Spring 2025

06 Static Traffic Assignment: Base Model II

CIVIL-477 Transportation network modeling & analysis

École polytechnique fédérale de Lausanne

Outline

- Congestion pricing
  - First-best vs second-best

- Likely path flow
  - Max-entropy method

- Dual formulation

Congestion pricing

- User equilibrium (UE) as a solution to optimization problem

$$\min_{\mathbf{f}} \quad z(\mathbf{x}) = \sum_a \int_0^{x_a} t_a(u)\,\mathrm{d}u$$

$$s.t. \quad \Lambda\mathbf{f} = \mathbf{q}$$

$$\Delta\mathbf{f} = \mathbf{x}$$

$$\mathbf{f} \geq \mathbf{0}$$

where

- $\mathbf{x}, \mathbf{f}$: link/path flow vector
- $\mathbf{q}$: OD demand vector
- Δ, Λ: link-path/OD-path incidence matrix
- $z(\mathbf{x})$: Beckmann's function

- ***Q: How about system optimum (SO)?***

Congestion pricing

- System optimum (SO) is naturally an optimization problem

$$\min_{\mathbf{f}} \; TT(\mathbf{x}) = \sum_a x_a t_a(x_a)$$

$$s.t. \; \Lambda\mathbf{f} = \mathbf{q}$$

$$\Delta\mathbf{f} = \mathbf{x}$$

$$\mathbf{f} \geq \mathbf{0}$$

where

- $\mathbf{x}, \mathbf{f}$: link/path flow vector
- $\mathbf{q}$: OD demand vector
- Δ, Λ: link-path/OD-path incidence matrix
- $TT(\mathbf{x})$: total travel time

- ***Q: How does it relate to UE?***

Congestion pricing

- System optimum (SO) is naturally an optimization problem

$$\min_{\mathbf{f}} \; TT(\mathbf{x}) = \sum_a x_a t_a(x_a) = \sum_a \int_0^{x_a} mt_a(u)\,\mathrm{d}u$$
$$s.t. \;\; \Lambda\mathbf{f} = \mathbf{q}$$
$$\Delta\mathbf{f} = \mathbf{x}$$
$$\mathbf{f} \geq \mathbf{0}$$

where

- $\mathbf{x}, \mathbf{f}$: link/path flow vector
- $\mathbf{q}$: OD demand vector
- Δ, Λ: link-path/OD-path incidence matrix
- $TT(\mathbf{x})$: total travel time
- $mt_a(x_a) = t_a(x_a) + x_a t_a'(x_a) = \frac{\partial(x_a t_a(x_a))}{\partial x_a}$: marginal link travel time

Congestion pricing

- Rationale of (first-best) congestion pricing

$$\min_{\mathbf{f}} \; TT(\mathbf{x}) = \sum_a \int_0^{x_a} (t_a(u) + \tau_a(u))\,\mathrm{d}u$$

$$s.t. \; \Lambda\mathbf{f} = \mathbf{q}$$

$$\Delta\mathbf{f} = \mathbf{x}$$

$$\mathbf{f} \geq \mathbf{0}$$

where

- $\mathbf{x}, \mathbf{f}$: link/path flow vector
- $\mathbf{q}$: OD demand vector
- Δ, Λ: link-path/OD-path incidence matrix
- $TT(\mathbf{x})$: total travel time
- $\tau_a(x_a) = mt_a(x_a) - t_a(x_a) = x_a t'_a(x_a)$: marginal pricing

First-best pricing

- Rationale of (first-best) congestion pricing
  - Flow-based toll added to all links $\tau_a(x_a) = x_a t'_a(x_a)$
    - ***negative externality*** caused by each additional traveler
      - not included in traveler's cost under UE (no regulation/intervention)
      - e.g., emission, noise, …

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      - e.g., emission, noise, …
    - in unit of time, transformed back to monetary value via value of time (VOT)
      - ***generalized link travel cost*** $gt_a(x_a) = \beta t_a(x_a) + \tilde{\tau}_a(x_a)$, where $\tilde{\tau}_a(x_a) = \beta\tau_a(x_a)$

First-best pricing

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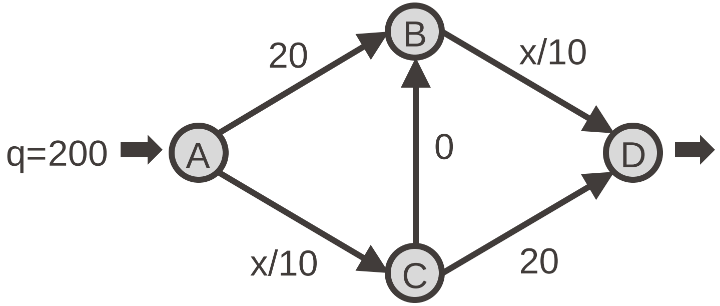
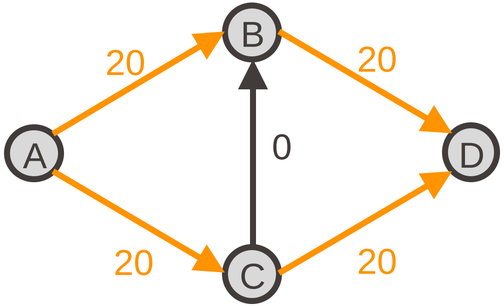
  - At SO, all used routes have equal and min marginal costs



