Spring 2025

04 Optimization Primer

CIVIL-477 Transportation network modeling & analysis

École polytechnique fédérale de Lausanne

Outline

- Linear programming
  - Shortest path problem
- Convex optimization
  - Shortest path problem
  - Traffic equilibrium
- Variational inequality
  - Traffic equilibrium

Linear programming

- Minimize a ***linear*** objective function of decision variables
  - Subject to ***linear*** equality and inequality constraints

- General formulation

$$\min_{x} \quad c^T x$$
$$s.t. \quad A_1 x \leq b_1$$
$$A_2 x = b_2$$

- Objective: $c \in \mathbb{R}^n, \ x \in \mathbb{R}^n$
- Inequality constraints: $A_1 \in \mathbb{R}^{m_1 \times n}, b_1 \in \mathbb{R}^{m_1}$
- Equality constraints: $A_2 \in \mathbb{R}^{m_2 \times n}, b_2 \in \mathbb{R}^{m_2}$

Linear programming

- Example I: Shipping goods
  - Optimize the shipping plan from n factories to m warehouses that minimizes the total shipping cost

$$\min_{x_{ij}} \quad \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}$$

$$s.t. \quad \sum_{j=1}^{m} x_{ij} \leq S_i, \qquad \forall i$$

$$\sum_{i=1}^{n} x_{ij} \geq D_j, \qquad \forall j$$

$$x_{ij} \geq 0, \qquad \forall i, j$$

- c_{ij} : shipping cost from factory i to warehouse j
- x_{ij} : shipping amount from factory i to warehouse j
- S_i : total supply of factory i
- D_j : total demand of warehouse j

- ***Q: How to rewrite the problem in the general form?***

Linear programming

- Example I: Shipping goods
  - Optimize the transport plan from n factories to m warehouses that minimizes the total shipping cost

$$\min_{x_{ij}} \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij}$$

$$s.t. \sum_{j=1}^{m} x_{ij} \leq S_i, \qquad \forall i$$

$$\sum_{i=1}^{n} x_{ij} \geq D_j, \qquad \forall j$$

$$x_{ij} \geq 0, \qquad \forall i,j$$

e.g., n = 2, m = 3

- Cost vector $\mathrm{c} = (c_{11}, c_{12}, \dots, c_{23})^T \in \mathbb{R}^6$
- Transport plan $\mathrm{x} = (x_{11}, x_{12}, \dots, x_{23})^T \in \mathbb{R}^6$
- Incidence matrix $A_1 = \begin{bmatrix} \Lambda \\ -I \end{bmatrix} \in \mathbb{R}^{5\times 6}$

$$\Lambda = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

- Demand/Supply/Feasibility vector

$$\mathrm{b}_1 = (S_1, S_2, -D_1, -D_2, -D_3, 0, \dots, 0)^T \in \mathbb{R}^{11}$$

Linear programming

- Example II: Shortest path problem
  - Send one unit of flow from origin r to destination s with minimum path cost

$$\min_{x_{ij}} \sum_{(i,j)} t_{ij} x_{ij}$$

$$s.t. \sum_{j \in N_i^+} x_{ij} - \sum_{j \in N_i^-} x_{ji} = \begin{cases} 1 & i = r \\ -1 & i = s \\ 0 & \text{otherwise} \end{cases}, \qquad \forall i \in N$$

$$x_{ij} \geq 0, \qquad \forall (i,j)$$

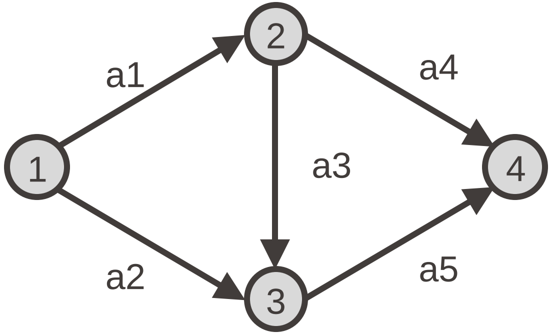
- x_{ij}: flow on link (i,j)
- t_{ij}: cost on link (i,j)
- N_i^-: upstream nodes of node i
- N_i^+: downstream nodes of node i

- ***Q: How to rewrite the problem in the general form?***

Linear programming

- Example II: Shortest path problem
  - Send one unit of flow from origin r to destination s with minimum path cost

$$\min_{\mathrm{x}} \quad \mathrm{t}^T \mathrm{x}$$
$$s.t.\ A_2\mathrm{x} = \mathrm{b}_2$$
$$\mathrm{x} \geq 0$$



- x: vector of link flows
- t: vector of link costs
- A_2: node-link matrix
- b_2: vector of node net flows
- $A_1 = -I$, $\mathrm{b}_1 = 0$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\mathrm{b}_2 = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$$

Linear programming

- Primal problem

$$\max_{x} \quad c^T x$$
$$s.t. \quad Ax \leq b$$
$$x \geq 0$$

  - $x, c \in \mathbb{R}^n$
  - $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

- Dual problem

$$\min_{y} \quad b^T y$$
$$s.t. \quad A^T y \geq c$$
$$y \geq 0$$

  - $y, b \in \mathbb{R}^m$
  - $A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n$

Linear programming

- Primal problem

$$\max_{x} \quad c^T x$$
$$s.t. \quad Ax \leq b$$
$$x \geq 0$$

- Dual problem

$$\min_{y} \quad b^T y$$
$$s.t. \quad A^T y \geq c$$
$$y \geq 0$$

- Weak duality
  - If x is a feasible solution to the primal problem and y is a feasible solution to the dual problem, then $c^T x \leq b^T y$

- ***Q: How to prove it?***

Linear programming

- Primal problem

$$\max_{x} \quad c^T x$$
$$s.t. \quad Ax \leq b$$
$$x \geq 0$$

- Dual problem

$$\min_{y} \quad b^T y$$
$$s.t. \quad A^T y \geq c$$
$$y \geq 0$$

- Weak duality
  - If x is a feasible solution to the primal problem and y is a feasible solution to the dual problem, then $c^T x \leq b^T y$
    - Proof. Directly from feasible constraints

$$c^T x \leq (A^T y)^T x = y^T (Ax) \leq y^T b = b^T y$$

- ***Q: What is the implication of weak duality?***

Linear programming

- Primal problem

$$\max_{x} \quad c^T x$$
$$s.t. \quad Ax \leq b$$
$$x \geq 0$$

- Dual problem

$$\min_{y} \quad b^T y$$
$$s.t. \quad A^T y \geq c$$
$$y \geq 0$$

- Weak duality
  - If x is a feasible solution to the primal problem and y is a feasible solution to the dual problem, then $c^T x \leq b^T y$
    - **Boundness:** Any feasible primal solution offers a lower bound of the dual problem, and vice versa.

- ***Q: What if $c^T x = b^T y$?***

Linear programming

- Primal problem

$$\max_{x} \quad c^T x$$
$$s.t. \quad Ax \le b$$
$$x \ge 0$$

- Dual problem

$$\min_{y} \quad b^T y$$
$$s.t. \quad A^T y \ge c$$
$$y \ge 0$$

- Weak duality
  - If x is a feasible solution to the primal problem and y is a feasible solution to the dual problem, then $c^T x \le b^T y$
    - **Boundness:** Any feasible primal solution offers a lower bound of the dual problem, and vice versa.
    - **Optimality:** x and y are both optimal solutions if $c^T x = b^T y$

- ***Q: What if there is no feasible primal/dual solution?***

Linear programming

- Primal problem

$$\max_{x} \quad c^T x$$
$$s.t. \quad Ax \leq b$$
$$x \geq 0$$

- Dual problem

$$\min_{y} \quad b^T y$$
$$s.t. \quad A^T y \geq c$$
$$y \geq 0$$

- Weak duality
  - If x is a feasible solution to the primal problem and y is a feasible solution to the dual problem, then $c^T x \leq b^T y$
    - **Boundness:** Any feasible primal solution offers a lower bound of the dual problem, and vice versa.
    - **Optimality:** x and y are both optimal solutions if $c^T x = b^T y$
    - **Unboundedness:** If the primal/dual problem is unbounded, then the dual/primal problem is infeasible

Linear programming

- Primal problem

$$\max_{x} \quad c^T x$$
$$s.t. \quad Ax \leq b$$
$$x \geq 0$$

- Dual problem

$$\min_{y} \quad b^T y$$
$$s.t. \quad A^T y \geq c$$
$$y \geq 0$$

- Strong duality
  - If the primal/dual problem has a finite optimal solution x^*/y^*, then the dual/primal problem also has a finite optimal solution y^*/x^*.
  - Further, the optimal objective value is the same, i.e., $c^T x^* = b^T y^*$.
    - Proof. Based on weak duality, but more complex

