EPFL

Spring 2025

07 Static Traffic Assignment: Extensions

CIVIL-477 Transportation network modeling & analysis

École polytechnique fédérale de Lausanne

Review

- Static traffic assignment (TA)
  - Predict aggregate traffic flows (link and path flows) on a given transportation network and demand pattern
  - Depending on the routing principle, lead to different type of equilibrium states (UE and SO) with diverse system efficiency (PoA)
  - The equilibrium states are transferable by changing the link cost function (congestion pricing)
- ***Q: What are the key assumptions?***

Review

- Assumptions and extensions of TA
  - Traffic flows are assigned at one shot
    - Dynamic traffic assignment (DTA) ⇒ **not covered in this course**
  - Travelers are rational and have perfect information about traffic conditions
    - Stochastic traffic assignment (STA) ⇒ **next lecture**
    - Day-to-day traffic assignment ⇒ **not covered in this course**
  - Travel demand is fixed
    - Elastic demand
  - Travelers are the same
    - Heterogeneous users
  - Link flows are unbounded
    - Side constraints

(Travel demand is fixed / Travelers are the same / Link flows are unbounded) ⇒ **this lecture**

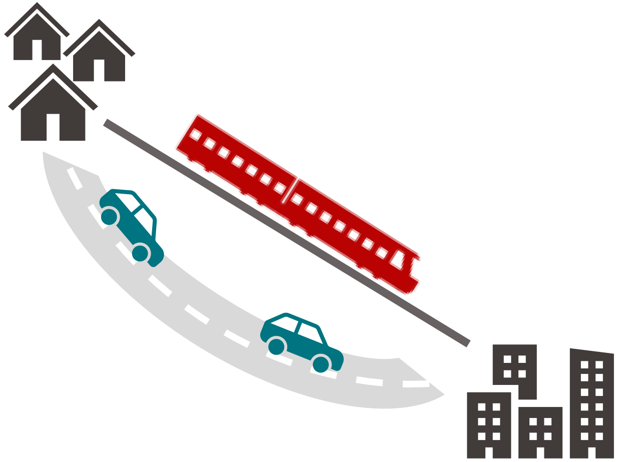
Outline

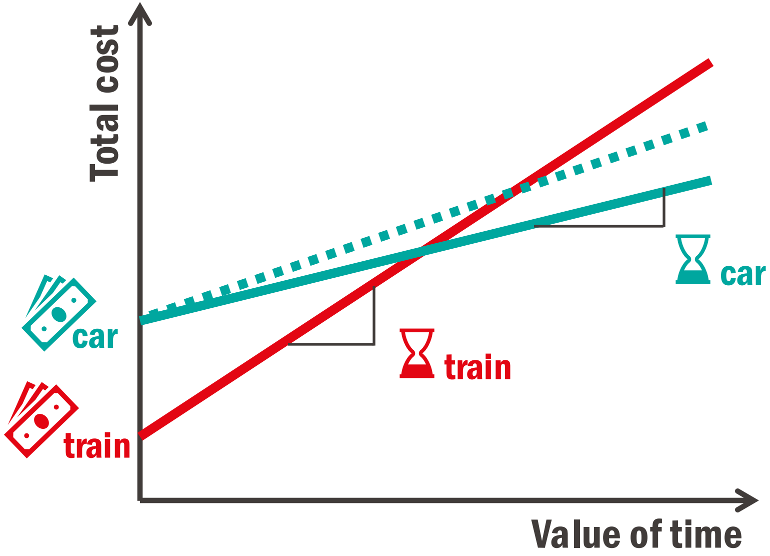
- Elastic demand
- Side constraints
- Heterogeneous users

Elastic demand

- Recall the mode choice problem of daily commute
  - When the highway gets congested, travelers would switch to train
  - Driving demand is elastic with respect to the traffic condition

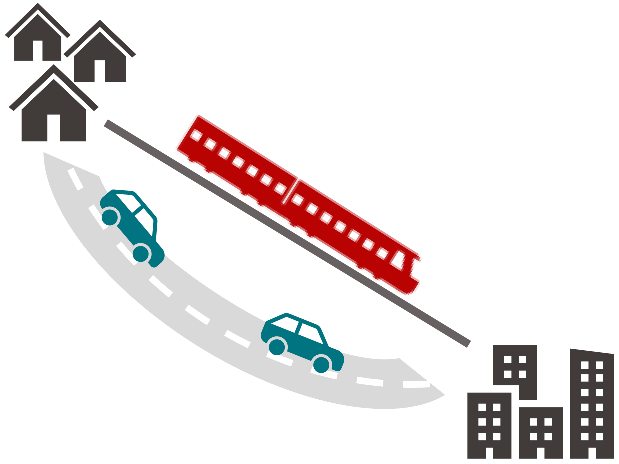
  - ***Q: What would happen if a toll is charged on highway?***

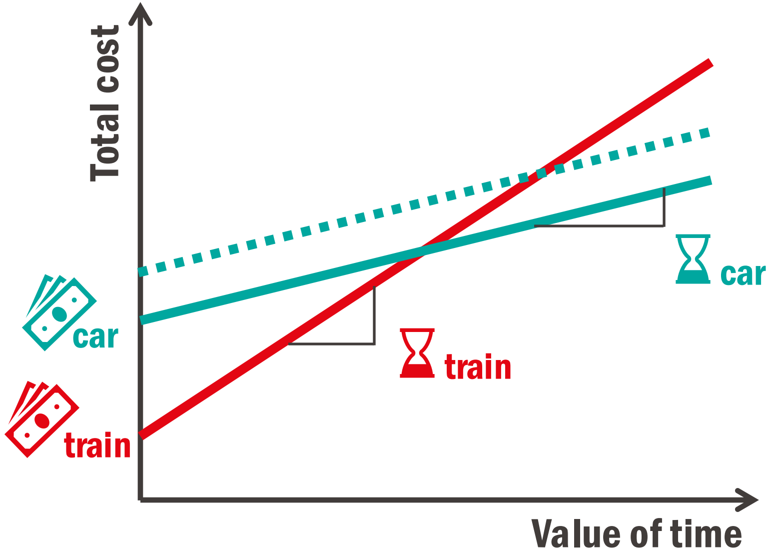




Elastic demand

- Recall the mode choice problem of daily commute
  - When the highway gets congested, travelers would switch to train
  - Driving demand is elastic with respect to the **total travel cost**



