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CIVIL 468: Dynamics of structures

Assignment #1

Notes:

1. You are allowed to work in groups of 2 people
2. Submissions shall be done through Moodle
3. Both students working in groups shall submit the same assignment in Moodle

Question 1 – (25 points) CORRECTED BY G. NAJID.

A rigid disk of mass m is mounted at the end of a flexible shaft as shown in Figure 1-1. Neglecting the weight of the shaft and neglecting damping, derive the equation of free torsional vibration of the disk. The shear modulus (of rigidity) of the shaft is G.

NOTE: You will have to review a bit your structural mechanics note on pure torsion.

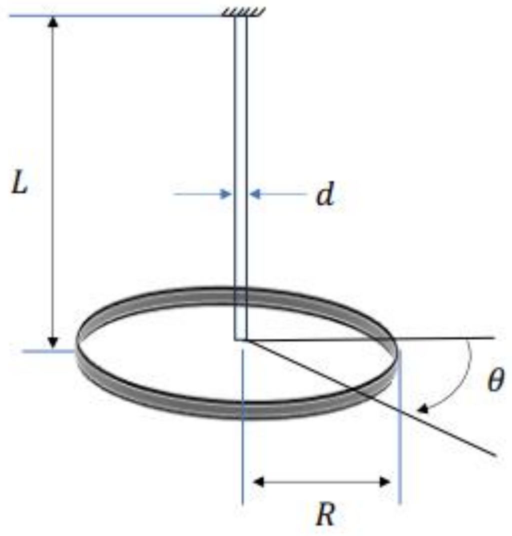


Figure 1-1 – Rigid disk and shaft in pure torsion

Solution

The forces acting on the disk are illustrated in the figure below.

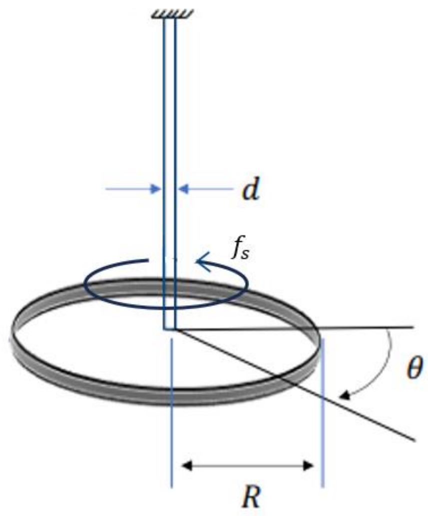


Figure 2: Forces acting on the disk

Writing the equation of motion using Newton's second law of motion in polar coordinates:

$$-f_s = I_0 \cdot \ddot{\theta}\ ; \qquad \text{(5 points)} \qquad (1)$$

Where:

$$I_0 = m \cdot \frac{R^2}{2}\ ; \qquad \text{(5 points)} \qquad (2)$$

Writing the torque-twist relation:

$$f_s = \left(\frac{GJ}{L}\right)\theta\ ; \qquad \text{(5 points)} \qquad (3)$$

Where:

$$J = \pi \cdot \frac{d^4}{32}\ ; \qquad (4)$$

Substituting Eq.(3) into Eq. (1):

$$I_0 \cdot \ddot{\theta} + \left(\frac{GJ}{L}\right)\theta = 0; \qquad \text{(5 points)} \qquad (5)$$

Or:

$$\left(m \cdot \frac{R^2}{2}\right)\ddot{\theta} + \left(\pi \cdot \frac{d^4}{32}\frac{G}{L}\right)\theta = 0; \qquad \text{(5 points)} \qquad (6)$$

Question 2 – (25 points) CORRECTED BY A. BONZLI

A heavy rigid platform of **weight w** is supported by four columns, **hinged at the top** and the **bottom**, and **braced** laterally in each side panel by two diagonal steel wires as shown in Figure 2-1. Each diagonal wire is **pretensioned** to a high stress; its cross-sectional **area is A** and elastic modulus is E. Neglecting the mass of the columns and wires, derive the equation of motion governing free vibration in (a) the x-direction, and (b) the y-direction.