SO marginal link travel time



First-best pricing

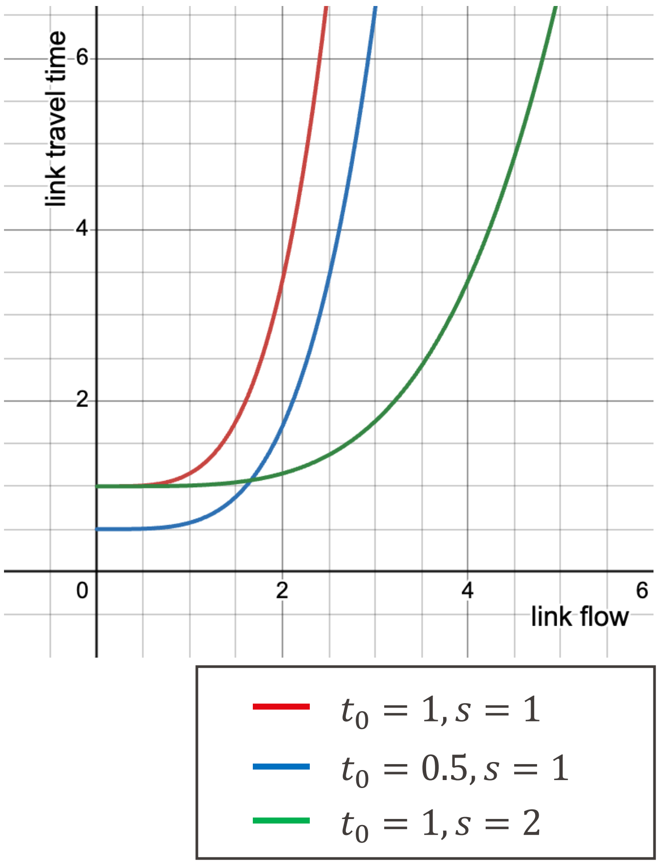
- First-best pricing under base BRP
  - Bureau of Public Roads (BPR) function

$$t(x) = t_0\left[1 + 0.15\left(\frac{x}{s}\right)^4\right]$$

where

- t_0: free-flow travel time (hr)
- s: saturation flow, or “capacity” (veh/hr)

- widely used in transportation planning
- ideal mathematical properties
  - e.g., strictly increasing



First-best pricing

- First-best pricing under base BRP
  - UE objective function

$$z(\mathbf{x}) = \sum_a \int_0^{x_a} t_a(u)\,\mathrm{d}u$$

    - integral of link travel time

$$\int_0^{x_a} t_a(u)\,\mathrm{d}u = \int_0^{x_a} t_0\left[1 + 0.15\left(\frac{u}{s}\right)^4\right]\mathrm{d}u$$

$$= t_0\left[x_a + 0.03s\left(\frac{x_a}{s}\right)^5\right] = x_a t_0\left[1 + 0.03\left(\frac{x_a}{s}\right)^4\right]$$

First-best pricing

- First-best pricing under base BRP
  - SO objective function

$$TT(\mathbf{x}) = \sum_a x_a t_a(x_a) = \sum_a x_a t_0 \left[1 + 0.15 \left(\frac{x_a}{s}\right)^4\right]$$

- marginal link cost

$$mt_a(x_a) = \frac{\partial\big(x_a t_a(x_a)\big)}{\partial x_a} = t_a(x_a) + x_a t_a'(x_a)$$

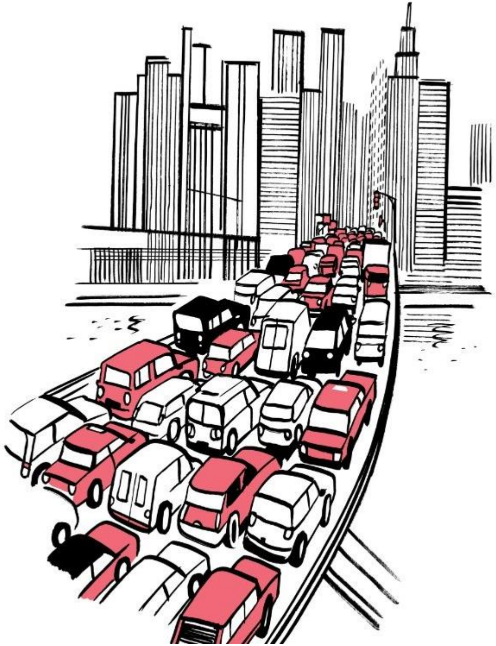
$$= t_0 \left[1 + 0.15 \left(\frac{x_a}{s}\right)^4\right] + x_a t_0 \left[\frac{0.6}{s} \left(\frac{x_a}{s}\right)^3\right]$$

$$= t_0 \left[1 + 0.75 \left(\frac{x_a}{s}\right)^4\right]$$

- ***Q: What is the first-best link toll? What are the key parameters?***

Second-best pricing

- Issues of first-best pricing



- ***Q: First-best pricing is theoretically optimal. Is it practical, why or why not?***

Second-best pricing

- Issues of first-best pricing

$$\tau_a(x_a) = x_a t_a'(x_a),\ \forall a \in A$$

  - Link cost function is usually unknown and even not fixed
    - influenced by weather, accidents, …
  - Adding tolls on all links and making them flow-dependent is not practical
    - challenges in traffic monitoring, toll collection, …
  - Travelers are not perfectly rational
    - possibly resolved in the era of AVs

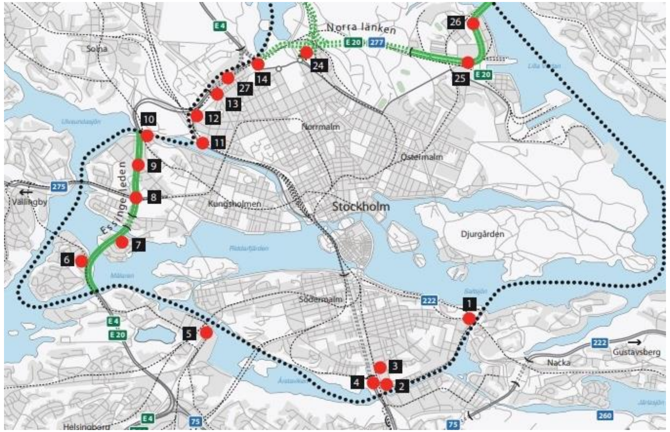
- ***Q: What are congestion pricing implemented in real practice?***