Linear programming

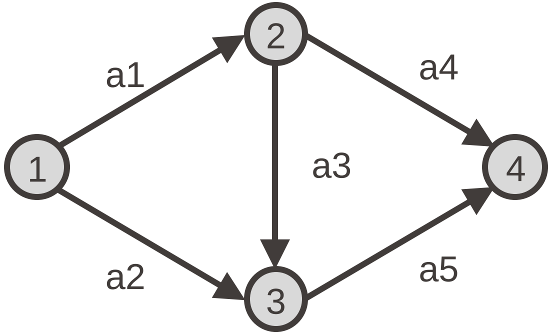
- General rules of conversion

Primal		Dual
max		min
objective coeff		RHS value
# vars		# constraints
= constraint		unrestricted var
≤ constraint	⇔	≥ 0 var
≥ constraint		≤ 0 var
≥ 0 var		≥ constraint
≤ 0 var		≤ constraint
unrestricted var		= constraint

Linear programming

- Dual of shortest path problem
  - Primal

$$\max_{x} \quad -t^T x$$
$$s.t.\ A_2 x = b_2$$
$$x \geq 0$$



$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$b_2 = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$$

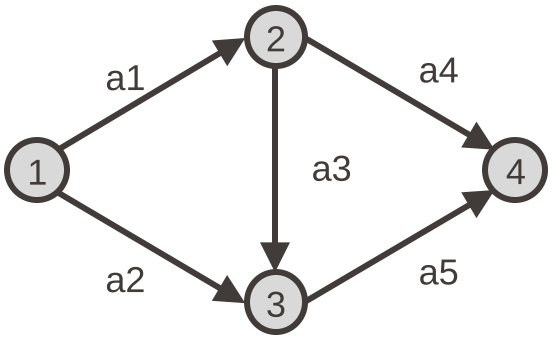
  - Dual

$$\min_{y} \quad b_2^T y$$
$$s.t.\ A_2^T y \geq -t$$

Linear programming

- Dual of shortest path problem
  - Primal

$$\max_{\mathrm{x}} \quad -\mathrm{t}^T\mathrm{x}$$
$$s.t.\ A_2\mathrm{x} = \mathrm{b}_2$$
$$\mathrm{x} \geq 0$$



  - Dual

$$\max_{\mathrm{u}} \quad \mathrm{b}_2^T\mathrm{u}$$
$$s.t.\ A_2^T\mathrm{u} \leq t$$

    - replace $\mathrm{y} = -\mathrm{u}$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\mathrm{b}_2 = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$$

Linear programming

- Dual of shortest path problem
  - Primal

$$\min_{x_{ij}} \sum_{(i,j)} t_{ij} x_{ij}$$

$$s.t. \sum_{j \in N_i^+} x_{ij} - \sum_{j \in N_i^-} x_{ji} = \begin{cases} 1 & i = r \\ -1 & i = s \\ 0 & \text{otherwise} \end{cases}, \qquad \forall i \in N$$

$$x_{ij} \geq 0, \qquad \forall (i,j)$$

  - Dual

$$\max_{u_i} \; u_r - u_s$$

$$s.t. \; u_i - u_j \leq t_{ij}, \qquad \forall i \in N$$

  - ***Q: What is the physical meaning of u_i^*?***

Linear programming

- Dual of shortest path problem
  - Primal

$$\min_{x_{ij}} \sum_{(i,j)} t_{ij}x_{ij}$$

$$s.t. \sum_{j\in N_i^+} x_{ij} - \sum_{j\in N_i^-} x_{ji} = \begin{cases} 1 & i = r \\ -1 & i = s \\ 0 & \text{otherwise} \end{cases}, \qquad \forall i \in N$$

$$x_{ij} \geq 0, \qquad \forall (i,j)$$

  - Dual

$$\max_{u_i} \; u_r - u_s$$

$$s.t. \; u_i - u_j \leq t_{ij}, \qquad \forall i \in N$$

    - when restricting $u_s = 0$, u_r^* is the min cost from origin r to destination s
    - ***Q: Does it remind you some shortest path algorithm?***

Questions?

Convex optimization

- Minimize a ***convex*** objective function of decision variables
  - Subject to ***convex*** constraints

- General formulation

$$\min_{\mathrm{x}} \quad f(\mathrm{x})$$

$$s.t. \quad g(\mathrm{x}) \leq 0$$

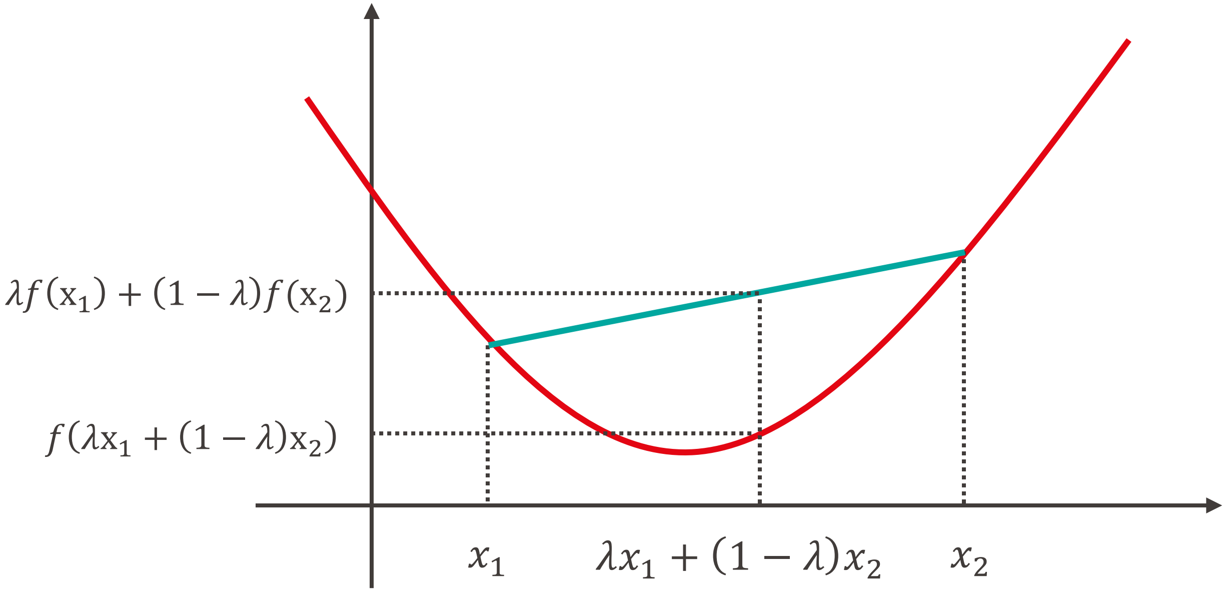
$$h(\mathrm{x}) = 0$$

- ***Q: What are convex function and convex set?***

Convex optimization

- Convex function
  - A function $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex if $\forall \mathrm{x}_1, \mathrm{x}_2 \in X$ and $\lambda \in [0, 1]$

$$f(\lambda \mathrm{x}_1 + (1-\lambda)\mathrm{x}_2) \leq \lambda f(\mathrm{x}_1) + (1-\lambda) f(\mathrm{x}_2)$$



Convex optimization

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$$f(\lambda \mathrm{x}_1 + (1-\lambda)\mathrm{x}_2) \leq \lambda f(\mathrm{x}_1) + (1-\lambda) f(\mathrm{x}_2)$$

- ***Q: Which of the following are convex?***
  - $f(x) = x$
  - $f(x) = x^2$
  - $f(x) = x^3$
  - $f(x) = e^x$
  - $f(x) = \log x$

Convex optimization

- Convex function
  - A function $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex if $\forall \mathrm{x}_1, \mathrm{x}_2 \in X$ and $\lambda \in [0,1]$

$$f(\lambda \mathrm{x}_1 + (1-\lambda)\mathrm{x}_2) \leq \lambda f(\mathrm{x}_1) + (1-\lambda) f(\mathrm{x}_2)$$