NOTE: Because of high pretension, all wires contribute to the structural stiffness (not the same with cables that in compression do not provide stiffness)

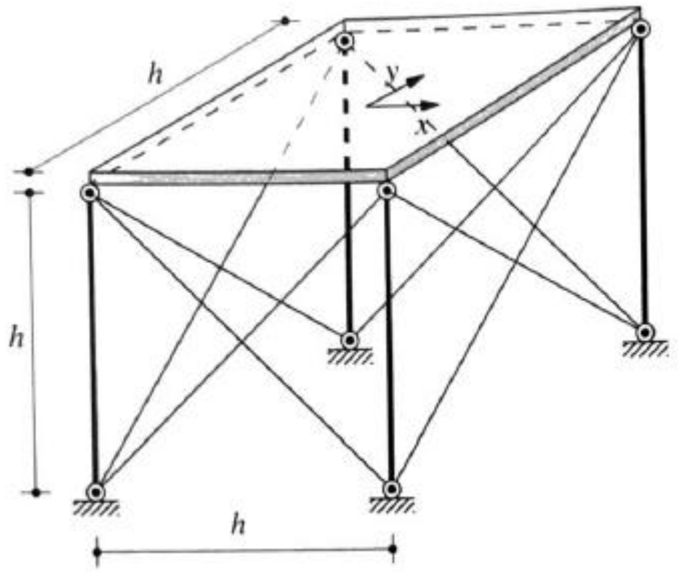


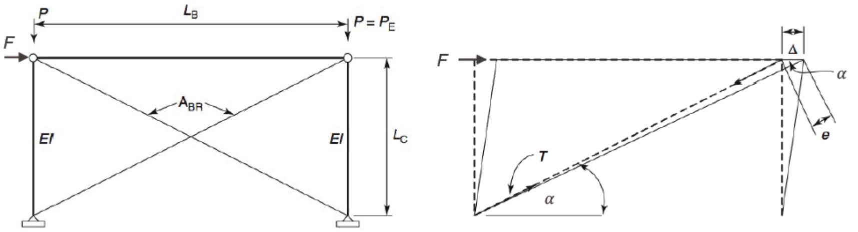
Figure 3-1 – single story building with wires

Solution

Part (a)
The first step to deriving the equation of motion is to determine the rigidity of the system. In the x-direction the stiffness is provided by the braces in tension as well as in compression. Since the braces are symmetrical, the stiffness from the brace in compression is the same as the stiffness of the brace in tension.

Note: In this case the braces working in compression add the same rigidity to the system as the braces in tension. This is only possible due to the prestressing. Without this prestressing the braces would start to buckle in compression and lose their resistance as soon as the buckling starts.

The stiffness of one frame with one brace is equal to:



Lateral Frame Stiffness,

$$K = \frac{F}{\Delta} = \frac{E A_{BR} L_B^2}{(L_B^2 + L_C^2)^{3/2}}$$

In our case this would give us:

$$k_{brace} = \frac{E \cdot A \cdot h^2}{(h^2+h^2)^{\frac{3}{2}}} = \frac{E \cdot A \cdot h^2}{(2 \cdot h^2)^{\frac{3}{2}}} = \frac{E \cdot A \cdot h^2}{2 \cdot \sqrt{2} \cdot h^3} = \frac{E \cdot A}{2 \cdot \sqrt{2} \cdot h} \qquad (2.1)$$

8 points for stiffness of one brace

In the x-direction we have four braces, therefore we have four times the rigidity (2.1):

$$k_x = 4 \cdot k_{brace} = \frac{4 \cdot E \cdot A}{2 \cdot \sqrt{2} \cdot h} = \frac{\sqrt{2} \cdot E \cdot A}{h} \qquad (2.2)$$

5 points for total stiffness

The second step is to determine the mass of the system. In this problem all weight from the structural elements is neglected. The only weight there is, is the weight of the platform. Weight is given in Newton/Kilonewton. To determine the mass of the platform, we need to divide it by the gravitational acceleration:

$$m = w/g \qquad (2.3)$$

2 points for transformation of weight in mass

Thus, the equation of motion governing free vibration in the x-direction can be written as:

$$m\ddot{u}_x + k_x u_x = \frac{w}{g}\ddot{u}_x + \frac{\sqrt{2} \cdot E \cdot A}{h} u_x = 0 \qquad (2.4)$$

5 points for total equation of motion

Part (b)
As can be seen in the figure above, the system has exactly same structural features as in x-direction. It has the same height, width, mass and number of braces, with the same cross section. Therefore, we can deduce:

$$k_y = k_x = \frac{\sqrt{2} \cdot E \cdot A}{h} \qquad (2.5)$$

And the equation of motion can be written as:

$$m\ddot{u}_y + k_y u_y = \frac{w}{g}\ddot{u}_y + \frac{\sqrt{2} \cdot E \cdot A}{h} u_y = 0 \qquad (2.6)$$

5 points for saying that system in y is the same.

