.

Problem 12.2.

Consider a linear code $\mathcal{C}$ on $\mathbb{F}_{13}$ defined by the following generator matrix:

$$
G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 12 & 4 & 2 & 6 \\ 0 & 1 & 3 & 4 & 10 \end{pmatrix}.
$$

1. Check that G is indeed a generator matrix.
2. Compute a parity-check matrix H for the generator matrix G.
3. Describe an algorithm to compute the minimum distance of this code. Is $\mathcal{C}$ an MDS code?
4. The goal of this question is to derive a decoder which allows to correct all the errors of weight 1.
   (a) Suppose we have received the word $\vec{y} \in \mathbb{F}_{13}^5$. Let $\vec{s} = \vec{y}H^T$ be its corresponding syndrome.
   Consider the following received words:
   $\vec{y}_1 = (0, 7, 11, 12, 10)$,
   $\vec{y}_2 = (8, 12, 11, 12, 10)$,
   $\vec{y}_3 = (0, 7, 8, 12, 10)$.
   Compute their corresponding syndromes. Are $\vec{y}_1, \vec{y}_2, \vec{y}_3$ valid codewords?

(b) Let $\mathcal{E}_1 = \{\vec{e} \in \mathbb{F}_{13}^5, w(\vec{e}) = 1\}$ be the set of vectors of weight 1. What is its cardinality?

(c) Is $\mathcal{E}_1$ a vector space?

(d) Consider the function

$$\begin{aligned} f : \mathcal{E}_1 &\to \mathbb{F}_{13}^2 \\ \vec{e} &\mapsto \vec{e}H^T \end{aligned}$$

Show that f is injective.

(e) Let $\mathcal{F}_1$ be the image of f. What is the cardinality of $\mathcal{F}_1$?

(f) Is f bijective?

(g) Suppose that $\vec{y}$ was obtained by transmitting a codeword $\vec{x}$ over an error channel. Let $\vec{s} = \vec{y}H^T$ be the corresponding syndrome and let $\vec{e} = \vec{y} - \vec{x}$ be the error vector. Show that $\vec{s} = \vec{e}H^T$. Show that if the error is of weight 1, then $\vec{s} \in \mathcal{F}_1$.

(h) Describe how to implement a decoder which is able to correct all errors of weight 1. *(Hint: you can make use of a lookup table.)*

(i) Suppose that $\vec{y}$ is obtained by transmitting a codeword $\vec{x}$ over an error channel. Suppose that the error $\vec{e} = \vec{y} - \vec{x}$ has weight 2. Is it possible that the syndrome $\vec{s} = \vec{y}H^T$ belongs to $\mathcal{F}_1$? If yes, given an example, or otherwise prove that it is impossible.

Problem 12.3.

1. Let G_1 be a generator matrix for a (n_1, k) linear code with $d_{\min} = d_1$ and G_2 be a generator matrix for a (n_2, k) linear code with $d_{\min} = d_2$.

   (a) Let $G = [G_1 || G_2]$ denote the horizontal concatenation of the two generator matrices. Show that G is a generator matrix for a linear code.

   (b) We denote by d_G the minimum distance of the code generated by G. How does d_G relate to d_1 and d_2?

2. Let G_1 be a generator matrix for a (n_1, k_1) linear code with $d_{\min} = d_1$ and G_2 be a generator matrix for a (n_2, k_2) linear code with $d_{\min} = d_2$.

   (a) Show that $G = \begin{bmatrix} G_1 & \mathbf{0} \\ \mathbf{0} & G_2 \end{bmatrix}$ is a generator matrix for a linear code.

   (b) We denote by d_G the minimum distance of the code generated by G. How does d_G relate to d_1 and d_2?

Problem 12.4.

1. Prove that if a linear (n, k) code $\mathcal{C}$ has minimum distance d_{min} and parity-check matrix H, then any set of $d_{min} - 1$ columns of H are linearly independent.

2. Consider a binary linear code $\mathcal{C}$ with the following parity-check matrix:

$$H = \begin{bmatrix} a & 1 & b & 1 & 0 & 0 & 0 \\ c & 0 & d & 0 & 1 & 0 & 0 \\ e & 1 & f & 0 & 0 & 1 & 0 \\ g & 1 & h & 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the missing entries of H knowing that the code has $d_{min} = 4$ and that $x = [1011010]$ is a codeword in $\mathcal{C}$.