Second-best pricing

- Practical implementations
  - Facility-based
    - when using some facility
    - e.g., highway, bridge

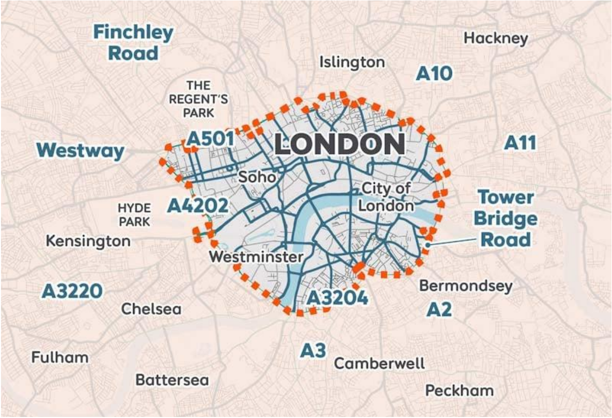
Second-best pricing

- Practical implementations
  - Facility-based
    - when using some facility
    - e.g., highway, bridge
  - Cordon-based
    - when passing some cordon/barrier
    - e.g., Stockholm, New York



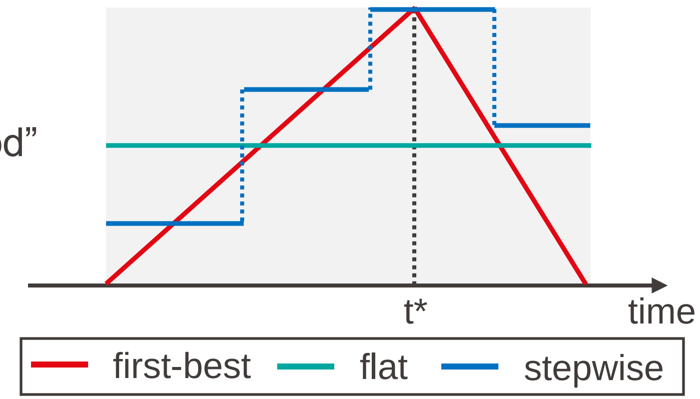
Second-best pricing

- Practical implementations
  - Facility-based
    - when using some facility
    - e.g., highway, bridge
  - Cordon-based
    - when passing some cordon/barrier
    - e.g., Stockholm, New York
  - Zone-based
    - when driving within some zone
    - e.g., London, Chicago

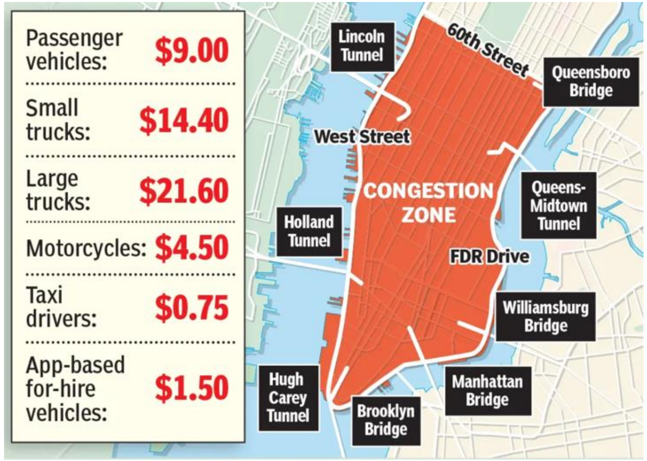


Second-best pricing

- Practical implementations
  - Temporal variation
    - flat rate: fixed price over a ”congestion period”
    - stepwise: constant within each time interval



  - Price discrimination
    - classified by vehicle and trip type
      - private car vs truck vs taxis
      - solo vs carpooling trips



Second-best pricing

- Optimize pricing ***objective*** subject to certain ***constraints***

- Application in traffic routing
  - Objective
    - minimize total travel time
  - Constraints
    - spatial: link, cordon, zone
    - temporal: flat, stepwise
    - scheme: vehicle-specific, trip-specific

Second-best pricing

- General framework

$$\min_{\tau} \; TT(\mathbf{x}^*, \tau)$$
$$s.t. \;\; \langle \mathbf{t}(\mathbf{x}^*, \tau), \mathbf{x} - \mathbf{x}^* \rangle \geq 0, \qquad \forall \mathbf{x} \in \Omega_{\mathbf{x}}$$
$$\tau \in \Omega_\tau$$

where

- $TT, \mathbf{t}$: total travel time and link cost function
- $\mathbf{x}^*, \mathbf{x}$: (equilibrium) link flow
- τ: link toll
- $\Omega_{\mathbf{x}}, \Omega_\tau$: feasible set of link flow and toll

Second-best pricing

- General framework

$$\min_{\tau} \quad TT(\mathbf{x}^*, \tau)$$
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$$\tau \in \Omega_{\tau}$$

Leader's problem

Follower's problem

where

- $TT, \mathbf{t}$: total travel time and link cost function
- $\mathbf{x}^*, \mathbf{x}$: (equilibrium) link flow
- τ: link toll
- $\Omega_{\mathbf{x}}, \Omega_{\tau}$: feasible set of link flow and toll

- Connection to Stackelberg game

Questions?