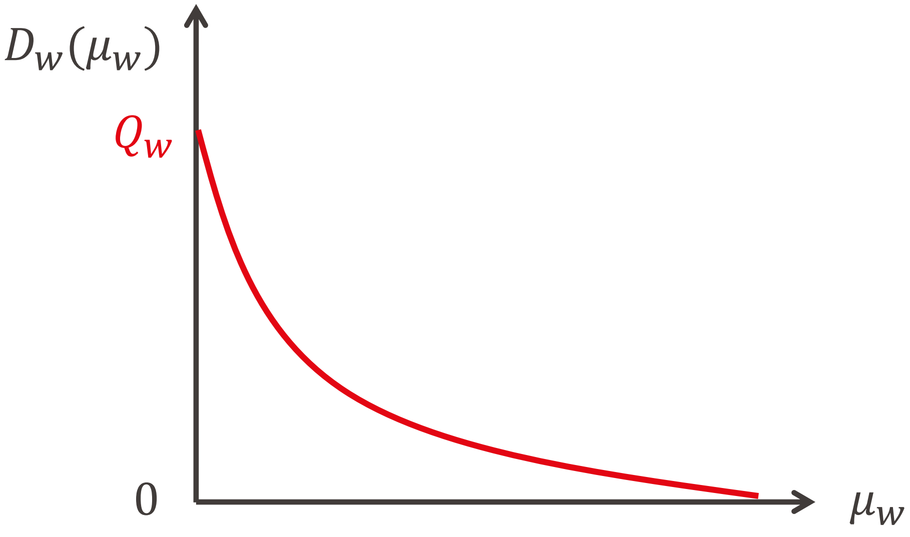


Elastic demand

- Let $D_w(\cdot)$ be the demand function of OD pair $w \in W$. Given the min travel cost μ_w, the travel demand is

$$q_w = D_w(\mu_w)$$

  - Bounded between 0 and max demand Q_w
  - Non-increasing with μ_w

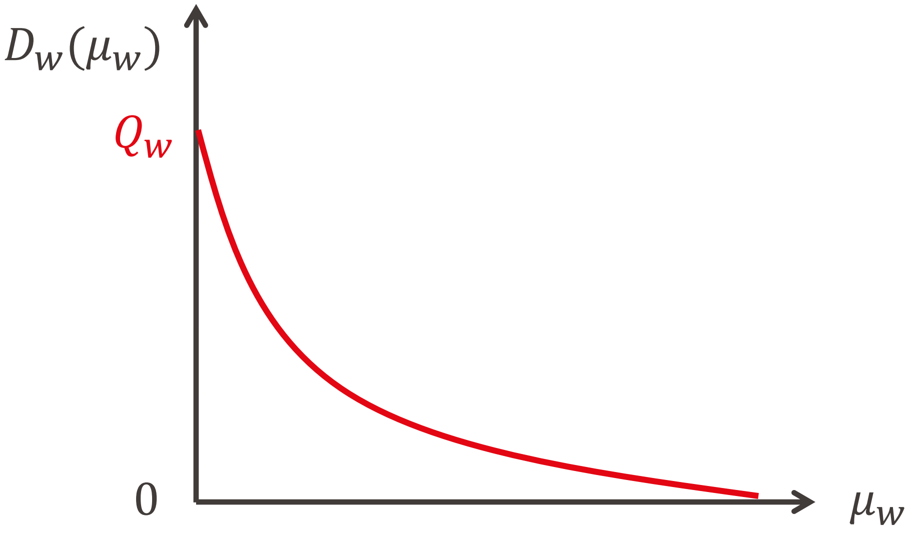


Elastic demand

- Let $D_w(\cdot)$ be the demand function of OD pair $w \in W$. Given the min travel cost μ_w, the travel demand is

$$q_w = D_w(\mu_w)$$

  - Bounded between 0 and max demand Q_w
  - **Strictly decreasing** with μ_w, then the inverse exists $\mu_w = D_w^{-1}(q_w)$



Elastic demand

- Traffic equilibrium conditions
  - A path flow $\mathbf{f}^* \in \Omega_{\mathbf{f}}$ such that $\forall w \in W, k \in P_w$,

$$f_k^*(c_k^* - \mu_w^*) = 0, \qquad c_k^* \geq \mu_w^*$$

where

- c_k^*: cost of path k at $\mathbf{f}^*$
- $\Omega_{\mathbf{f}}$: set of feasible paths

- ***Q: How to integrate elastic demand into the equilibium conditions?***

Elastic demand

- Traffic equilibrium conditions with elastic demand
  - A path flow $\mathbf{f}^* \in \Omega_{\mathbf{f}}$ such that $\forall w \in W$,
    - if $q_w > 0$, then $\mu_w^* = D_w^{-1}(q_w)$ and $\forall k \in P_w$

$$f_k^*(c_k^* - \mu_w^*) = 0, \qquad c_k^* \geq \mu_w^*$$

    - otherwise, $\mu_w^* > D_w^{-1}(0)$ and $f_k^* = 0, \forall k \in P_w$

where

- c_k^*: cost of path k at $\mathbf{f}^*$
- $\Omega_{\mathbf{f}}$: set of feasible paths

- ***Q: Is there an equivalent optimization as the classic model?***

Elastic demand

- Traffic equilibrium conditions with elastic demand
  - A path flow $\mathbf{f}^* \in \Omega_\mathbf{f}$ such that $\forall w \in W$,
    - if $q_w > 0$, then $\mu_w^* = D_w^{-1}(q_w)$ and $\forall k \in P_w$

$$f_k^*(c_k^* - \mu_w^*) = 0, \qquad c_k^* \geq \mu_w^*$$

    - otherwise, $\mu_w^* > D_w^{-1}(0)$ and $f_k^* = 0, \forall k \in P_w$

- Equivalent KKT conditions

$$\begin{aligned} (\mathbf{f}^*)^T(\mathbf{c}^* - \Lambda^T \boldsymbol{\mu}^*) &= 0 \\ \mathbf{c}^* - \Lambda^T \boldsymbol{\mu}^* &\geq \mathbf{0} \\ \Lambda \mathbf{f}^* &= \mathbf{q}^* \\ \mathbf{f}^* &\geq \mathbf{0} \\ (\mathbf{q}^*)^T(D^{-1}(\mathbf{q}^*) - \boldsymbol{\mu}^*) &= 0 \\ \boldsymbol{\mu}^* - D^{-1}(\mathbf{q}^*) &\geq \mathbf{0} \\ \mathbf{q}^* &\geq 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \nabla_\mathbf{f} \mathcal{L} = \mathbf{c}^* - \boldsymbol{\lambda}^* - \Lambda^T \boldsymbol{\mu}^* &= 0 \\ \boldsymbol{\lambda}^* \geq 0, \qquad (\boldsymbol{\lambda}^*)^T \mathbf{f}^* &= 0 \\ \Lambda \mathbf{f}^* &= \mathbf{q}^* \\ \mathbf{f}^* &\geq \mathbf{0} \\ \nabla_\mathbf{q} \mathcal{L} = -D^{-1}(\mathbf{q}^*) - \boldsymbol{\rho}^* + \boldsymbol{\mu}^* &= 0 \\ \boldsymbol{\rho}^* \geq 0, \qquad (\boldsymbol{\rho}^*)^T \mathbf{q}^* &= 0 \end{aligned}$$