Question 3 – (25 points) CORRECTED BY A. BONZLI

A machine weighting $1kN$ is mounted on a supporting system consisting of four springs and four dampers (i.e., device that supplements high levels of damping ratio). The vertical deflection of the supporting system under the weight of the machine is measured as $20mm$. The dampers are designed to reduce the amplitude of vertical vibration to one-eighth of the initial amplitude after two complete cycles of free vibration. Answer the following questions:

3.1. Calculate the undamped natural frequency.
3.2. Calculate the damping ratio.
3.3. Calculate the damped natural frequency.
3.4. Comment on the effect of damping on the natural frequency.

5 points for having consistent units.

Solution

Part 3.1 6 points

The first step is to calculate the natural frequency. We know that:

$$\omega_n = \sqrt{\frac{k}{m}} \tag{3.1}$$

Therefore, we need to determine the seismic mass, as well as the rigidity of the system. We know that the machine weights $w = 1\ kN$ and we neglect the weight of the structural system. We can calculate the seismic mass as:

$$m = \frac{w}{g} = \frac{1000\ N}{9.81\ m/s^2} = 101.94\ kg \tag{3.2}$$

2 / 6

To determine the total rigidity of the springs we can compare the deflection due to the weight of the machine to the weight of the machine:

$$k = \frac{1000\ N}{0.02\ m} = 50'000\ N/m \tag{3.3}$$

2 / 6

Finally, we can calculate the circular natural frequency as in formula (3.1)

2 / 6

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{50'000\ N/m}{101.94\ kg}} = \sqrt{\frac{50'000\ \cancel{kg} * \cancel{m}/s^2/\cancel{m}}{101.94\ \cancel{kg}}} = \sqrt{490.48/s^2} = 22.15\ rad/s$$

The natural frequency then is: $f_n = \frac{\omega_n}{2\pi} = 3.52\ /s$

Note: Make sure the units cancel each other out under the square root. It does not matter what units are used if you are consistent.

Part 3.2 6 points

The next step is to determine the damping ratio of the system. We are given the information that the dampers reduce the amplitude of the vibration to 1/8 of the initial amplitude within two complete cycles. Assuming small damping, we can use the following relation:

$$\ln\left(\frac{u_1}{u_{j+1}}\right) \approx 2\pi \cdot \zeta \cdot j \tag{3.4}$$

2 / 6

Knowing that in two cycles the amplitude reduces to 1/8 we can determine $j = 2$ and $u_3 = \frac{u_1}{8}$. Thus, we find:

2 / 6

$$\zeta = \frac{\ln(8)}{2\pi \cdot 2} = 0.165 = 16.5\ \%$$

2 / 6 for assuming the right j and u3

Note: It is also right to assume the exact relationship between damping and amplitude. The formula would be the following:

$$\ln\left(\frac{u_1}{u_{j+1}}\right) = \frac{2\pi \cdot \zeta \cdot j}{\sqrt{1-\zeta^2}} \tag{3.5}$$

Solving the equation with the same j and u_3 we get:

$$\zeta = 0.163 = 16.3\ \%$$

Part 3.3 4 points

Next, we want to know the damped circular natural frequency of the system. The damped natural frequency is defined as:

$$\omega_D = \omega_n \cdot \sqrt{1-\zeta^2} \tag{3.6}$$

Implementing it numerically we get:

$$\omega_D = \omega_n \cdot \sqrt{1-\zeta^2} = 22.15 \cdot \sqrt{1-0.165^2} = 21.84/sec$$

Part 3.4 4 points

By comparing the natural frequency ω_n to the damped natural frequency ω_D, we can see that the damping slightly reduces the natural frequency of vibration of the system. This effect can be seen by comparing the curves of a damped to an undamped system as in the figure below. Some people call this damped period also "pseudo-period" as it is not the natural period of the system but the one measurable. Usually, the difference between the two periods is not very big and oftentimes we can assume $\omega_D \approx \omega_n$.