Likely path flow

- Non-uniqueness of equilibrium path flows

$$\min_{\mathbf{f}} \quad z(\mathbf{x}) = \sum_a \int_0^{x_a} t_a(u)\, \mathrm{d}u$$
$$s.t. \quad \Lambda\mathbf{f} = \mathbf{q}$$
$$\Delta\mathbf{f} = \mathbf{x}$$
$$\mathbf{f} \geq \mathbf{0}$$

- Suppose t_a is differentiable and strictly increasing, then
  - there exists unique UE link flows $\mathbf{x}^*$
  - any path flows $\mathbf{f}^*$ that satisfies $\mathbf{x}^* = \Delta\mathbf{f}^*$ is UE path flows

Likely path flow

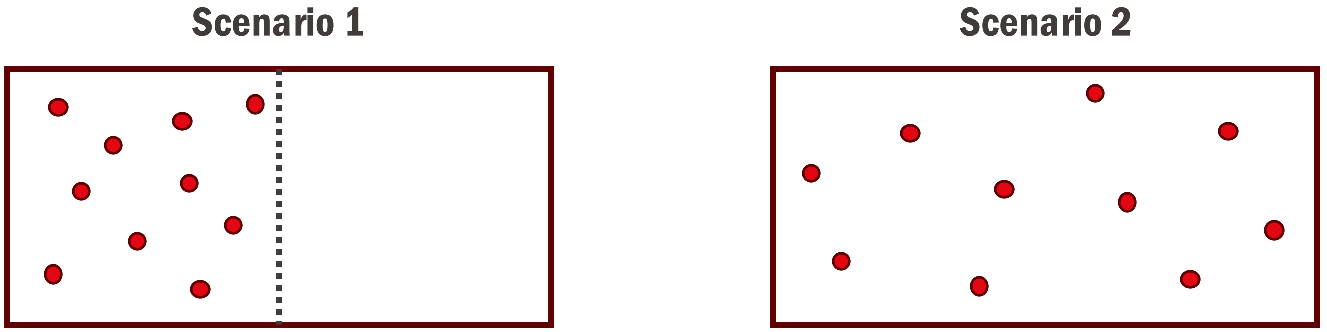
- Non-uniqueness of equilibrium path flows
  - If the primary goal is to predict congestion, then solving $\mathbf{x}^*$ is sufficient.

- Path flows are needed to
  - Answer who are traveling a particular link
    - e.g., equity-related analysis

  - Design path-based incentives
    - e.g., emission-based pricing

Likely path flow

- Rationale of ***most likely*** path flow
  - Suppose there are 10 points scattering in a two-dimension region.



  - ***Q: Which scenario is more likely to happen, and why?***

Likely path flow

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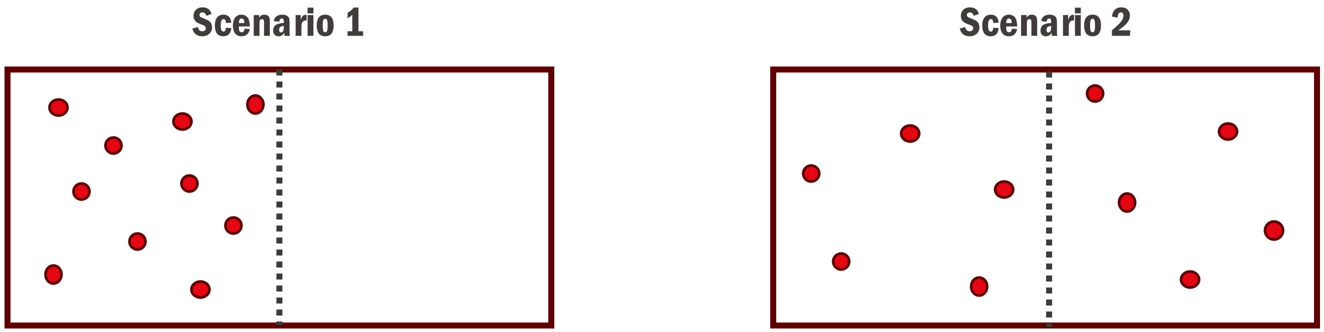

- ***Q: Which scenario is more likely to happen, and why?***
  - Probability that N_{left} points fall in the left half of the region

$$\text{Prob}\,[N_{\text{left}} = 10] = \left(\frac{1}{2}\right)^{10} < 0.001$$

$$\text{Prob}\,[N_{\text{left}} = 5] = C(10,5)\left(\frac{1}{2}\right)^{5}\left(\frac{1}{2}\right)^{5} \approx 0.25$$

***combination $C(n,k) = \frac{n!}{k!(n-k)!}$**

Likely path flow

- Rationale of ***most likely*** path flow
  - Suppose there are q travelers choosing between K paths
  - Path flow $\mathbf{f} = (f_1, \dots, f_K)$, $\sum_k f_k = q$ with probability

$$\text{Prob}\,[\mathbf{f}] = \frac{q!}{f_1! \dots f_K!} \Pi_k \left(\frac{1}{K}\right)^{f_k}$$
$$= \frac{q!}{f_1! \dots f_K!} \left(\frac{1}{K}\right)^{\sum_k f_k} = \frac{q!}{f_1! \dots f_k!} \left(\frac{1}{K}\right)^{q}$$

  - Most likely path flow

$$\mathbf{f}^* = \arg\max_{\mathbf{f}} \text{Prob}\,[\mathbf{f}] = \arg\max_{\mathbf{f}} \frac{q!}{f_1! \dots f_K!}$$

Likely path flow

- Rationale of ***most likely*** path flow
  - Suppose there are q travelers choosing between K paths
  - Most likely path flow

$$\mathbf{f}^* = \arg\max_{\mathbf{f}} \frac{q!}{f_1! \dots f_K!}$$

log is strictly increasing

$$= \arg\max_{\mathbf{f}} \log\left(\frac{q!}{f_1! \dots f_K!}\right) = \log q! - \sum_k \log f_k!$$