Elastic demand

- Equivalent optimization problem

$$\min_{\mathbf{f},\mathbf{q}} \; Z(\mathbf{f},\mathbf{q}) = \sum_{a\in A}\int_0^{x_a} t_a(u)\,\mathrm{d}u - \sum_{w\in W}\int_0^{q_w} D_w^{-1}(u)\,\mathrm{d}u$$

$$s.t. \quad \Lambda\mathbf{f} = \mathbf{q}$$

$$\Delta\mathbf{f} = \mathbf{x}$$

$$\mathbf{f},\mathbf{q} \geq \mathbf{0}$$

- ***Q: Can you prove it yields the same KKT conditions?***

Elastic demand

- Alternative formulation I
  - Similar to link flow $\mathbf{x}$, demand flow $\mathbf{q}$ is an intermediate variable that can be represented as a linear combination of path flow $\mathbf{f}$

$$\min_{\mathbf{f}} \; Z(\mathbf{f}) = \sum_{a \in A} \int_0^{\sum_{k \in P} \delta_{ak} f_k} t_a(u)\, \mathrm{d}u - \sum_{w \in W} \int_0^{\sum_{k \in P_w} f_k} D_w^{-1}(u)\, \mathrm{d}u$$

$$s.t. \qquad \mathbf{f} \geq \mathbf{0}$$

where

- P_w: set of paths connecting OD pair $w \in W$
- P: set of all paths
- δ_{ak}: binary indicator of link-path relationship (element in Δ)

Elastic demand

- Alternative formulation II
  - A VI formulation also exists

Find $(\mathbf{x}^*, \mathbf{q}^*) \in \Omega_{\mathbf{x}} \times \Omega_{\mathbf{q}}$ such that

$$\langle \mathbf{t}(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle - \langle D^{-1}(\mathbf{q}^*), \mathbf{q} - \mathbf{q}^* \rangle \geq 0, \qquad \forall \mathbf{x} \in \Omega_{\mathbf{x}}, \mathbf{q} \in \Omega_{\mathbf{q}}$$

where

- $\Omega_{\mathbf{x}} = \{\mathbf{x} | \Delta\mathbf{f} = \mathbf{x}, \Lambda\mathbf{f} = \mathbf{q}, \mathbf{f} \geq 0\}$: feasible set of link flows
- $\Omega_{\mathbf{q}} = \{\mathbf{q} | \mathbf{0} \leq \mathbf{q} \leq \mathbf{Q}\}$: feasible set of OD demand

Elastic demand

- Alternative formulation III
  - Instead of solving the realized demand $\mathbf{q}$, solving the residual demand

$$\mathbf{e} = \mathbf{Q} - \mathbf{q},$$

where $\mathbf{Q}$ is the max demand flow

$$\min_{\mathbf{f},\mathbf{e}} \; Z(\mathbf{f},\mathbf{e}) = \sum_{a \in A} \int_0^{x_a} t_a(u)\,\mathrm{d}u - \sum_{w \in W} \int_0^{Q_w - e_w} D_w^{-1}(u)\,\mathrm{d}u$$

$$s.t. \quad \Lambda \mathbf{f} + \mathbf{e} = \mathbf{Q}$$

$$\Delta \mathbf{f} = \mathbf{x}$$

$$\mathbf{f}, \mathbf{e} \geq \mathbf{0}$$

- ***Q: How to present the second integral in a different way?***

Elastic demand

- Alternative formulation III
  - Instead of solving the realized demand $\mathbf{q}$, solving the residual demand

$$\mathbf{e} = \mathbf{Q} - \mathbf{q},$$

where $\mathbf{Q}$ is the max demand flow

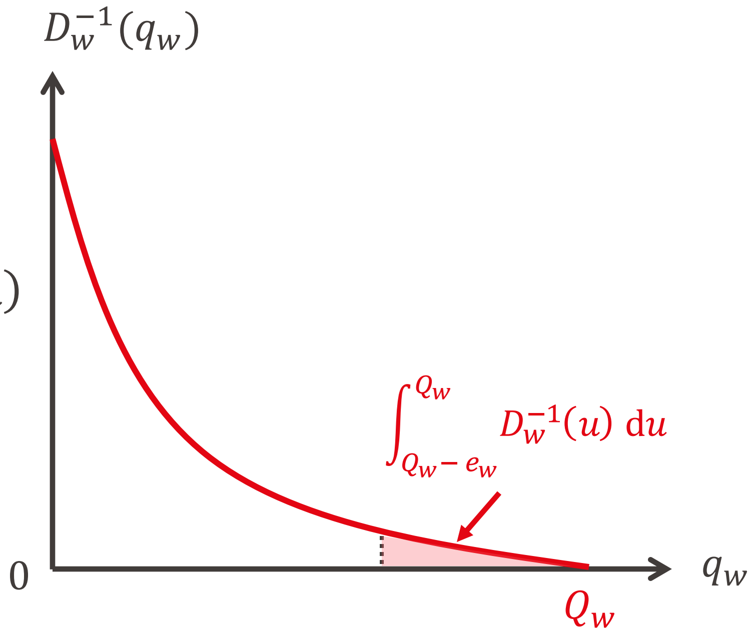
$$\int_0^{Q_w - e_w} D_w^{-1}(u)\,\mathrm{d}u$$

$$= \int_0^{Q_w} D_w^{-1}(u)\,\mathrm{d}u - \int_{e_w}^{0} D_w^{-1}(Q_w - u)\,\mathrm{d}(-u)$$

$$= \int_0^{Q_w} D_w^{-1}(u)\,\mathrm{d}u - \int_0^{e_w} \tilde{t}_w(u)\,\mathrm{d}u$$

where $\tilde{t}_w(u) = D_w^{-1}(Q_w - u)$

- strictly increasing when D_w^{-1} is strictly decreasing



Elastic demand

- Alternative formulation III
  - Instead of solving the realized demand $\mathbf{q}$, solving the residual demand