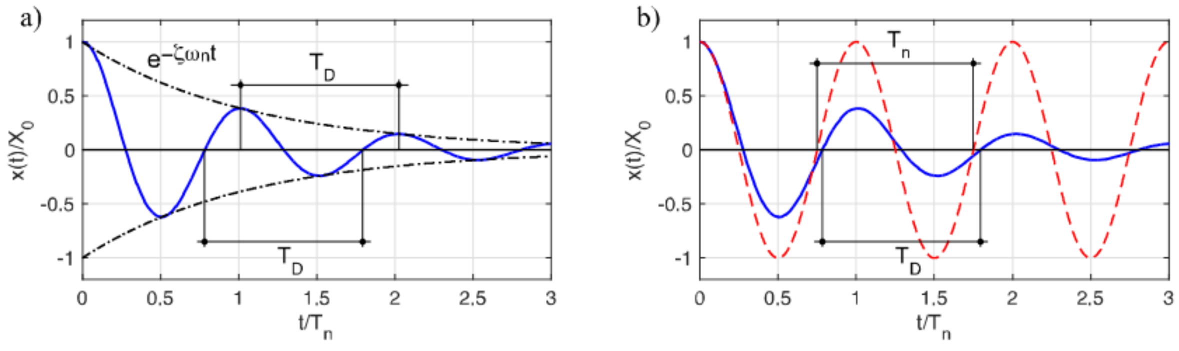


Figure 3-1: period of vibration with/without damping
(Figure from the book "Dynamique des Structures" de P. Lestuzzi et I.F.C. Smith)

Question 4 – (25 points) CORRECTED BY G. NAJID.

An undamped single-degree-of-freedom system is subjected to the triangular pulse in Figure 4-1. Answer to the following questions:

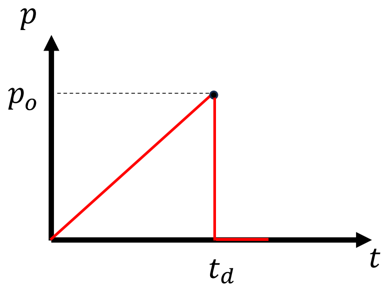


Figure 4-1 – Triangular pulse

4.1. Show that the displacement response is as follows:

$$\frac{u(t)}{(u_{st})_o} = \begin{cases} \frac{t}{t_d} - \frac{1}{2\pi}\left(\frac{T_n}{t_d}\right)\sin\left(\frac{2\pi t}{T_n}\right), & 0 \leq t \leq t_d \\ \cos\frac{2\pi}{T_n}(t - t_d) + \frac{1}{2\pi}\left(\frac{T_n}{t_d}\right) sin\frac{2\pi}{T_n}(t - t_d) - \frac{1}{2\pi}\left(\frac{T_n}{t_d}\right) sin\frac{2\pi t}{T_n}, & t \geq t_d \end{cases}$$

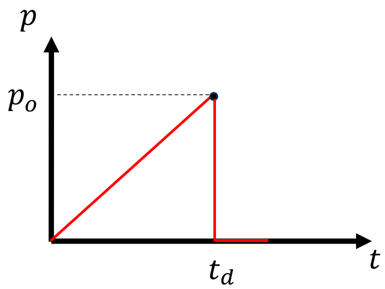
Plot the response for two values of $t_d/T_n = 1/2$ and 2.

4.2. Derive the equations for the dynamic response factor R_d during (i) the forced vibration phase and (ii) the free vibration phase.

4.3. Plot R_d for the two phases against t_d/T_n. Also plot the shock spectrum (overall maximum response from both the forced and free vibration phases).

4.4. During the lecture series and the in-class exercise of Week #3, we showed that for small values of t_d/T_n, the shock spectrum can be determined by treating the acting force as an impulse of a certain amplitude I. Determine the shock spectrum by this approach and superimpose it on the plot from Question 4.3. Determine the error in this approximate result for $t_d/T_n = 1/4$?

Solution



4.1. **(10 points)**

The equation of motion is given as:

$$m\ddot{u} + k \cdot u = \begin{cases} p_0(t/t_d); & 0 \leq t \leq t_d \\ 0 & t \geq t_d \end{cases} \quad (1)$$

i- Forced vibration phase: (4 points)

The response is given by:

$$u(t) = (u_{st})_0 \left(\frac{t}{t_d} - \frac{\sin(\omega_n t)}{\omega_n \cdot t_d}\right) \; ; \; t \leq t_d \quad (2)$$

(2/4)

Rewriting in terms of $\frac{t_d}{T_n}$ gives:

$$\frac{u(t)}{(u_{st})_0} = \frac{t}{t_d} - \frac{\sin\left(\frac{2\pi}{T_n}t\right)}{\frac{2\pi}{T_n} \cdot t_d} = \frac{t}{t_d} - \frac{1}{2\pi}\left(\frac{T_n}{t_d}\right)\sin\left(\frac{2\pi}{T_n}t\right) \; ; \; t \leq t_d \quad (3)$$