  - $f(x) = x$
  - $f(x) = x^2$
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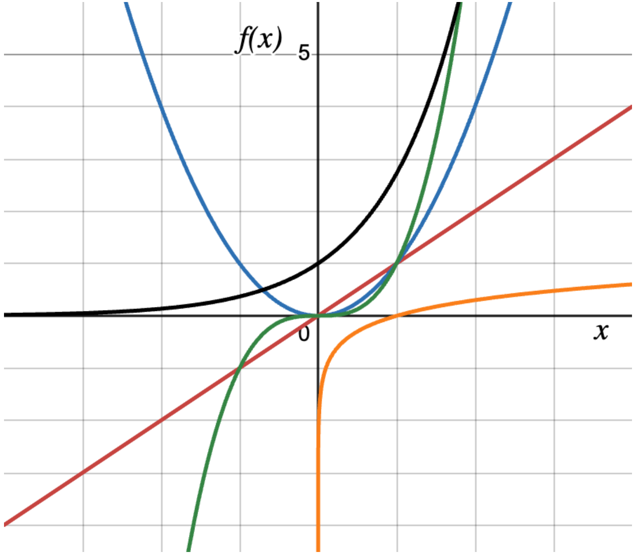


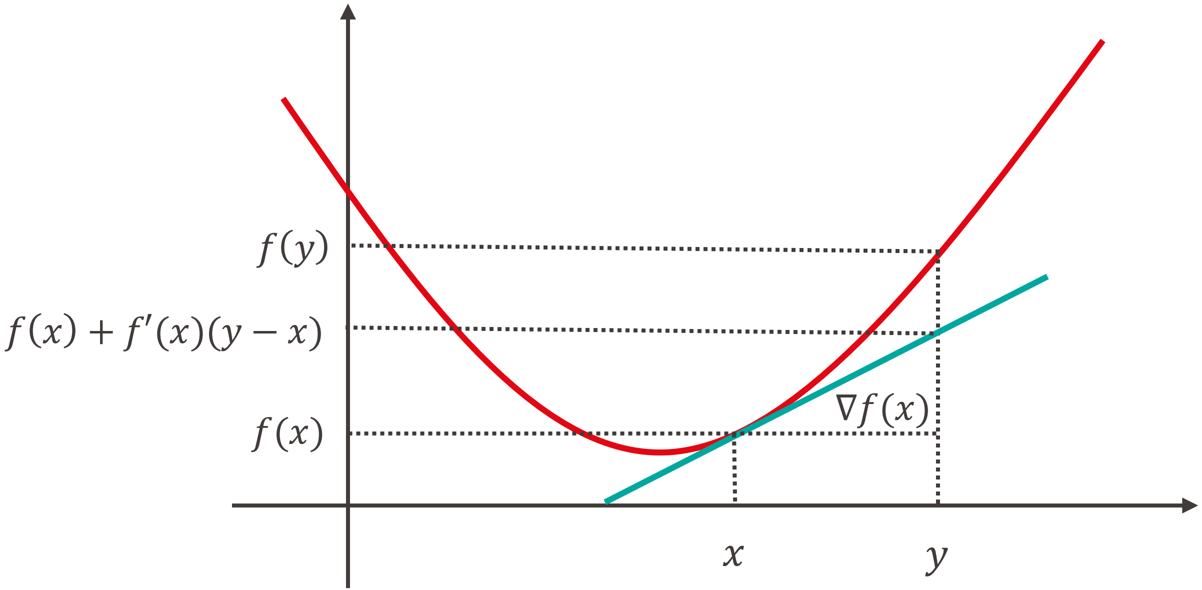
- ***Q: Is there an easier way to check convexity?***

Convex optimization

- Convex function
  - A ***differentiable*** function $f\colon X \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex iff $\forall \mathrm{x}, \mathrm{y} \in X$

$$f(\mathrm{y}) \geq f(\mathrm{x}) + \nabla f(\mathrm{x})^T(\mathrm{y} - \mathrm{x})$$

where $\nabla f(\mathrm{x}) = \left[\frac{\partial f(\mathrm{x})}{\partial \mathrm{x}_1}, \dots, \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}_\mathrm{n}}\right] \in \mathbb{R}^n$ is the gradient of f at x



Convex optimization

- Convex function
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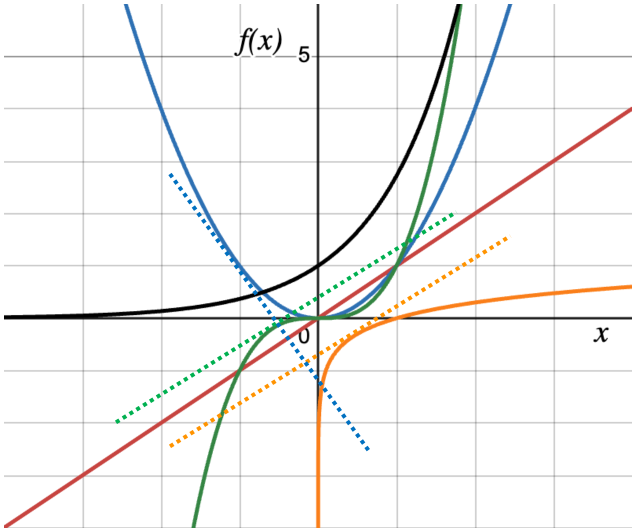
  - $f(x) = x$
  - $f(x) = x^2$
  - $f(x) = x^3$
  - $f(x) = e^x$
  - $f(x) = \log x$



- ***Q: What if $f(x)$ is twice-differentiable?***

Convex optimization

- Convex function
  - A ***twice-differentiable*** function $f: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is convex iff $\forall \mathrm{x} \in X$, Hessian matrix

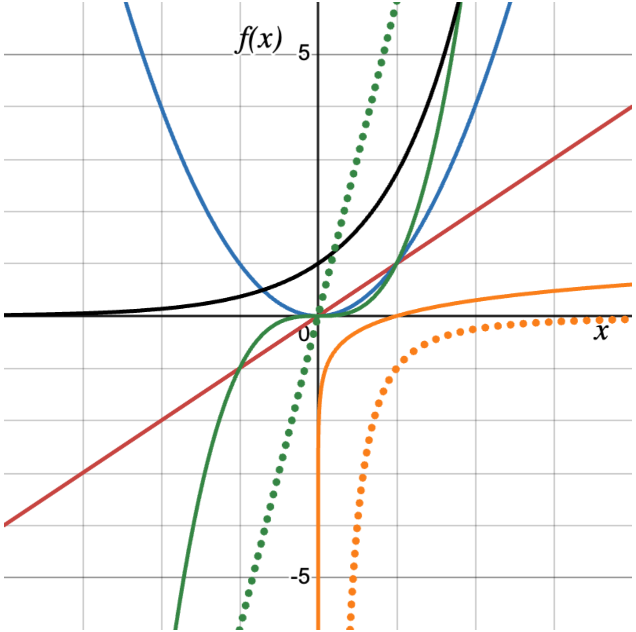
$$\nabla^2 f(\mathrm{x}) = \begin{bmatrix} \frac{\partial^2 f(\mathrm{x})}{\partial x_1^2}, \frac{\partial^2 f(\mathrm{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(\mathrm{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathrm{x})}{\partial x_n \partial x_1}, \frac{\partial^2 f(\mathrm{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(\mathrm{x})}{\partial x_n^2} \end{bmatrix} \succcurlyeq 0$$

  - Matrix A is called ***positive semidefinite***, i.e., $A \succcurlyeq 0$, if $\mathrm{x}^T A \mathrm{x} \geq 0, \forall \mathrm{x} \neq 0$

Convex optimization

- Convex function
  - A ***twice-differentiable*** function $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex iff $\forall \mathrm{x}, \mathrm{y} \in X$, Hessian matrix $\nabla^2 f(\mathrm{x}) \succcurlyeq 0$

- $f(x) = x$ $\Rightarrow f''(x) = 0$
- $f(x) = x^2$ $\Rightarrow f''(x) = 2$
- $f(x) = x^3$ $\Rightarrow f''(x) = 6x$
- $f(x) = e^x$ $\Rightarrow f''(x) = e^x$
- $f(x) = \log x$ $\Rightarrow f''(x) = -\frac{1}{x^2}$