$$\mathbf{e} = \mathbf{Q} - \mathbf{q},$$

where $\mathbf{Q}$ is the max demand flow

$$\min_{\mathbf{f},\mathbf{e}} \; Z(\mathbf{f},\mathbf{e}) = \sum_{a\in A} \int_0^{x_a} t_a(u)\,\mathrm{d}u - \sum_{w\in W} \left(\int_0^{Q_w} D_w^{-1}(u)\,\mathrm{d}u - \int_0^{e_w} \tilde{t}_w(u)\,\mathrm{d}u \right)$$

constant, thus safely dropped

$$s.t. \quad \Lambda\mathbf{f} + \mathbf{e} = \mathbf{Q}$$

$$\Delta\mathbf{f} = \mathbf{x}$$

$$\mathbf{f},\mathbf{e} \geq \mathbf{0}$$

Elastic demand

- Alternative formulation III
  - Instead of solving the realized demand $\mathbf{q}$, solving the residual demand

$$\mathbf{e} = \mathbf{Q} - \mathbf{q},$$

where $\mathbf{Q}$ is the max demand flow

$$\min_{\mathbf{f},\mathbf{e}} \; Z(\mathbf{f},\mathbf{e}) = \sum_{a\in A} \int_0^{x_a} t_a(u)\,\mathrm{d}u + \sum_{w\in W} \int_0^{e_w} \tilde{t}_w(u)\,\mathrm{d}u$$

$$s.t. \quad \Lambda\mathbf{f} + \mathbf{e} = \mathbf{Q}$$

$$\Delta\mathbf{f} = \mathbf{x}$$

$$\mathbf{f},\mathbf{e} \geq \mathbf{0}$$

- ***Q: What is the physical meaning of $\tilde{t}_w$?***

Elastic demand

- Alternative formulation III
  - Instead of solving the realized demand $\mathbf{q}$, solving the residual demand

$$\mathbf{e} = \mathbf{Q} - \mathbf{q},$$

where $\mathbf{Q}$ is the max demand flow

$$\min_{\mathbf{f},\mathbf{e}} \; Z(\mathbf{f}, \mathbf{e}) = \sum_{a \in A} \int_0^{x_a} t_a(u)\,\mathrm{d}u + \sum_{w \in W} \int_0^{e_w} \tilde{t}_w(u)\,\mathrm{d}u$$

$$s.t. \quad \Lambda \mathbf{f} + \mathbf{e} = \mathbf{Q}$$

$$\Delta \mathbf{f} = \mathbf{x}$$

$$\mathbf{f}, \mathbf{e} \geq \mathbf{0}$$

  - Equivalent to create a “virtual” link between each OD pair and link cost $\tilde{t}_w$
    - reduce to classic TA with extended link and path sets and link cost functions

Elastic demand

- Alternative formulation III

$$\min_{\mathbf{x}} \quad z(\mathbf{x}) = \sum_{a \in A} \int_0^{x_a} t_a(u)\,\mathrm{d}u + \sum_{w \in W} \int_0^{x_w} \tilde{t}_w(u)\,\mathrm{d}u$$

$$s.t. \quad \mathbf{x} \in \Omega_{\mathbf{x}} = \{\mathbf{x} | \widetilde{\Delta}\tilde{\mathbf{f}} = \mathbf{x}, \widetilde{\Lambda}\tilde{\mathbf{f}} = \mathbf{Q}, \tilde{\mathbf{f}} \geq 0\}$$

where

- $\tilde{\mathbf{f}}$: extended definition of path flow
- $\widetilde{\Delta}$, $\widetilde{\Lambda}$: extended definitions of link-path and OD-path matrices

- ***Q: Is this optimization problem convex? Is the solution unique?***

Elastic demand

- Alternative formulation III

$$\min_{\mathbf{x}} \; z(\mathbf{x}) = \sum_{a\in A} \int_0^{x_a} t_a(u)\,\mathrm{du} + \sum_{w\in W} \int_0^{x_w} \tilde{t}_w(u)\,\mathrm{du}$$

$$s.t. \quad \mathbf{x} \in \Omega_{\mathbf{x}} = \{\mathbf{x} | \widetilde{\Delta}\tilde{\mathbf{f}} = \mathbf{x}, \widetilde{\Lambda}\tilde{\mathbf{f}} = \mathbf{Q}, \tilde{\mathbf{f}} \geq 0\}$$

  - Feasible set $\Omega_{\mathbf{x}}$ is convex
  - Hessian matrix $H_{\mathbf{x}} = \nabla^2 Z(\mathbf{x})$ is definite positive
    - diagonal matrix with all positive elements

$$\frac{\partial^2}{\partial x_w^2} Z(\mathbf{x}) = \frac{\partial}{\partial x_w} D_w^{-1}(Q_w - x_w) = -\frac{\partial D_w^{-1}(q_w)}{\partial q_w} > 0$$

$$\frac{\partial^2}{\partial x_w x_{w'}} Z(\mathbf{x}) = \frac{\partial^2}{\partial x_w x_a} Z(\mathbf{x}) = 0$$

Elastic demand

- Solution method
  - Given the similarity of link-based formulation (Alt. III), same solution algorithms for standard traffic assignment (e.g., FW, PG) can be utilized
  - PG can be more easily implemented with path-based formulation (Alt. I) as there is no more demand conservation constraint
  - Revised FW algorithm with demand update
    - Update direction $(\mathbf{y}, \mathbf{p})$
      - Solve shortest-path between each OD $w \in W$ and get min path cost $\mu_w, w \in W$
      - Compute $p_w = D_w^{-1}(\mu_w), w \in W$ to get targe demand $\mathbf{p}$
      - Perform all-or-nothing assignment with demand $\mathbf{p}$ to get target link flow $\mathbf{y}$

Questions?