$\log n! \approx n \log n - n$
when $n \gg 0$

$$\approx (q \log q \ - q) - \sum_k (f_k \log f_k - f_k)$$

$$= q \log q - \sum_k f_k \log f_k$$

$q = \sum_k f_k$

$$= -\sum_k f_k \log\left(\frac{f_k}{q}\right)$$

Likely path flow

- Rationale of ***most likely*** path flow
  - Suppose there are q travelers choosing between K paths
  - Most likely path flow

$$\mathbf{f}^* = \arg\max_{\mathbf{f}} -\sum_k f_k \log\left(\frac{f_k}{q}\right)$$

$$= \arg\max_{\mathbf{f}} -\sum_k \left(\frac{f_k}{q}\right) \log\left(\frac{f_k}{q}\right)$$

$$= \arg\max_{\mathbf{f}} -\sum_k p_k \log p_k$$

Shannon entropy

* probability of choosing path
$p_k = f_k/q$

Likely path flow

- Max-entropy method
  - Find the max-entropy path flow $\mathbf{f}^*$ that leads to the UE link flow. $\mathbf{x}^*$ and maximizes entropy

$$\max_{\mathbf{f}} \quad -\sum_{w\in W}\sum_{k\in P_w} f_k \log\left(\frac{f_k}{q_w}\right)$$

$$s.t. \quad \Lambda\mathbf{f} = \mathbf{q}$$

$$\Delta\mathbf{f} = \mathbf{x}^*$$

$$\mathbf{f} \geq \mathbf{0}$$

  - ***Q: Is there a unique solution to this problem?***

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$$s.t. \quad \Lambda\mathbf{f} = \mathbf{q}$$

$$\Delta\mathbf{f} = \mathbf{x}^*$$

$$\mathbf{f} \geq \mathbf{0}$$

  - ***Q: Is there a unique solution to this problem?***
    - Yes, because the objective function is strictly convex, and the feasible set is convex. How to prove it?

Likely path flow

- Max-entropy method
  - Find the max-entropy path flow $\mathbf{f}^*$ that leads to the UE link flow. $\mathbf{x}^*$ and maximizes entropy

$$\max_{\mathbf{f}} \quad -\sum_{w \in W} \sum_{k \in P_w} f_k \log\left(\frac{f_k}{q_w}\right)$$

$$s.t. \quad \Lambda \mathbf{f} = \mathbf{q}$$

$$\Delta \mathbf{f} = \mathbf{x}^*$$

$$\mathbf{f} \geq \mathbf{0}$$

  - ***Q: Is it really the "most likely" path flow?***
    - Only when we do not have any prior information
      - e.g., historical trajectories

Questions?

Dual formulation

- Link-based formulation: origin-based

$$\min_{\mathbf{x}} \quad z(\mathbf{x}) = \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(u)\,\mathrm{d}u$$

$$s.t. \quad \sum_{j\in N_i^+} x_{ij}^r - \sum_{j\in N_i^-} x_{ji}^r = q_i^r = \begin{cases} \sum_s q_{rs} & i = r \\ -q_{rs} & i = s \\ 0 & \text{otherwise} \end{cases}, \qquad \forall i \in N, r \in R$$

$$\sum_{r\in R} x_{ij}^r = x_{ij}, \qquad \forall (i,j) \in A$$

$$x_{ij}^r \geq 0, \qquad \forall (i,j) \in A$$

where N

- N, R: set of nodes and origins
- N_i^-, N_i^+: set of upstream/downstream nodes of node i
- A: set of links

Dual formulation

- Link-based formulation: destination-based

$$\min_{\mathbf{x}} \quad z(\mathbf{x}) = \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(u)\,\mathrm{d}u$$

$$s.t. \quad \sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s = q_i^s = \begin{cases} q_{rs} & i = r \\ -\sum_{r\in R} q_{rs} & i = s \\ 0 & \text{otherwise} \end{cases}, \qquad \forall i \in N, s \in S$$

$$\sum_{s\in R} x_{ij}^s = x_{ij}, \qquad \forall (i,j) \in A$$

$$x_{ij}^s \geq 0, \qquad \forall (i,j) \in A$$

where N

- N, S: set of nodes and destinations
- N_i^-, N_i^+: set of upstream/downstream nodes of node i
- A: set of links

Dual formulation

- Link-based formulation: destination-based

$$\mathcal{L}(\mathbf{x},\mu,\lambda) = \sum_{(i,j)\in A} \int_0^{\sum_{r\in R} x_{ij}^s} t_{ij}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} \mu_i^s \Big(\sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s - q_i^s \Big) - \sum_{s\in S}\sum_{(i,j)\in A} \lambda_{ij}^s x_{ij}^s$$

- KKT conditions
  - Stationarity $$\frac{\partial}{\partial x_{ij}^s}\mathcal{L}(\mathbf{x},\mu,\lambda) = t_{ij}(x_{ij}) - \mu_i^s + \mu_j^s - \lambda_{ij}^s = 0, \qquad \forall s\in S,(i,j)\in A$$
  - Complementary $$\lambda_{ij}^s x_{ij}^s = 0, \qquad \forall s\in S,(i,j)\in A$$
  - Primal feasibility $$\sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s = q_i^s, \qquad \forall s\in S, i\in N$$ $$x_{ij}^s \geq 0, \qquad \forall s\in S,(i,j)\in A$$
  - Dual feasibility $$\lambda_{ij}^s \geq 0, \qquad \forall s\in S,(i,j)\in A$$