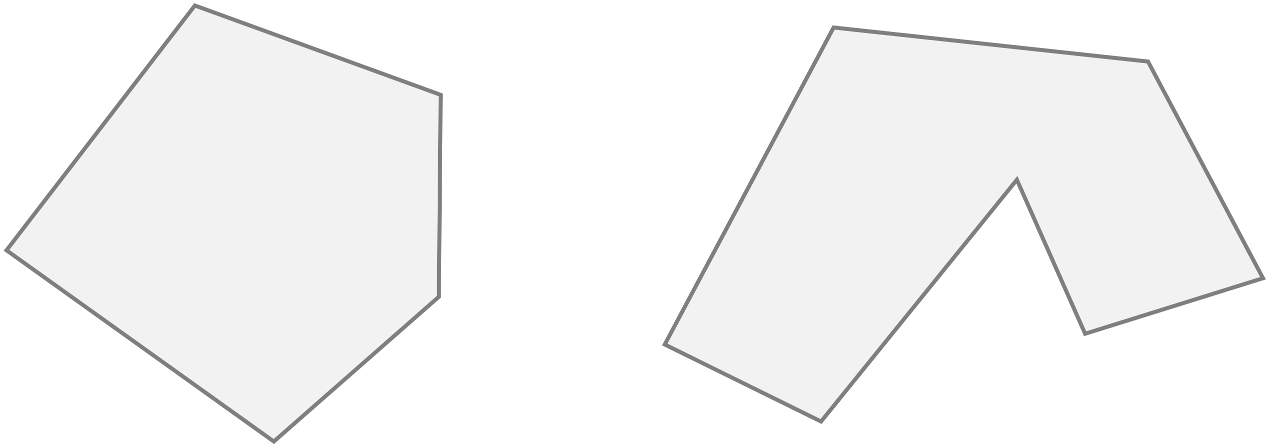


Convex optimization

- Convex set
  - A set $X \subseteq \mathbb{R}^n$ is convex iff $\forall \mathrm{x}_1, \mathrm{x}_2 \in X$ and $\lambda \in [0,1]$,

$$\lambda \mathrm{x}_1 + (1-\lambda) f(\mathrm{x}_2) \in X$$

    - a ***convex combination*** of any two points in the set also belongs to the set



- ***Q: Are these convex sets? Why and why not?***

Convex optimization

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  - A set $X \subseteq \mathbb{R}^n$ is convex iff $\forall \mathrm{x}_1, \mathrm{x}_2 \in X$ and $\lambda \in [0, 1]$,

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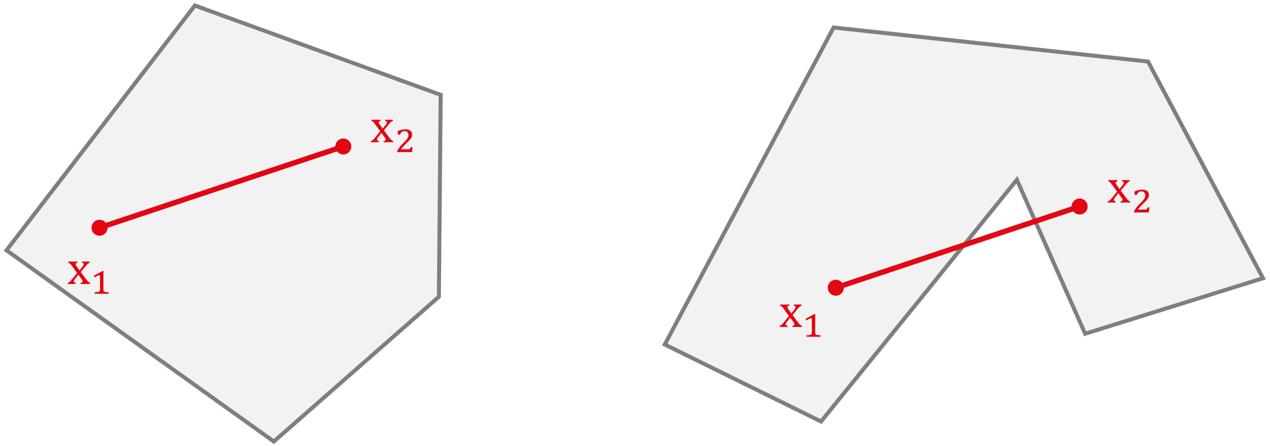

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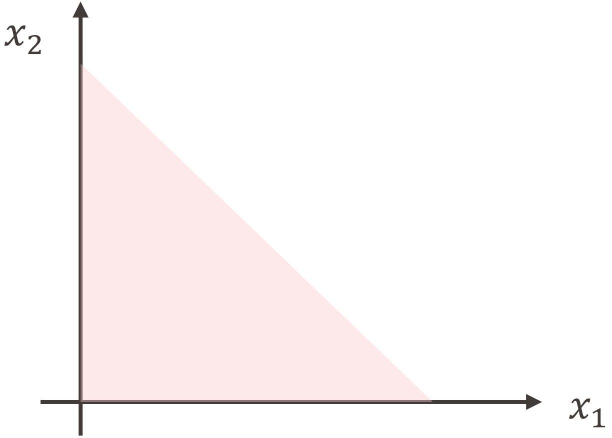
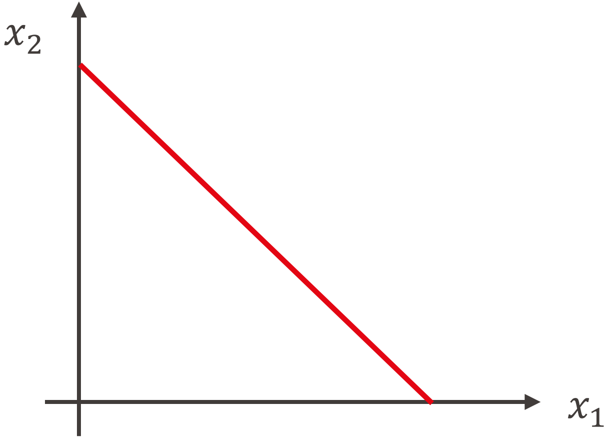


$X = \{(x_1, x_2) | x_1 + x_2 \leq 1, x_1, x_2 \geq 0\}$



$X = \{(x_1, x_2) | x_1 + x_2 = 1, \}$

- ***Q: Are these convex sets? Why and why not?***

Questions?

Convex optimization

- Minimize a ***convex*** objective function of decision variables
  - Subject to ***convex*** constraints

- General formulation

$$\min_{x} \; f(x)$$
$$s.t. \quad g(x) \leq 0$$
$$h(x) = 0$$

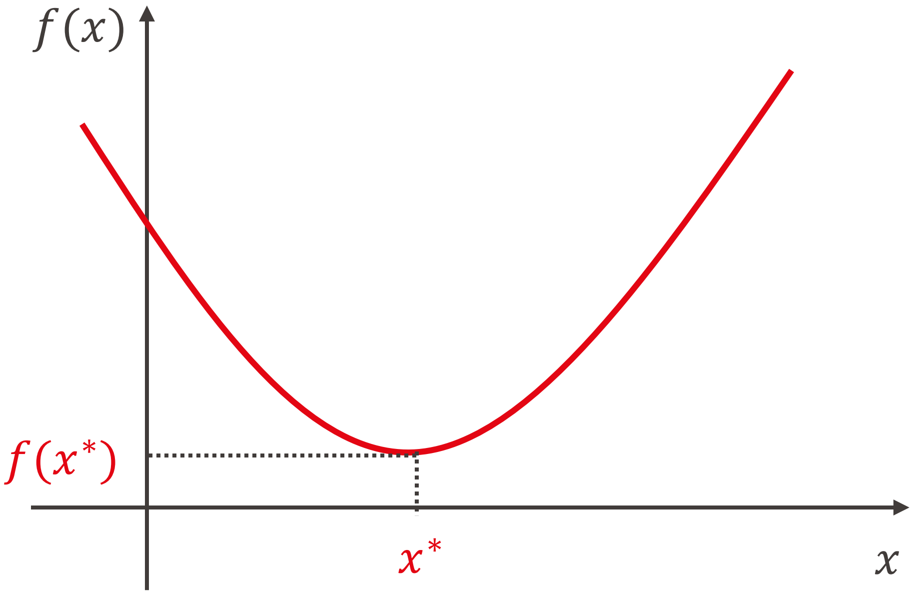
$$\min_{x} \; c^T x$$
$$s.t. \quad A_1 x \leq b_1$$
$$A_2 x = b_2$$

- ***Q: Does linear programming belong to convex optimization?***

Convex optimization

- How to solve a convex optimization problem?
  - Single variable x
  - Differentiable objective $f(x)$

$$\min_{x} \quad f(x)$$
$$s.t. \quad x \in \mathbb{R}$$



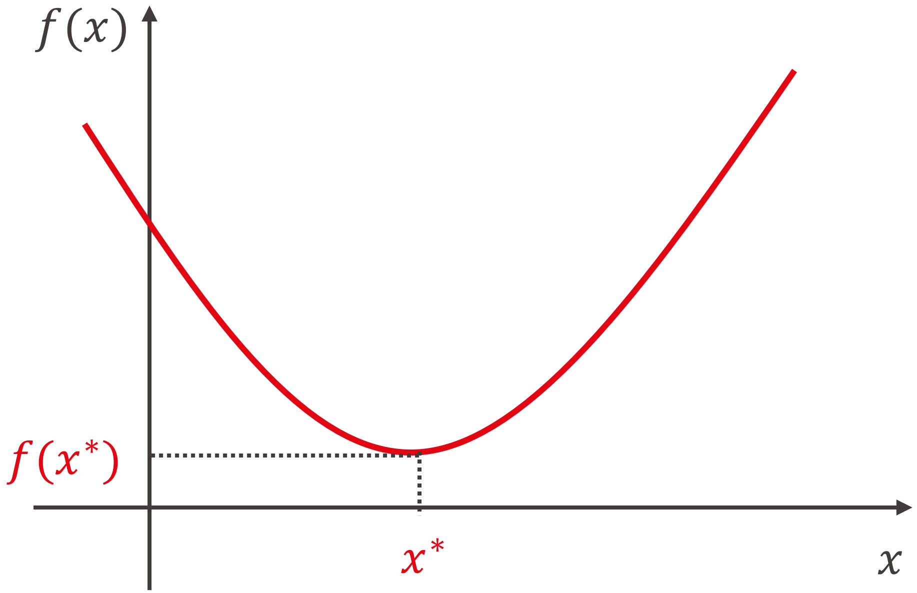
- ***Q: What is the optimal solution and its property?***