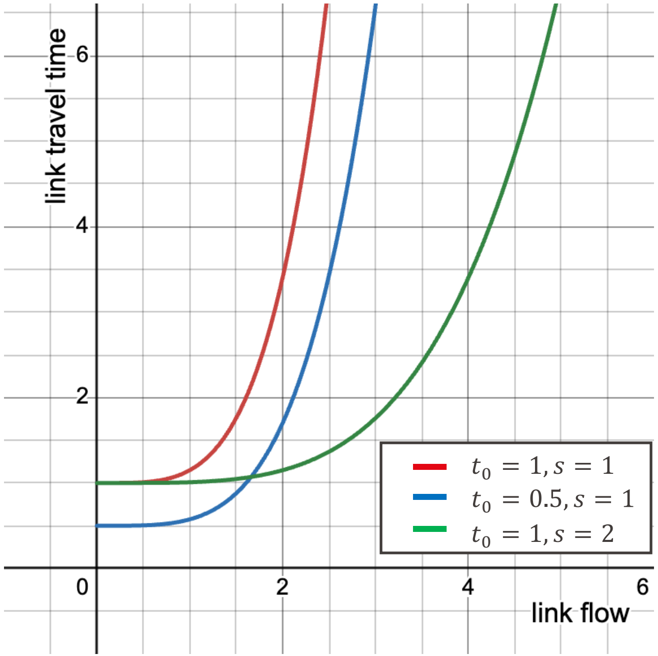
Side constraints

- Recall the link cost function (e.g., BPR)
  - Link flow only has a lower bound 0 that corresponds to the free-flow travel time

$$t(x) = t_0\left[1 + 0.15\left(\frac{x}{s}\right)^4\right]$$

where

- t_0: free-flow travel time (hr)
- s: saturation flow, or "capacity" (veh/hr)



  - Roads do have physical capacities
  - Jointly capacity constraints also exist
    - e.g., intersections

Side constraints

- Capacitated traffic assignment (CAP)
  - Consider independent link capacity $\mathbf{C} = \{C_a\}_{\forall a \in A}$

$$
\begin{aligned}
\min_{\mathbf{f}} \quad & Z(\mathbf{f}) \\
s.t. \quad & \Lambda\mathbf{f} = \mathbf{q} \\
& \Delta\mathbf{f} \leq \mathbf{C} \\
& \mathbf{f} \geq \mathbf{0}
\end{aligned}
$$

- ***Q: What are the KKT conditions of this problem?***

Side constraints

- Capacitated traffic assignment (CAP)
  - Consider independent link capacity $\mathbf{C} = \{C_a\}_{\forall a \in A}$

- Lagrangian $\mathcal{L}(\mathbf{f}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) = Z(\mathbf{f}) - \boldsymbol{\lambda}^T \mathbf{f} - \boldsymbol{\mu}^T (\Lambda \mathbf{f} - \mathbf{q}) + \boldsymbol{\nu}^T (\Delta \mathbf{f} - \mathbf{C})$

- KKT conditions

$$
\begin{aligned}
\nabla_{\mathbf{f}} \mathcal{L} = \mathbf{c}^* - \boldsymbol{\lambda}^* - \Lambda^T \boldsymbol{\mu}^* + \Delta^T \boldsymbol{\nu}^* &= 0 \\
\boldsymbol{\lambda}^* \geq 0, \qquad (\boldsymbol{\lambda}^*)^T \mathbf{f}^* &= 0 \\
\Lambda \mathbf{f}^* &= \mathbf{q} \\
\mathbf{f}^* &\geq \mathbf{0} \\
\boldsymbol{\nu}^* \geq 0, \qquad (\boldsymbol{\nu}^*)^T (\Delta \mathbf{f}^* - \mathbf{C}) &= 0 \\
\Delta \mathbf{f} &\leq \mathbf{C}
\end{aligned}
$$

- ***Q: How to interpret dual variable ν^*?***

Side constraints

- Capacitated traffic assignment (CAP)
  - Consider independent link capacity $\mathbf{C} = \{C_a\}_{\forall a \in A}$

- Lagrangian $\mathcal{L}(\mathbf{f}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) = Z(\mathbf{f}) - \boldsymbol{\lambda}^T\mathbf{f} - \boldsymbol{\mu}^T(\Lambda\mathbf{f} - \mathbf{q}) + \boldsymbol{\nu}^T(\Delta\mathbf{f} - \mathbf{C})$

- KKT conditions
  - Consider v_a as queueing delay at link $a \in A$
  - Define $\bar{\mathbf{c}} = \mathbf{c} + \Delta^T\boldsymbol{\nu}$ as general path costs

$$\nabla_{\mathbf{f}}\mathcal{L} = \bar{\mathbf{c}}^* - \boldsymbol{\lambda}^* - \Lambda^T\boldsymbol{\mu}^* = 0 \qquad \Longleftrightarrow \qquad (\mathbf{f}^*)^T(\bar{\mathbf{c}}^* - \Lambda^T\boldsymbol{\mu}^*) = 0$$

$$\boldsymbol{\lambda}^* \geq 0, \qquad (\boldsymbol{\lambda}^*)^T\mathbf{f}^* = 0 \qquad \qquad \bar{\mathbf{c}}^* \geq \Lambda^T\boldsymbol{\mu}^*$$

equilibrium conditions based on $\bar{\mathbf{c}}$

$$\Lambda\mathbf{f}^* = \mathbf{q}$$

$$\mathbf{f}^* \geq \mathbf{0}$$

$$\boldsymbol{\nu}^* \geq 0, \qquad (\boldsymbol{\nu}^*)^T(\Delta\mathbf{f}^* - \mathbf{C}) = 0 \qquad \Longleftrightarrow$$

queue only emerges if flow reaches capacity

$$\Delta\mathbf{f} \leq \mathbf{C}$$

Side constraints

- Solution method
  - Although the solution properties remain the same as the addition constraints are also linear, solving the problem becomes more challenging
    - e.g., all-or-nothing assignment in FW may not ensure a feasible target link flow

$$\min_{\mathbf{f}} \; Z(\mathbf{f})$$
$$s.t. \; \Lambda\mathbf{f} = \mathbf{q}$$
$$\Delta\mathbf{f} \leq \mathbf{C}$$
$$\mathbf{f} \geq \mathbf{0}$$

  - Main idea: relax constraints and introduce penalty
    - Barrier method
    - Lagrange Multiplier method

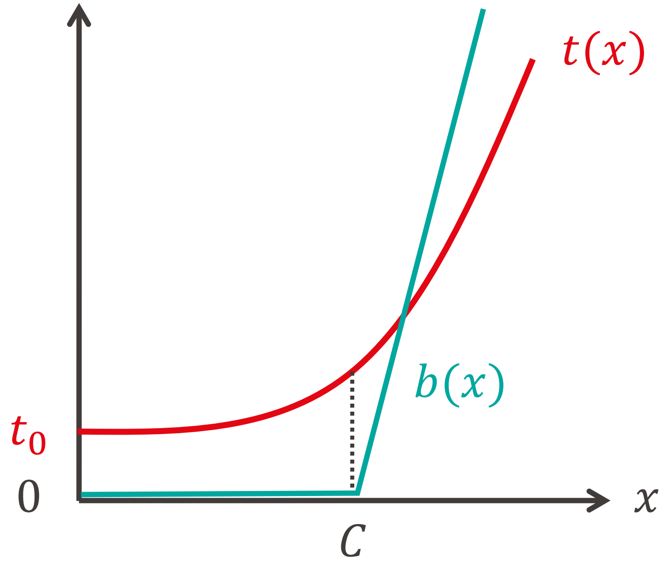
Side constraints

- Barrier method
  - Add a barrier function to each link cost function such that link cost approaches to infinity when link flow exceeds capacity
  - The solution algorithms for standard TA problem can be utilized to solve the relaxed problem and the solution converges to (CAP) as $\gamma \rightarrow 0$

$$\hat{t}(x) = t(x) + \gamma b(x, C)$$

where

- $b(\cdot)$: barrier function
- γ: weighting parameter



Side constraints

- Lagrange Multiplier method
  - Solve the relaxed problem with fixed queuing delay $\boldsymbol{\nu}$ and update $\boldsymbol{\nu}$ based on constraint deviations