Dual formulation

- Link-based formulation: destination-based

$$\mathcal{L}(\mathbf{x},\mu,\lambda) = \sum_{(i,j)\in A} \int_0^{\sum_{r\in R} x_{ij}^s} t_{ij}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} \mu_i^s \Big(\sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s - q_i^s \Big) - \sum_{s\in S}\sum_{(i,j)\in A} \lambda_{ij}^s x_{ij}^s$$

- KKT conditions
  - Stationarity $\quad \dfrac{\partial}{\partial x_{ij}^s}\mathcal{L}(\mathbf{x},\mu,\lambda) = t_{ij}(x_{ij}) - \mu_i^s + \mu_j^s - \lambda_{ij}^s = 0, \quad \forall s\in S, (i,j)\in A$
  - Complementary $\quad \lambda_{ij}^s x_{ij}^s = 0, \quad \forall s\in S, (i,j)\in A$
  - Primal feasibility $\quad \sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s = q_i^s, \quad \forall s\in S, i\in N$

    $x_{ij}^s \geq 0, \quad \forall s\in S, (i,j)\in A$
  - Dual feasibility $\quad \lambda_{ij}^s \geq 0, \quad \forall s\in S, (i,j)\in A$

Dual formulation

- Link-based formulation : destination-based

$$\mathcal{L}(\mathbf{x},\mu,\lambda) = \sum_{(i,j)\in A} \int_0^{\sum_{r\in R} x_{ij}^s} t_{ij}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} \mu_i^s \Big(\sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s - q_i^s \Big) - \sum_{s\in S}\sum_{(i,j)\in A} \lambda_{ij}^s x_{ij}^s$$

- KKT conditions
  - Stationarity $t_{ij} - \mu_i^s + \mu_j^s \geq 0, \quad \forall s \in S, (i,j) \in A$
  - Complementary $(\mu_i^s - t_{ij} - \mu_j^s) x_{ij}^s = 0, \quad \forall s \in S, (i,j) \in A$
  - Primal feasibility $\sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s = q_i^s, \quad \forall s \in S, i \in N$

    $x_{ij}^s \geq 0, \quad \forall s \in S, (i,j) \in A$

- KKT conditions of another problem with decision variables $\mathbf{t} = (t_{ij})_{(i,j)\in A}$, $\mu = (\mu_i^s)_{s\in S, i\in N}$ and dual variables $\mathbf{x} = (x_{ij}^s)_{r\in s,(i,j)\in A}$

Dual formulation

- Dual formulation: destination-based

$$\min_{\mathbf{t},\mu} \quad z_D(\mathbf{t},\mu) = \sum_{(i,j)\in A} \int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} q_i^s \mu_i^s$$

$$s.t. \quad \mu_i^s \leq t_{ij} + \mu_j^s\,, \qquad \forall s \in S, (i,j) \in A$$

$$t_{ij} \geq t_{ij}^0, \qquad \forall (i,j) \in A$$

$$\mu_s^s = 0, \qquad \forall s \in S$$

where N

- N, S: set of nodes and destinations
- A: set of links
- ℓ_{ij}^{-1}: inverse of link cost function, i.e., $t_{ij} = \ell(x_{ij})$
- t_{ij}^0: free-flow link travel time
- μ_i^s: travel time to destination

Dual formulation

- Dual formulation: destination-based

$$\mathcal{L}_D(\mathbf{t},\mu,\mathbf{x},\mathbf{y}) = \sum_{(i,j)\in A}\int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} q_i^s\mu_i^s + \sum_{s\in S}\sum_{(i,j)\in A} x_{ij}^s\left(\mu_i^s - t_{ij} - \mu_j^s\right) - \sum_{(i,j)\in A} y_{ij}(t_{ij} - t_{ij}^0)$$

- KKT conditions
  - Stationarity

$$\frac{\partial}{\partial t_{ij}}\mathcal{L}_D(\mathbf{t},\mu,\mathbf{x},\mathbf{y}) = \ell_{ij}^{-1}\left(t_{ij}\right) - \sum_{s\in S} x_{ij}^s - y_{ij} = 0, \qquad \forall (i,j)\in A$$

$$\frac{\partial}{\partial \mu_i^s}\mathcal{L}_D(\mathbf{t},\mu,\mathbf{x},\mathbf{y}) = -q_i^s + \sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s = 0, \qquad \forall s,\in S,(i,j)\in A$$

  - Complementary

$$x_{ij}^s(\mu_i^s - t_{ij} - \mu_j^s) = 0, \qquad \forall s,\in S,(i,j)\in A$$

$$y_{ij}\left(t_{ij} - t_{ij}^0\right) = 0, \qquad \forall (i,j)\in A$$

  - Primal feasibility

$$\mu_i^s \le t_{ij} + \mu_j^s, \qquad \forall s\in S,(i,j)\in A$$

$$t_{ij} \ge t_{ij}^0, \qquad \forall (i,j)\in A \qquad \mu_s^s = 0, \qquad \forall s\in S$$

  - Dual feasibility

$$x_{ij}^s \ge 0, \qquad \forall s\in S,(i,j)\in A \qquad y_{ij}\ge 0, \qquad \forall (i,j)\in A$$

Dual formulation

- Dual formulation: destination-based

$$\mathcal{L}_D(\mathbf{t},\mu,\mathbf{x},\mathbf{y}) = \sum_{(i,j)\in A}\int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} q_i^s \mu_i^s + \sum_{s\in S}\sum_{(i,j)\in A} x_{ij}^s\left(\mu_i^s - t_{ij} - \mu_j^s\right) - \sum_{(i,j)\in A} y_{ij}(t_{ij} - t_{ij}^0)$$