Convex optimization

- How to solve a convex optimization problem?
  - Single variable x
  - Differentiable objective $f(x)$

$$\min_x \quad f(x)$$
$$s.t. \quad x \in \mathbb{R}$$

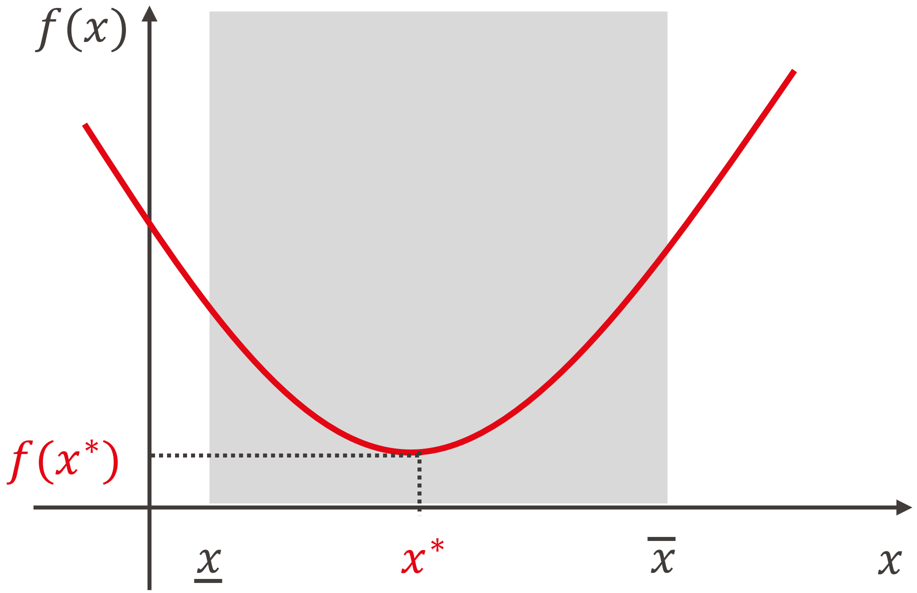
$$\nabla f(x^*) = 0$$



Convex optimization

- How to solve a convex optimization problem?
  - Single variable x
  - Differentiable objective $f(x)$

$$\min_{x} \quad f(x)$$
$$s.t. \quad x \in [\underline{x}, \overline{x}]$$



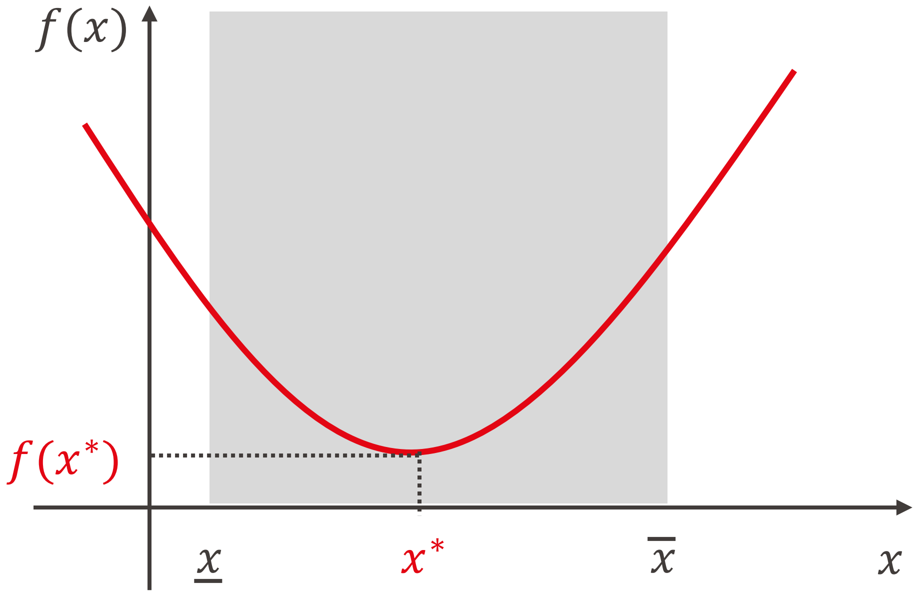
- ***Q: Where is the optimal solution?***

Convex optimization

- How to solve a convex optimization problem?
  - Single variable x
  - Differentiable objective $f(x)$

$$\min_{x} \quad f(x)$$
$$s.t. \quad x \in [\underline{x}, \overline{x}]$$

$\nabla f(x^*) = 0$ if $x^* \in [\underline{x}, \overline{x}]$

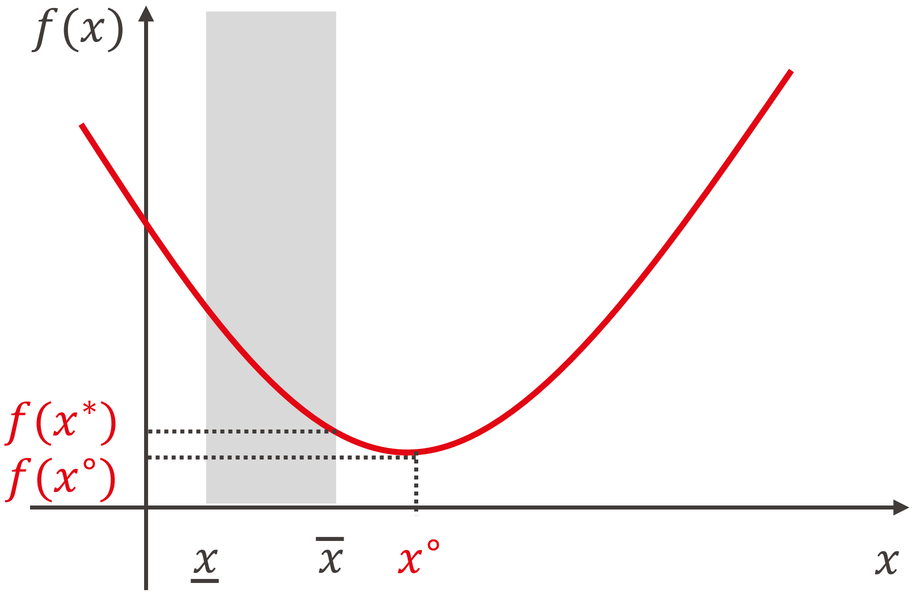


Convex optimization

- How to solve a convex optimization problem?
  - Single variable x
  - Differentiable objective $f(x)$

$$\min_{x} \quad f(x)$$
$$s.t. \quad x \in [\underline{x}, \overline{x}]$$

$$\nabla f(x^{\circ}) = 0$$
$$x^{*} = \overline{x} \text{ if } x^{\circ} \geq \overline{x}$$