  - At iteration n,
    - Solve relaxed TA with $\boldsymbol{\nu}^n$

$$\mathbf{x}^{n+1} = \arg\max_{\mathbf{x}\in\Omega_{\mathbf{x}}} \mathcal{L}(\mathbf{x}, \boldsymbol{\nu}^n) = z(\mathbf{x}) + (\boldsymbol{\nu}^n)^T(\mathbf{x} - \mathbf{C})$$

      - unique solution exists as $\mathcal{L}(\mathbf{x}, \boldsymbol{\nu}^n)$ is strictly convex
    - Update queueing delay based on $\mathbf{x}^{n+1}$

$$\boldsymbol{\nu}^{n+1} = \boldsymbol{\nu}^n + \eta[\mathbf{x}^{n+1} - \mathbf{C}]_+$$

      - increase ν_a if $x_a^{n+1} > C_a$ by the product of parameter $\eta > 0$ and exceed flow $x_a^{n+1} - C_a$

Questions?

Heterogeneous users

- Recall standard TA assumes homogenous users with
  - Same contribution to traffic congestion (e.g., car vs truck)
  - Same travel preference (e.g., VOT)
  - Same routing principle (e.g., selfish vs selfless)

Heterogeneous users

- Multiple vehicle type
  - Assume trucks (r) occupy twice road space as cars (v) and thus have doubled impacts on the link travel time

$$t(x_v, x_r) = t_0 \left[1 + 0.15 \left(\frac{x_v + 2x_r}{s} \right)^4 \right]$$

  - Asymmetric marginal impact

$$\frac{\partial}{\partial x_v} t(x_v, x_r) = \frac{0.6 t_0}{s} \left(\frac{x_v + 2x_r}{s} \right)^3$$

$$\frac{\partial}{\partial x_r} t(x_v, x_r) = \frac{1.2 t_0}{s} \left(\frac{x_v + 2x_r}{s} \right)^3$$

- ***Q: Is there an equivalent optimization as the standard TA?***

Heterogeneous users

- Multiple vehicle type
  - Assume trucks (r) occupy twice road space as cars (v) and thus have doubled impacts on the link travel time

$$t(x_v, x_r) = t_0\left[1 + 0.15\left(\frac{x_v + 2x_r}{s}\right)^4\right]$$

  - Presumed equivalent problem

$$\min_{\mathbf{f}_v,\mathbf{f}_r} Z(\mathbf{f}_v, \mathbf{f}_r) = \sum_{a\in A} \int_0^{x_{v,a}} t_a(u, x_{r,a})\,\mathrm{du} + \int_0^{x_{r,a}} t_a(x_{v,a}, u)\,\mathrm{du}$$

$$\begin{aligned} s.t. \quad \Delta(\mathbf{f}_v + \mathbf{f}_r) &= \mathbf{x} \\ \Lambda\mathbf{f}_v &= \mathbf{q}_v \\ \Lambda\mathbf{f}_r &= \mathbf{q}_r \\ \mathbf{f}_v, \mathbf{f}_r &\geq \mathbf{0} \end{aligned}$$

Heterogeneous users

- Multiple vehicle type
  - Assume trucks (r) occupy twice road space as cars (v) and thus have doubled impacts on the link travel time

$$t(x_v, x_r) = t_0\left[1 + 0.15\left(\frac{x_v + 2x_r}{s}\right)^4\right]$$

  - Presumed equivalent problem

$$\begin{aligned}
\nabla_{\mathbf{f}_v}\mathcal{L} = \nabla_{\mathbf{f}_v}Z - \boldsymbol{\lambda}_v^* - \Lambda^T\boldsymbol{\mu}_v^* &= 0\\
\boldsymbol{\lambda}_v^* \geq 0, \qquad (\boldsymbol{\lambda}_v^*)^T\mathbf{f}_v^* &= 0\\
\nabla_{\mathbf{f}_r}\mathcal{L} = \nabla_{\mathbf{f}_r}Z - \boldsymbol{\lambda}_r^* - \Lambda^T\boldsymbol{\mu}_r^* &= 0\\
\boldsymbol{\lambda}_r^* \geq 0, \qquad (\boldsymbol{\lambda}_r^*)^T\mathbf{f}_r^* &= 0\\
\Lambda\mathbf{f}_v^* = \mathbf{q}_v^*, \qquad \Lambda\mathbf{f}_r^* &= \mathbf{q}_r^*\\
\mathbf{f}_v^*, \mathbf{f}_r^* &\geq \mathbf{0}
\end{aligned}$$

equivalent to multi-class traffic equilibrium if

$$\nabla_{\mathbf{f}_v}Z(\mathbf{f}_v, \mathbf{f}_r) = \mathbf{c}$$
$$\nabla_{\mathbf{f}_r}Z(\mathbf{f}_v, \mathbf{f}_r) = \mathbf{c}$$

- ***Q: Does these equalities hold?***

Heterogeneous users

- Multiple vehicle type
  - Assume trucks (r) occupy twice road space as cars (v) and thus have doubled impacts on the link travel time

$$t(x_v, x_r) = t_0 \left[1 + 0.15 \left(\frac{x_v + 2x_r}{s} \right)^4 \right]$$

  - Presumed equivalent problem

$$\frac{\partial Z(\mathbf{f}_v, \mathbf{f}_r)}{\partial f_{v,k}} = \sum_{a \in A} \frac{\partial}{\partial f_{v,k}} \int_0^{\sum_k \delta_{ak} f_{v,k}} t_a(u, x_{r,a})\, \mathrm{d}u + \int_0^{x_{r,a}} \frac{\partial}{\partial f_{v,k}} t_a(x_{v,a}, u)\, \mathrm{d}u$$

$$= \sum_{a \in A} \delta_{ak} t_a(x_{v,a}, x_{r,a}) + \int_0^{x_{r,a}} \delta_{ak} \frac{\partial t_a(x_{v,a}, u)}{\partial x_{v,a}} \mathrm{d}u$$

$$= c_k + \sum_{a \in A} \delta_{ak} \int_0^{x_{r,a}} \frac{\partial t_a(x_{v,a}, u)}{\partial x_{v,a}} \mathrm{d}u$$