- KKT conditions
  - Stationarity

$$\frac{\partial}{\partial t_{ij}}\mathcal{L}_D(\mathbf{t},\mu,\mathbf{x},\mathbf{y}) = \ell_{ij}^{-1}(t_{ij}) - \sum_{s\in S} x_{ij}^s - y_{ij} = 0, \qquad \forall (i,j)\in A$$

$$\frac{\partial}{\partial \mu_i^s}\mathcal{L}_D(\mathbf{t},\mu,\mathbf{x},\mathbf{y}) = -q_i^s + \sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s = 0, \qquad \forall s,\in S,(i,j)\in A$$

  - Complementary

$$x_{ij}^s(\mu_i^s - t_{ij} - \mu_j^s) = 0, \qquad \forall s,\in S,(i,j)\in A$$

$$y_{ij}\left(t_{ij} - t_{ij}^0\right) = 0, \qquad \forall (i,j)\in A$$

  - Primal feasibility

$$\mu_i^s \le t_{ij} + \mu_j^s, \qquad \forall s\in S,(i,j)\in A$$

$$t_{ij} \ge t_{ij}^0, \qquad \forall (i,j)\in A \qquad \mu_s^s = 0, \qquad \forall s\in S$$

  - Dual feasibility

$$x_{ij}^s \ge 0, \qquad \forall s\in S,(i,j)\in A \qquad y_{ij}\ge 0, \qquad \forall (i,j)\in A$$

Dual formulation

- Dual formulation: destination-based

$$\mathcal{L}_D(\mathbf{t},\mu,\mathbf{x},\mathbf{y}) = \sum_{(i,j)\in A}\int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{du} - \sum_{s\in S}\sum_{i\in N} q_i^s \mu_i^s + \sum_{s\in S}\sum_{(i,j)\in A} x_{ij}^s\left(\mu_i^s - t_{ij} - \mu_j^s\right) - \sum_{(i,j)\in A} y_{ij}(t_{ij} - t_{ij}^0)$$

- KKT conditions
  - Stationarity

$$\left(\ell_{ij}^{-1}(t_{ij}) - \sum_{s\in S} x_{ij}^s\right)(t_{ij} - t_{ij}^0) = 0, \qquad \forall (i,j)\in A$$

$$\sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s = q_i^s, \qquad \forall s,\in S,(i,j)\in A$$

  - Complementary

$$x_{ij}^s(\mu_i^s - t_{ij} - \mu_j^s) = 0, \qquad \forall s,\in S,(i,j)\in A$$

  - Primal feasibility

$$\mu_i^s \le t_{ij} + \mu_j^s, \qquad \forall s\in S,(i,j)\in A$$

$$\mu_s^s = 0, \qquad \forall s\in S$$

  - Dual feasibility

$$x_{ij}^s \ge 0, \qquad \forall s\in S,(i,j)\in A$$

Dual formulation

- Dual formulation: destination-based

$$\mathcal{L}_D(\mathbf{t},\mu,\mathbf{x},\mathbf{y}) = \sum_{(i,j)\in A}\int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} q_i^s\mu_i^s + \sum_{s\in S}\sum_{(i,j)\in A} x_{ij}^s\left(\mu_i^s - t_{ij} - \mu_j^s\right) - \sum_{(i,j)\in A} y_{ij}(t_{ij}-t_{ij}^0)$$

- KKT conditions
  - Stationarity

$$x_{ij} = \begin{cases}\sum_{s\in S} x_{ij}^s, & \text{if } t_{ij} > t_{ij}^0 \\ 0, & \text{otherwise}\end{cases}, \qquad \forall (i,j)\in A$$

$$\sum_{j\in N_i^+} x_{ij}^s - \sum_{j\in N_i^-} x_{ji}^s = q_i^s, \qquad \forall s,\in S,(i,j)\in A$$

  - Complementary

$$x_{ij}^s(\mu_i^s - t_{ij} - \mu_j^s) = 0, \qquad \forall s,\in S,(i,j)\in A$$

  - Primal feasibility

$$\mu_i^s \le t_{ij} + \mu_j^s, \qquad \forall s\in S,(i,j)\in A$$

$$\mu_s^s = 0, \qquad \forall s\in S$$

  - Dual feasibility

$$x_{ij}^s \ge 0, \qquad \forall s\in S,(i,j)\in A$$

Dual formulation

	Primal	Dual
$\mu_i^s \le t_{ij} + \mu_j^s$	Stationary	Primal feasibility
$(\mu_i^s - t_{ij} - \mu_j^s) x_{ij}^s = 0$	Complementary	Complementary
$\sum_{j \in N_i^+} x_{ij}^s - \sum_{j \in N_i^-} x_{ji}^s = q_i^s$	Primal feasibility	Stationary
$\sum_{s \in R} x_{ij}^s = x_{ij}$	Primal feasibility	Stationary
$x_{ij}^s \ge 0$	Primal feasibility	Dual feasibility
$t_{ij} \ge t_{ij}^0$	Definition	Primal feasibility
$\mu_s^s = 0$	N/A	Primal feasibility

* Recall we've also imposed $\mu_s^s = 0$ in the dual LP for shortest path

Dual formulation

- Primal problem
  - Solve destination-based link flows $\mathbf{x}$
  - Dim = (# link)*(# destination) = $\mathcal{O}(|N|^3)$

- Dual problem
  - Solve link cost $\mathbf{t}$ and cost-to-destination μ
  - Dim = # link + (# node)*(# destination) = $\mathcal{O}(2|N|^2)$
    - further reduced to $\mathcal{O}(|N|^2)$ by solving optimal μ given $\mathbf{t}$

Dual formulation

- Reduced form

$$\min_{\mathbf{t},\mu} \quad z_D(\mathbf{t},\mu) = \sum_{(i,j)\in A} \int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} q_i^s \mu_i^s$$

$$\Leftrightarrow \quad \min_{\mathbf{t}} \quad z_D(\mathbf{t},\mu^*(\mathbf{t})) = \sum_{(i,j)\in A} \int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{d}u - \max_{\mu} \sum_{s\in S}\sum_{i\in N} q_i^s \mu_i^s$$