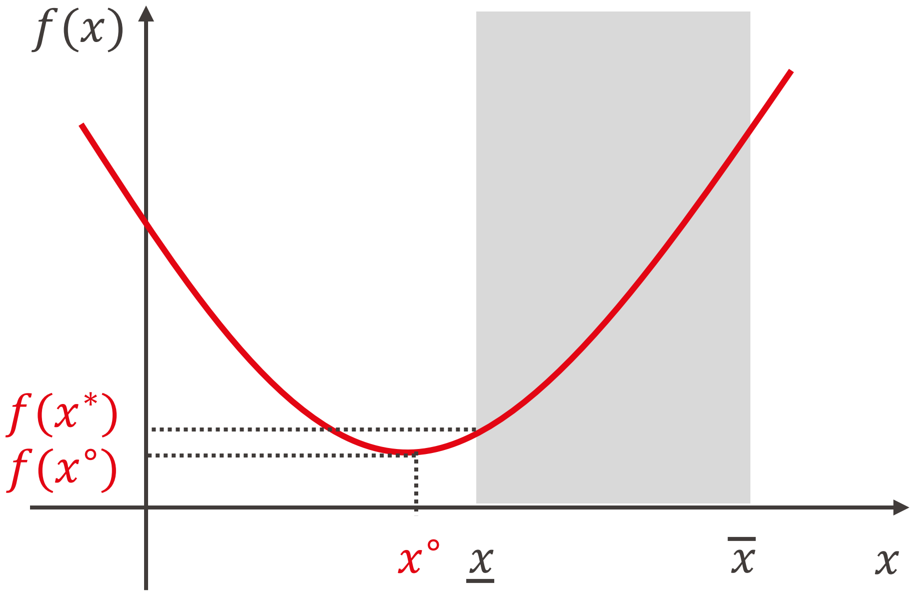


Convex optimization

- How to solve a convex optimization problem?
  - Single variable x
  - Differentiable objective $f(x)$

$$\min_{x} \quad f(x)$$
$$s.t. \quad x \in [\underline{x}, \overline{x}]$$

$$\nabla f(x^{\circ}) = 0$$
$$x^{*} = \underline{x} \text{ if } x^{\circ} \leq \underline{x}$$

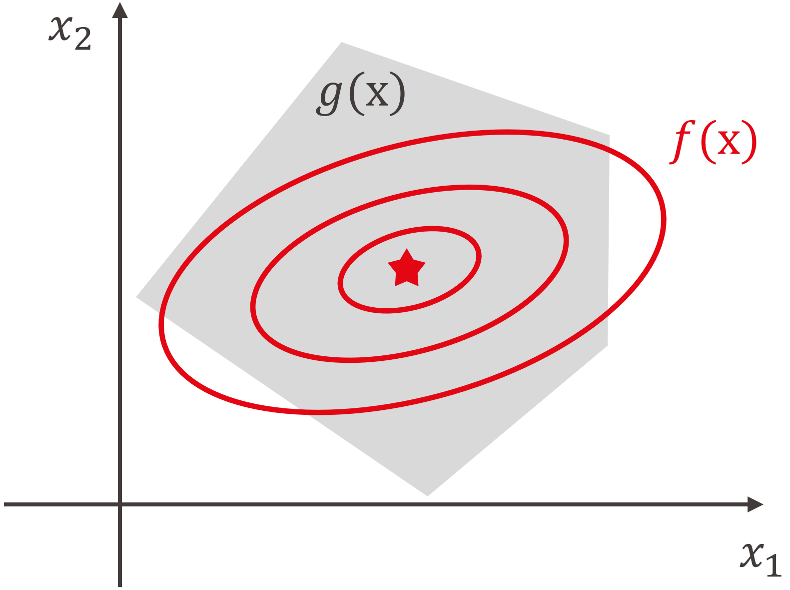


Convex optimization

- How to solve a convex optimization problem?
  - Multi-variable $\mathrm{x} = (x_1, \ldots, x_n)^T$
  - Differentiable objective $f(\mathrm{x})$

$$\min_{\mathrm{x}} \quad f(\mathrm{x})$$
$$s.t. \quad g(\mathrm{x}) \leq 0$$

$\nabla f(\mathrm{x}^*) = 0$ if $g(\mathrm{x}^*) \leq 0$



Convex optimization

- How to solve a convex optimization problem?
  - Multi-variable $\mathrm{x} = (x_1, \dots, x_n)^T$
  - Differentiable objective $f(\mathrm{x})$

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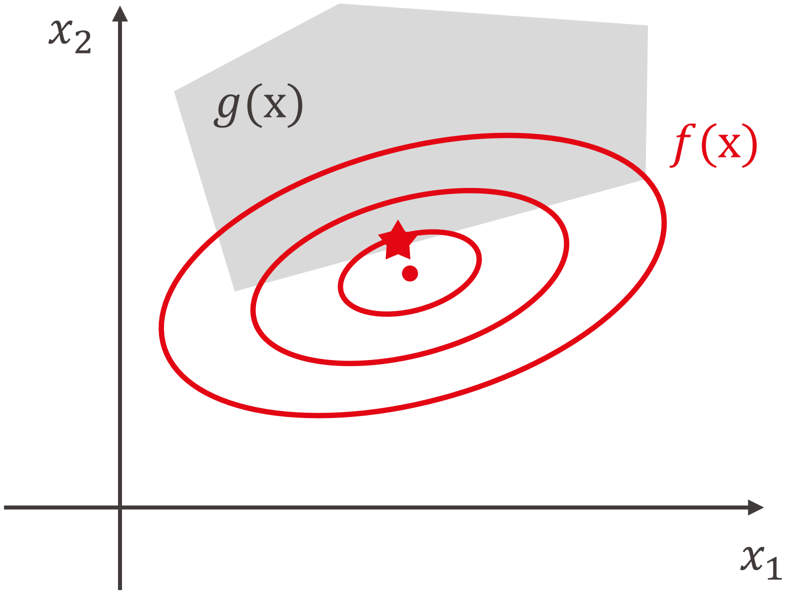

- ***Q: What if x° s.t. $\nabla f(x^{\circ}) = 0$ does not satisfy $g(x^{\circ}) \leq 0$***

Convex optimization

- Lagrangian function

$$\mathcal{L}(\mathrm{x}, \lambda, \mu) = f(\mathrm{x}) + \lambda^T g(x) + \mu^T h(x)$$

where λ, μ are called Lagrange multipliers (dual variables)

- Primal problem

$$\min_{\mathrm{x}} \quad f(\mathrm{x})$$
$$s.t. \quad g(\mathrm{x}) \leq 0$$
$$h(\mathrm{x}) = 0$$

  - Feasible set

$$X = \{\mathrm{x} | g(\mathrm{x}) \leq 0, h(\mathrm{x}) = 0\}$$

- Dual problem

$$\max_{\lambda,\mu} \mathcal{F}(\lambda, \mu) = \inf_{\mathrm{x} \in X} \mathcal{L}(\mathrm{x}, \lambda, \mu)$$
$$s.t. \quad \lambda \geq 0$$

  - Lower bound of primal problem

$$\mathcal{F}(\lambda, \mu) \leq \mathcal{L}(\mathrm{x}, \lambda, \mu) \leq f(\mathrm{x}), \ \forall \mathrm{x} \in X$$

Convex optimization

- Primal problem

$$\min_{x} \quad f(x)$$
$$s.t. \quad g(x) \leq 0$$
$$h(x) = 0$$

- Dual problem

$$\max_{\lambda,\mu} \mathcal{F}(\lambda,\mu) = \inf_{x \in X} \mathcal{L}(x,\lambda,\mu)$$
$$s.t. \quad \lambda \geq 0$$

- Weak duality
  - Suppose f^* and $\mathcal{F}^*$ are the optimal values of primal and dual problems, then $\mathcal{F}^* \leq f^*$.
- Strong duality
  - Suppose f^* and $\mathcal{F}^*$ are the optimal values of primal and dual problems, then $\mathcal{F}^* = f^*$.

- ***Q: When do weak and strong duality hold?***

Convex optimization

- Primal problem

$$\min_{\mathrm{x}} \quad f(\mathrm{x})$$
$$s.t. \quad g(\mathrm{x}) \leq 0$$
$$h(\mathrm{x}) = 0$$

- Dual problem

$$\max_{\lambda,\mu} \mathcal{F}(\lambda,\mu) = \inf_{\mathrm{x}\in X} \mathcal{L}(\mathrm{x},\lambda,\mu)$$
$$s.t. \quad \lambda \geq 0$$

- Weak duality (always true)
  - Suppose f^* and $\mathcal{F}^*$ are the optimal values of primal and dual problems, then $\mathcal{F}^* \leq f^*$.
- Strong duality (require some constraint qualifications)
  - Suppose f^* and $\mathcal{F}^*$ are the optimal values of primal and dual problems, then $\mathcal{F}^* = f^*$.