Heterogeneous users

- Multiple vehicle type
  - Assume trucks (r) occupy twice road space as cars (v) and thus have doubled impacts on the link travel time

$$t(x_v, x_r) = t_0\left[1 + 0.15\left(\frac{x_v + 2x_r}{s}\right)^4\right]$$

  - There is no equivalent optimization problem, but the equivalent VI problem always exits
    - Path-based

      Find feasible path flows $(\mathbf{f}_v^*, \mathbf{f}_r^*)$ such that

$$\langle \mathbf{c}(\mathbf{f}_v^*, \mathbf{f}_r^*), \mathbf{f}_v - \mathbf{f}_v^* \rangle + \langle \mathbf{c}(\mathbf{f}_v^*, \mathbf{f}_r^*), \mathbf{f}_t - \mathbf{f}_t^* \rangle \geq 0, \qquad \forall \mathbf{f}_v \in \Omega_{\mathbf{f}_v}, \mathbf{f}_r \in \Omega_{\mathbf{f}_r}$$

    - Link-based

      Find feasible link flows $(\mathbf{x}_v^*, \mathbf{x}_t^*)$ such that

$$\langle \mathbf{t}(\mathbf{x}_v^*, \mathbf{x}_r^*), \mathbf{x}_v - \mathbf{x}_v^* \rangle + \langle \mathbf{t}(\mathbf{x}_v^*, \mathbf{x}_r^*), \mathbf{x}_r - \mathbf{x}_r^* \rangle \geq 0, \qquad \forall \mathbf{x}_v \in \Omega_{\mathbf{x}_v}, \mathbf{x}_r \in \Omega_{\mathbf{x}_r}$$

Heterogeneous users

- General result
  - Consider K classes of users, each with link flows $\mathbf{x}_k$ and link costs $\mathbf{t}_k(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K) \in \mathbb{R}^{Km}, m = |A|$.
  - Multi-class traffic equilibrium has an equivalent optimization problem iff the Jacobian matrix of link costs $\nabla\mathbf{t}(\mathbf{x})$ is symmetric

$$\nabla\mathbf{t}(\mathbf{x}) = \begin{bmatrix} \frac{\partial t_{1,1}(\mathbf{x})}{\partial x_{1,1}} & \frac{\partial t_{1,1}(\mathbf{x})}{\partial x_{1,2}} & \cdots & \frac{\partial t_{1,1}(\mathbf{x})}{\partial x_{2,1}} & \cdots & \frac{\partial t_{1,1}(\mathbf{x})}{\partial x_{k,m}} \\ \frac{\partial t_{1,2}(\mathbf{x})}{\partial x_{1,1}} & \frac{\partial t_{1,2}(\mathbf{x})}{\partial x_{1,2}} & \cdots & \frac{\partial t_{1,2}(\mathbf{x})}{\partial x_{2,1}} & \cdots & \frac{\partial t_{1,2}(\mathbf{x})}{\partial x_{k,m}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial t_{2,1}(\mathbf{x})}{\partial x_{1,1}} & \frac{\partial t_{2,1}(\mathbf{x})}{\partial x_{1,2}} & \cdots & \frac{\partial t_{2,1}(\mathbf{x})}{\partial x_{2,1}} & \cdots & \frac{\partial t_{2,1}(\mathbf{x})}{\partial x_{k,m}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial t_{k,m}(\mathbf{x})}{\partial x_{1,1}} & \frac{\partial t_{k,m}(\mathbf{x})}{\partial x_{1,2}} & \cdots & \frac{\partial t_{k,m}(\mathbf{x})}{\partial x_{2,1}} & \cdots & \frac{\partial t_{k,m}(\mathbf{x})}{\partial x_{m}} \end{bmatrix}$$

$$\frac{\partial t_i(\mathbf{x})}{\partial x_j} = \frac{\partial t_j(\mathbf{x})}{\partial x_i}, \qquad \forall i, j$$

Heterogeneous users

- General result
  - Consider K classes of users, each with link flows $\mathbf{x}_k$ and link costs $\mathbf{t}_k(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K) \in \mathbb{R}^{Km}, m = |A|$.
  - Multi-class traffic equilibrium has an equivalent optimization problem iff the Jacobian matrix of link costs $\nabla\mathbf{t}(\mathbf{x})$ is symmetric
    - Quiz 1: What is the Jacobian in the car-truck example?

Heterogeneous users

- General result
  - Consider K classes of users, each with link flows $\mathbf{x}_k$ and link costs $\mathbf{t}_k(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K) \in \mathbb{R}^{Km}, m = |A|$.
  - Multi-class traffic equilibrium has an equivalent optimization problem iff the Jacobian matrix of link costs $\nabla \mathbf{t}(\mathbf{x})$ is symmetric
    - Quiz 2: If cars and trucks have equal contribution to traffic, does the symmetry condition hold?

$$t(x_v, x_r) = t_0 \left[1 + 0.15 \left(\frac{x_v + x_r}{s}\right)^4\right]$$

Heterogeneous users

- General result
  - Consider K classes of users, each with link flows $\mathbf{x}_k$ and link costs $\mathbf{t}_k(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K) \in \mathbb{R}^{Km}, m = |A|$.

  - Multi-class traffic equilibrium has an equivalent optimization problem iff the Jacobian matrix of link costs $\nabla\mathbf{t}(\mathbf{x})$ is symmetric
    - Quiz 3: If cars and trucks have different impacts on traffic and also experience different travel times such that

$$t_v(x_v, x_r) = t_0\left[1 + 0.15\left(\frac{x_v + 2x_r}{s}\right)^4\right]$$

$$t_r(x_v, x_r) = 2t_0\left[1 + 0.15\left(\frac{x_v + 2x_r}{s}\right)^4\right]$$

Heterogeneous users

- General result
  - Consider K classes of users, each with link flows $\mathbf{x}_k$ and link costs $\mathbf{t}_k(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K) \in \mathbb{R}^{Km}, m = |A|$.
  - Multi-class traffic equilibrium has an equivalent optimization problem iff the Jacobian matrix of link costs $\nabla\mathbf{t}(\mathbf{x})$ is symmetric
    - Quiz 3: If cars and trucks have different impacts on traffic and also experience different travel times such that

$$t_v(x_v, x_r) = t_0\left[1 + 0.15\left(\frac{x_v + 2x_r}{s}\right)^4\right]$$

$$t_r(x_v, x_r) = 2t_0\left[1 + 0.15\left(\frac{x_v + 2x_r}{s}\right)^4\right]$$

$$\frac{\partial t_v(x_v, x_r)}{\partial x_r} = \frac{\partial t_r(x_v, x_t)}{\partial x_v} = \frac{1.2t_0}{s}\left(\frac{x_v + 2x_r}{s}\right)^3$$