- Observation: μ only appears in the second part of objective
  - Solve $\mu^*(\mathbf{t})$ given $\mathbf{t}$, then optimize $\mathbf{t}$

Dual formulation

- Reduced form

$$\min_{\mathbf{t},\mu} \quad z_D(\mathbf{t},\mu) = \sum_{(i,j)\in A} \int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} q_i^s \mu_i^s$$

$$\Leftrightarrow \quad \min_{\mathbf{t}} \quad z_D(\mathbf{t},\mu^*(\mathbf{t})) = \sum_{(i,j)\in A} \int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{d}u - \max_{\mu} \sum_{s\in S}\sum_{i\in N} q_i^s \mu_i^s$$

$$s.t. \quad \mu_i^s \le t_{ij} + \mu_j^s, \qquad \forall s \in S, (i,j) \in A$$

$$t_{ij} \ge t_{ij}^0, \qquad \forall (i,j) \in A$$

$$\mu_s^s = 0, \qquad \forall s \in S$$

- ***Q: Does this subproblem remind you another problem?***

Dual formulation

- Reduced form

$$\min_{\mathbf{t}} \quad z_D(\mathbf{t}) = \sum_{(i,j)\in A} \int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} q_i^s \mu_i^{s*}(\mathbf{t})$$

$$s.t. \quad t_{ij} \geq t_{ij}^0, \qquad \forall (i,j) \in A$$

where

- $\mu_i^{s*}(\mathbf{t})$: node labels obtained by solving the shortest path from all nodes to destination s given link cost $\mathbf{t}$

- This dual problem can be solved in the similar way as the primal problem

Dual formulation

- Gradient projection
  - Primal problem

$$\min_{\mathbf{f}} \ z(\Delta\mathbf{f})$$
$$s.t. \ \Lambda\mathbf{f} = \mathbf{q}$$
$$\mathbf{f} \geq \mathbf{0}$$

  - At each iteration,
    - Step 1: Find descent direction $\mathbf{d}^n = -\nabla z(\Delta\mathbf{f}^n) = -\mathbf{c}(\mathbf{f}^n)$
    - Step 2: Compute candidate path flow $\mathbf{y}^n = \mathbf{f}^n + \alpha\mathbf{d}^n$
    - Step 3: Project back to feasible set $\mathbf{f}^{n+1} = \arg\min_{\mathbf{f}\in\Omega_\mathbf{f}} \ ||\mathbf{f} - \mathbf{y}^n||^2$
    - Step 4: Convergence check
      - Terminate if $\mathbf{f}^{n+1} = \mathbf{f}^n$ and return $\mathbf{f}^* = \mathbf{f}^n$

*key challenge: projection onto feasible path set

Dual formulation

- Gradient projection
  - Primal problem

$$\min_{\mathbf{f}} \; z(\Delta\mathbf{f})$$
$$s.t. \;\; \Lambda\mathbf{f} = \mathbf{q}$$
$$\mathbf{f} \geq \mathbf{0}$$

  - Dual problem

$$\min_{\mathbf{t}} \; z_D(\mathbf{t})$$
$$s.t. \;\; \mathbf{t} \geq \mathbf{t}^0$$

  - At each iteration,
    - Step 1: Find descent direction $\mathbf{d}^n = -\nabla z(\Delta\mathbf{f}^n) = -\mathbf{c}(\mathbf{f}^n)$ — $\mathbf{d}^n = -\nabla z_D(\mathbf{t}^n)$
    - Step 2: Compute candidate path flow $\mathbf{y}^n = \mathbf{f}^n + \alpha\mathbf{d}^n$ — $\mathbf{y}^n = \mathbf{t}^n + \alpha\mathbf{d}^n$
    - Step 3: Project back to feasible set $\mathbf{f}^{n+1} = \arg\min_{\mathbf{f}\in\Omega_{\mathbf{f}}} \left\|\mathbf{f} - \mathbf{y}^n\right\|^2$ — $\mathbf{t}^{n+1} = \max(\mathbf{t}^0, \mathbf{y}^n)$
    - Step 4: Convergence check
      - Terminate if $\mathbf{f}^{n+1} = \mathbf{f}^n$ and return $\mathbf{f}^* = \mathbf{f}^n$
  - Projection is no longer a challenge, but $\nabla z_D(\mathbf{t}^n)$ becomes hard to compute

Dual formulation

- Gradient projection
  - Primal problem

$$\min_{\mathbf{f}} \ z(\Delta\mathbf{f})$$
$$s.t. \ \Lambda\mathbf{f} = \mathbf{q}$$
$$\mathbf{f} \geq \mathbf{0}$$

  - Dual problem

$$\min_{\mathbf{t}} \ z_D(\mathbf{t})$$
$$s.t. \ \mathbf{t} \geq \mathbf{t}^0$$

  - In fact, $z_D(\mathbf{t})$ is not even smooth, i.e., continuously differentiable, due to $\mu_i^{s*}(\mathbf{t})$

$$z_D(\mathbf{t}) = \sum_{(i,j)\in A} \int_{t_{ij}^0}^{t_{ij}} \ell_{ij}^{-1}(u)\,\mathrm{d}u - \sum_{s\in S}\sum_{i\in N} q_i^s \mu_i^{s*}(\mathbf{t})$$

$$\Rightarrow \ \mathbf{d}^n \in -\ell^{-1}(\mathbf{t}^n) + \partial_t \mu_i^{s*}(\mathbf{t}^n)\mathbf{q}$$

***subgradient of μ_i^{s*} at $\mathbf{t}^n$**

Questions?