Convex optimization

- Suppose strong duality holds, then

$$
\begin{aligned}
f(\mathrm{x}^*) &= \mathcal{F}(\lambda^*, \mu^*) \\
&= \inf_{\mathrm{x}\in X} \mathcal{L}(\mathrm{x}, \lambda^*, \mu^*) = \inf_{\mathrm{x}\in X}\left(f(\mathrm{x}) + (\lambda^*)^T g(\mathrm{x}) + (\mu^*)^T h(\mathrm{x})\right) \\
&\le f(\mathrm{x}^*) + (\lambda^*)^T g(\mathrm{x}^*) + \boxed{(\mu^*)^T h(\mathrm{x}^*)} \;= 0
\end{aligned}
$$

$$\Rightarrow \quad (\lambda^*)^T g(\mathrm{x}^*) \ge 0$$

Convex optimization

- Suppose strong duality holds, then

$$
\begin{aligned}
f(\mathrm{x}^*) &= \mathcal{F}(\lambda^*, \mu^*) \\
&= \inf_{\mathrm{x} \in X} \mathcal{L}(\mathrm{x}, \lambda^*, \mu^*) = \inf_{\mathrm{x} \in X} \left(f(\mathrm{x}) + (\lambda^*)^T g(\mathrm{x}) + (\mu^*)^T h(\mathrm{x}) \right) \\
&\leq f(\mathrm{x}^*) + (\lambda^*)^T g(\mathrm{x}^*)
\end{aligned}
$$

$$
\Rightarrow \quad (\lambda^*)^T g(\mathrm{x}^*) \geq 0
$$

- Primal and dual feasibility requires $g(\mathrm{x}^*) \leq 0, \lambda^* \geq 0 \Rightarrow (\lambda^*)^T g(\mathrm{x}^*) \leq 0$

Convex optimization

- Suppose strong duality holds, then

$$f(\mathrm{x}^*) = \mathcal{F}(\lambda^*, \mu^*)$$

$$= \inf_{\mathrm{x}\in X} \mathcal{L}(\mathrm{x}, \lambda^*, \mu^*) = \inf_{\mathrm{x}\in X} \left(f(\mathrm{x}) + (\lambda^*)^T g(\mathrm{x}) + (\mu^*)^T h(\mathrm{x}) \right)$$

$$\leq f(\mathrm{x}^*) + (\lambda^*)^T g(\mathrm{x}^*)$$

$$\Rightarrow \quad (\lambda^*)^T g(\mathrm{x}^*) \geq 0$$

- Primal and dual feasibility requires $g(\mathrm{x}^*) \leq 0, \lambda^* \geq 0 \Rightarrow (\lambda^*)^T g(\mathrm{x}^*) \leq 0$
- Complementary conditions

$$(\lambda^*)^T g(\mathrm{x}^*) = 0$$

- ***Q: What does it imply?***

Convex optimization

- Suppose strong duality holds, then

$$
\begin{aligned}
f(\mathrm{x}^*) &= \mathcal{F}(\lambda^*, \mu^*) \\
&= \inf_{\mathrm{x}\in X} \mathcal{L}(\mathrm{x}, \lambda^*, \mu^*) = \inf_{\mathrm{x}\in X}\Big(f(\mathrm{x}) + (\lambda^*)^T g(\mathrm{x}) + (\mu^*)^T h(\mathrm{x})\Big) \\
&\leq f(\mathrm{x}^*) + (\lambda^*)^T g(\mathrm{x}^*)
\end{aligned}
$$

$$
\Rightarrow \quad (\lambda^*)^T g(\mathrm{x}^*) \geq 0
$$

- Primal and dual feasibility requires $g(\mathrm{x}^*) \leq 0, \lambda^* \geq 0 \Rightarrow (\lambda^*)^T g(\mathrm{x}^*) \leq 0$
- Complementary conditions

$$
(\lambda^*)^T g(\mathrm{x}^*) = 0
$$

  - If $\lambda^* > 0$, then $g(\mathrm{x}^*) = 0$, and if $g(\mathrm{x}^*) < 0$, then $\lambda^* = 0$
  - ***Q: Does it remind you the traffic equilibrium conditions?***

Convex optimization

- Karush-Kuhn-Tucker (KKT) conditions
  - Necessary conditions for the optimal solution to the primal and dual problem
  - Suppose strong duality holds, then x^* and λ^*, μ^* must satisfy
    - Primal feasibility $\quad g(\mathrm{x}^*) \leq 0, \quad h(\mathrm{x}^*) = 0$
    - Dual feasibility $\quad \lambda^* \geq 0$
    - Complementary $\quad (\lambda^*)^T g(\mathrm{x}^*) = 0$
    - Stationarity $\quad \nabla_{\mathrm{x}} \mathcal{L}(\mathrm{x}^*, \lambda^*, \mu^*) = 0$
- ***Q: Why the last condition hold at optimal solutions?***

Convex optimization

- Karush-Kuhn-Tucker (KKT) conditions
  - Necessary conditions for the optimal solution to the primal and dual problem
  - Suppose strong duality holds, then x^* and λ^*, μ^* must satisfy
    - Primal feasibility $g(\mathrm{x}^*) \leq 0, \quad h(\mathrm{x}^*) = 0$
    - Dual feasibility $\lambda^* \geq 0$
    - Complementary $(\lambda^*)^T g(\mathrm{x}^*) = 0$
    - Stationarity $\nabla_{\mathrm{x}} \mathcal{L}(\mathrm{x}^*, \lambda^*, \mu^*) = 0$

Proof. Due to strong duality,

$$f(\mathrm{x}^*) = \mathcal{F}(\lambda^*, \mu^*) = \inf_{\mathrm{x} \in X} \mathcal{L}(\mathrm{x}, \lambda^*, \mu^*)$$

Hence, x^* minimizes $\mathcal{L}(\mathrm{x}, \lambda^*, \mu^*)$, an unconstrained optimization, which implies $\nabla_{\mathrm{x}} \mathcal{L}(\mathrm{x}^*, \lambda^*, \mu^*) = 0$.

Convex optimization

- Karush-Kuhn-Tucker (KKT) conditions
  - Necessary conditions for the optimal solution to the primal and dual problem
    - ***also sufficient when f, g, h are all differentiable and convex***
  - Suppose strong duality holds, then x^* and λ^*, μ^* must satisfy
    - Primal feasibility $g(\mathrm{x}^*) \leq 0, \quad h(\mathrm{x}^*) = 0$
    - Dual feasibility $\lambda^* \geq 0$
    - Complementary $(\lambda^*)^T g(\mathrm{x}^*) = 0$
    - Stationarity $\nabla_{\mathrm{x}} \mathcal{L}(\mathrm{x}^*, \lambda^*, \mu^*) = 0$

Questions?

Convex optimization

- KKT conditions of shortest path

$$\min_{x_{ij}} \sum_{(i,j)} t_{ij} x_{ij}$$

$$s.t. \sum_{j \in N_i^+} x_{ij} - \sum_{j \in N_i^-} x_{ji} = b_i = \begin{cases} 1 & i = r \\ -1 & i = s \\ 0 & \text{otherwise} \end{cases}, \qquad \forall i \in N$$

$$x_{ij} \geq 0, \qquad \forall (i,j)$$

- Lagrangian

$$\mathcal{L}(\mathrm{x}, \lambda, \mu) = \sum_{(i,j)} t_{ij} x_{ij} + \sum_{(i,j)} \lambda_{ij}(-x_{ij}) + \sum_{i} \mu_i \left(\sum_{j \in N_i^+} x_{ij} - \sum_{j \in N_i^-} x_{ji} - b_i \right)$$

Convex optimization

- KKT conditions of shortest path
  - Stationarity

$$\frac{\partial \mathcal{L}}{\partial x_{ij}} = t_{ij} - \lambda_{ij} + \mu_i - \mu_j = 0 \quad \Rightarrow \quad \lambda_{ij} = t_{ij} + \mu_i - \mu_j$$

  - Primal feasibility $M\mathrm{x} = \mathrm{b}, \mathrm{x} \geq 0$
  - Dual feasibility $\lambda_{ij} \geq 0$
  - Complementary $\lambda_{ij} x_{ij} = 0$

- ***Q: How these conditions relate to the shortest path?***

Convex optimization

- KKT conditions of shortest path
  - Stationarity

$$\frac{\partial \mathcal{L}}{\partial x_{ij}} = t_{ij} - \lambda_{ij} + \mu_i - \mu_j = 0 \;\Rightarrow\; \lambda_{ij} = t_{ij} + \mu_i - \mu_j$$

  - Primal feasibility $M\mathrm{x} = \mathrm{b}, \mathrm{x} \geq 0$
  - Dual feasibility $\lambda_{ij} \geq 0$
  - Complementary $\lambda_{ij} x_{ij} = 0$

forward Bellman optimality condition

- Link (i, j) is on the shortest path $\Rightarrow$ $x_{ij} = 1 \;\Rightarrow\; \lambda_{ij} = 0 \Rightarrow$ $\mu_j = \mu_i + t_{ij}$
  - μ_i as the shortest distance from origin to node i

Questions?