    - ***Q: What is the objective function?***

Heterogeneous users

- General result
  - Consider K classes of users, each with link flows $\mathbf{x}_k$ and link costs $\mathbf{t}_k(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K) \in \mathbb{R}^{Km}, m = |A|$.
  - Multi-class traffic equilibrium has an equivalent optimization problem iff the Jacobian matrix of link costs $\nabla\mathbf{t}(\mathbf{x})$ is symmetric
    - Recall the candidate objective in the car-truck example

$$Z(\mathbf{f}_v, \mathbf{f}_r) = \sum_{a \in A} \int_0^{x_{v,a}} t_{v,a}(u, x_{r,a})\,\mathrm{d}u + \int_0^{x_{r,a}} t_{r,a}(x_{v,a}, u)\,\mathrm{d}u$$

$$\begin{aligned}\frac{\partial Z(\mathbf{f}_v, \mathbf{f}_r)}{\partial f_{v,k}} &= c_k + \sum_{a \in A} \delta_{ak} \int_0^{x_{r,a}} \frac{\partial t_{r,a}(x_{v,a}, u)}{\partial x_{v,a}}\,\mathrm{d}u \\ &= c_k + \sum_{a \in A} \delta_{ak} \int_0^{x_{r,a}} \frac{\partial t_{v,a}(x_{v,a}, u)}{\partial x_{r,a}}\,\mathrm{d}u \\ &= c_k + \sum_{a \in A} \delta_{ak} \left(t_{v,a}(x_{v,a}, x_{r,a}) - t_{v,a}(x_{v,a}, 0) \right)\end{aligned}$$

symmetric condition

Heterogeneous users

- General result
  - Consider K classes of users, each with link flows $\mathbf{x}_k$ and link costs $\mathbf{t}_k(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K) \in \mathbb{R}^{Km}, m = |A|$.
  - Multi-class traffic equilibrium has an equivalent optimization problem iff the Jacobian matrix of link costs $\nabla\mathbf{t}(\mathbf{x})$ is symmetric
    - Recall the candidate objective in the car-truck example

$$Z(\mathbf{f}_v, \mathbf{f}_r) = \sum_{a \in A} \int_0^{x_{v,a}} t_{v,a}(u, x_{r,a})\,\mathrm{d}u + \int_0^{x_{r,a}} t_{r,a}(x_{v,a}, u)\,\mathrm{d}u$$

$$\frac{\partial Z(\mathbf{f}_v, \mathbf{f}_r)}{\partial f_{v,k}} = 2c_k - \sum_{a \in A} \delta_{ak} t_{v,a}(x_{v,a}, 0)$$

    - ***Q: How to construct an objective function that get rids of factor 2 and the second term?***

Heterogeneous users

- General result
  - Consider K classes of users, each with link flows $\mathbf{x}_k$ and link costs $\mathbf{t}_k(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K) \in \mathbb{R}^{Km}, m = |A|$.
  - Multi-class traffic equilibrium has an equivalent optimization problem iff the Jacobian matrix of link costs $\nabla\mathbf{t}(\mathbf{x})$ is symmetric

$$\min_{\mathbf{x}} \quad z(\mathbf{x}) = \frac{1}{K}\sum_{a\in A}\sum_{k=1}^{K}\left[\int_0^{x_{k,a}} t_{k,a}\left(u, \mathbf{x}_{-k,a}\right) \mathrm{d}u + (K-1)\int_0^{x_{k,a}} t_{k,a}(u, \mathbf{0})\, \mathrm{d}u\right]$$

$$s.t. \quad \mathbf{x}_k \in \Omega_{\mathbf{x}_k}, \qquad \forall k$$

where $\mathbf{x}_{-k,a}$ is flows of user classes other than k on link a

Heterogeneous users

- General result
  - Consider K classes of users, each with link flows $\mathbf{x}_k$ and link costs $\mathbf{t}_k(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K) \in \mathbb{R}^{Km}, m = |A|$.
  - Multi-class traffic equilibrium has an equivalent optimization problem iff the Jacobian matrix of link costs $\nabla \mathbf{t}(\mathbf{x})$ is symmetric

$$\min_{\mathbf{x}} \quad z(\mathbf{x}) = \frac{1}{K} \sum_{a \in A} \sum_{k=1}^{K} \left[\int_0^{x_{k,a}} t_{k,a}\left(u, \mathbf{x}_{-k,a}\right) \mathrm{d}u + (K-1) \int_0^{x_{k,a}} t_{k,a}(u, \mathbf{0})\, \mathrm{d}u \right]$$

$$s.t. \quad \mathbf{x}_k \in \Omega_{\mathbf{x}_k}, \qquad \forall k$$

  - No matter whether symmetry condition holds, the equivalent VI problem always exists

    Find feasible link flows $\mathbf{x}^*$ such that

$$\sum_{k=1}^{K} \langle \mathbf{t}_k(\mathbf{x}^*), \mathbf{x}_k - \mathbf{x}_k^* \rangle \geq 0, \qquad \forall \mathbf{x}_k \in \Omega_{\mathbf{x}_k}, \forall k$$

Heterogeneous users

- Solution method
  - Symmetric: Due to the similar formulation, same solution algorithms for standard traffic assignment (e.g., FW, PG) can be utilized
  - Asymmetric: Revised FW algorithm with diagonalization

    only diagonal elements in $\nabla \mathbf{t}(\mathbf{x})$ are considered

    - Line search α
      - With target link flow $\mathbf{y}$ and current link flow $\mathbf{x}$, solve step size α such that

$$\sum_{a \in A} \sum_{k=1}^{K} t_{k,a}\left(\alpha y_{k,a} + (1-\alpha) x_{k,a}, \mathbf{x}_{-k,a}\right)\left(y_{k,a} - x_{k,a}\right) = 0$$

      - It corresponds to a relaxed objective function without link interactions

$$\min_{\alpha} \sum_{a \in A} \sum_{k=1}^{K} \int_{0}^{\alpha y_{k,a} + (1-\alpha) x_{k,a}} t_{k,a}\left(u, \mathbf{x}_{-k,a}\right) \mathrm{d}u$$

should have changed with α but kept constant

Questions?