Convex optimization

- KKT conditions vs traffic equilibrium
  - Complementary conditions for deterministic UE

$$f_k^*(c_k^* - \mu_w^*) = 0, \qquad \forall k \in P_w, w \in W$$

$$c_k^* \geq \mu_w^*, \qquad \forall k \in P_w, w \in W$$

where

- f_k^*, c_k^*: equilibrium flow and cost of path k
- μ_w^*: equilibrium min path cost between OD pair w
- P_w, W: set of path between OD pair w and set of OD pairs

Convex optimization

- KKT conditions vs traffic equilibrium
  - Complementary conditions for deterministic UE

$$(\mathrm{f}^*)^T(\mathrm{c}^* - \Lambda^T\mu^*) = 0$$

$$\mathrm{c}^* - \Lambda^T\mu^* \geq 0$$

$$\Lambda\mathrm{f}^* = \mathrm{q}$$

$$\mathrm{f}^* \geq 0$$

- f^*: equilibrium path flow
- c^*: equilibrium path cost
- μ^*: equilibrium min path cost
- Λ: OD-path incidence matrix
- q: demand vector

  - KKT conditions

$$\nabla_{\mathrm{x}}\mathcal{L}(\mathrm{x}^*, \lambda^*, \mu^*) = 0$$

$$g(\mathrm{x}^*) \leq 0$$

$$h(\mathrm{x}^*) = 0$$

$$\lambda^* \geq 0$$

$$(\lambda^*)^T g(\mathrm{x}^*) = 0$$

- Replace x by f, $g(\mathrm{x}) = -\mathrm{x}$, and $h(\mathrm{x}) = \mathrm{q} - \Lambda\mathrm{f}$

Convex optimization

- KKT conditions vs traffic equilibrium
  - Complementary conditions for deterministic UE

$$(\mathrm{f}^*)^T(\mathrm{c}^* - \Lambda^T\mu^*) = 0$$

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- c^*: equilibrium path cost
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- Λ: path-OD incidence matrix
- q: demand vector

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$$\nabla_{\mathrm{f}}\mathcal{L}(\mathrm{f}^*, \lambda^*, \mu^*) = 0$$

$$\mathrm{f}^* \geq 0$$

$$\Lambda \mathrm{f}^* = \mathrm{q}$$

$$\lambda^* \geq 0$$

$$(\lambda^*)^T \mathrm{f}^* = 0$$

- Replace x by f, $g(\mathrm{x}) = -\mathrm{x}$, and $h(\mathrm{x}) = \mathrm{q} - \Lambda \mathrm{f}$
- Set $\nabla_{\mathrm{f}}\mathcal{L}(\mathrm{f}^*, \lambda^*, \mu^*) = c^* - \lambda^* - \Lambda^T\mu^*$

Convex optimization

- KKT conditions vs traffic equilibrium
  - Complementary conditions for deterministic UE

$$(\mathrm{f}^*)^T(\mathrm{c}^* - \Lambda^T\mu^*) = 0$$

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$$\mathrm{f}^* \geq 0$$

- f^*: equilibrium path flow
- c^*: equilibrium path cost
- μ^*: equilibrium min path cost
- Λ: path-OD incidence matrix
- q: demand vector

  - KKT conditions

$$c^* - \Lambda^T\mu^* = \lambda^*$$

$$\mathrm{f}^* \geq 0$$

$$\Lambda \mathrm{f}^* = \mathrm{q}$$

$$\lambda^* \geq 0$$

$$(\lambda^*)^T\mathrm{f}^* = 0$$

- Replace x by f, $g(\mathrm{x}) = -\mathrm{x}$, and $h(\mathrm{x}) = \mathrm{q} - \Lambda\mathrm{f}$
- Set $\nabla_{\mathrm{f}}\mathcal{L}(\mathrm{f}^*, \lambda^*, \mu^*) = c^* - \lambda^* - \Lambda^T\mu^*$
- Combine 1st and 4th condition and plug 1st condition into 5th condition

Convex optimization

- KKT conditions vs traffic equilibrium
  - Complementary conditions for deterministic UE

$$(\mathrm{f}^*)^T(\mathrm{c}^* - \Lambda^T\mu^*) = 0$$

$$\mathrm{c}^* - \Lambda^T\mu^* \geq 0$$

$$\Lambda\mathrm{f}^* = \mathrm{q}$$

$$\mathrm{f}^* \geq 0$$

- f^*: equilibrium path flow
- c^*: equilibrium path cost
- μ^*: equilibrium min path cost
- Λ: path-OD incidence matrix
- q: demand vector

  - KKT conditions

$$c^* - \Lambda^T\mu^* \geq 0$$

$$\mathrm{f}^* \geq 0$$

$$\Lambda\mathrm{f}^* = \mathrm{q}$$

$$(c^* - \Lambda^T\mu^*)^T\mathrm{f}^* = 0$$

- Replace x by f, $g(\mathrm{x}) = -\mathrm{x}$, and $h(\mathrm{x}) = \mathrm{q} - \Lambda\mathrm{f}$
- Set $\nabla_{\mathrm{f}}\mathcal{L}(\mathrm{f}^*, \lambda^*, \mu^*) = c^* - \lambda^* - \Lambda^T\mu^*$
- Combine 1st and 4th condition and plug 1st condition into 5th condition

Convex optimization

- KKT conditions vs traffic equilibrium
  - Complementary conditions for deterministic UE

$$(\mathrm{f}^*)^T(\mathrm{c}^* - \Lambda^T\mu^*) = 0$$

$$\mathrm{c}^* - \Lambda^T\mu^* \geq 0$$

$$\Lambda \mathrm{f}^* = \mathrm{q}$$

$$\mathrm{f}^* \geq 0$$

- f^*: equilibrium path flow
- c^*: equilibrium path cost
- μ^*: equilibrium min path cost
- Λ: path-OD incidence matrix
- q: demand vector

- ***Q: What does this equivalence imply?***
  - We can construct an optimization problem whose optimal solution must satisfy the equilibrium conditions

Questions?

Variational inequality

- Consider a convex optimization problem

$$\min_{\mathrm{x}} \quad f(\mathrm{x})$$
$$s.t. \quad \mathrm{x} \leq X$$

- $f(\cdot)$: differentiable convex function with gradient $F(\mathrm{x}) = \nabla f(\mathrm{x})$
- X: convex set

- Equivalent variational inequality (VI) problem
  - Find $\mathrm{x}^* \in X$ such that

$$\langle F(\mathrm{x}^*), \mathrm{x} - \mathrm{x}^* \rangle \geq 0, \qquad \forall \mathrm{x} \in X$$

  - ***Q: What does this equivalent imply?***

Variational inequality

- Consider a convex optimization problem

$$\min_{\mathrm{x}} \quad f(\mathrm{x})$$
$$s.t. \quad \mathrm{x} \leq X$$

- $f(\cdot)$: differentiable convex function with gradient $F(\mathrm{x}) = \nabla f(\mathrm{x})$
- X: convex set

- Equivalent variational inequality (VI) problem
  - Find $\mathrm{x}^* \in X$ such that

$$\langle F(\mathrm{x}^*), \mathrm{x} - \mathrm{x}^* \rangle \geq 0, \qquad \forall \mathrm{x} \in X$$

  - All differentiable convex programs have corresponding VI formulations
  - However, the reverse only holds under certain conditions

Variational inequality

- VI conditions vs traffic equilibrium
  - Find $\mathrm{x}^* \in X$ such that

$$\langle F(\mathrm{x}^*), \mathrm{x} - \mathrm{x}^* \rangle \geq 0, \qquad \forall \mathrm{x} \in X$$

  - Replace x by f, $F(\mathrm{x})$ by $\mathrm{c}(\mathrm{f})$, and specify $X = \{\mathrm{f} \,|\, \Lambda \mathrm{f} = \mathrm{q}, \mathrm{f} \geq 0\}$

$$\langle \mathrm{c}(\mathrm{f}^*), \mathrm{f} - \mathrm{f}^* \rangle \geq 0, \qquad \forall \mathrm{f} \in X$$

Variational inequality

- VI conditions vs traffic equilibrium
  - Find $x^* \in X$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in X$$

    - Replace x by f, $F(x)$ by $c(f)$, and specify $X = \{f \,|\, \Lambda f = q, f \geq 0\}$

$$\langle c(f^*), f - f^* \rangle \geq 0, \qquad \forall f \in X$$

$$\Rightarrow c(f^*)^T f \geq c(f^*)^T f^*, \qquad \forall f \in X$$

    - The inequality conditions implies that, given the equilibrium path cost $c(f^*)$, the equilibrium path flows f^* lead to the minimum total cost
    - Meanwhile, the total cost is minimized when all travelers take the shortest paths
    - Hence, f^* is the best response for all travelers, which implies equilibrium

  - ***Q: Does the reverse also hold? How to prove it?***

Questions?