(2/4)

ii- Free vibration phase: (4 points)

The free vibration resulting from $u(t_d)$ and $\dot{u}(t_d)$ is:

$$u(t) = u(t_d)\cos[\omega_n(t - t_d)] + \frac{\dot{u}(t_d)}{\omega_n}\sin[\omega_n(t - t_d)] \quad (4)$$

(1/4)

From equation (3), $u(t_d)$ and $\dot{u}(t_d)$ are determined:

$$u(t_d) = (u_{st})_0\left(1 - \frac{\sin(\omega_n t)}{\omega_n \cdot t_d}\right) \quad (5)$$

(0.5/4)

$$\dot{u}(t_d) = (u_{st})_0 \frac{1}{t_d}(1 - \cos(\omega_n t)) \tag{6}$$

(0.5/4)

Substituting equation (5) and (6) in equation (4) gives:

$$\frac{u(t)}{(u_{st})_0} = \cos[\omega_n(t - t_d)] - \frac{\sin(\omega_n t)}{\omega_n \cdot t_d} + \frac{\sin[\omega_n(t-t_d)]}{\omega_n t_d} \;;\; t \geq t_d \tag{7}$$

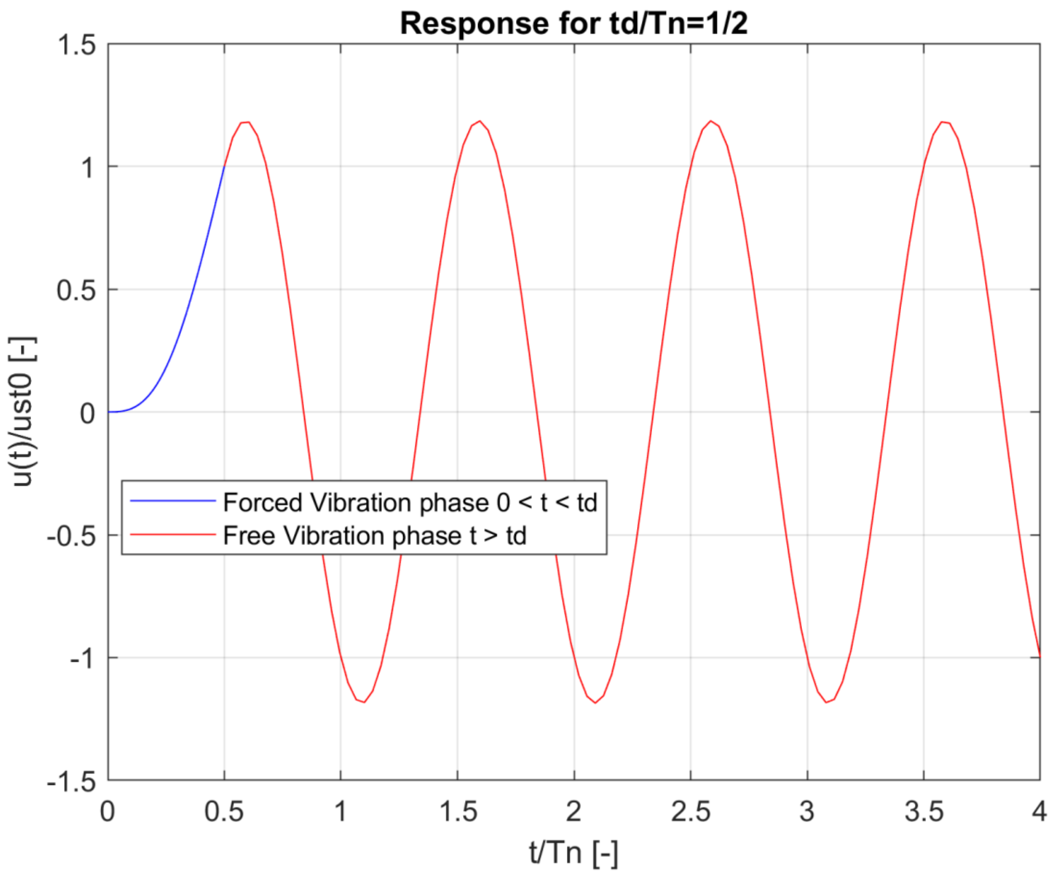
(1/4)

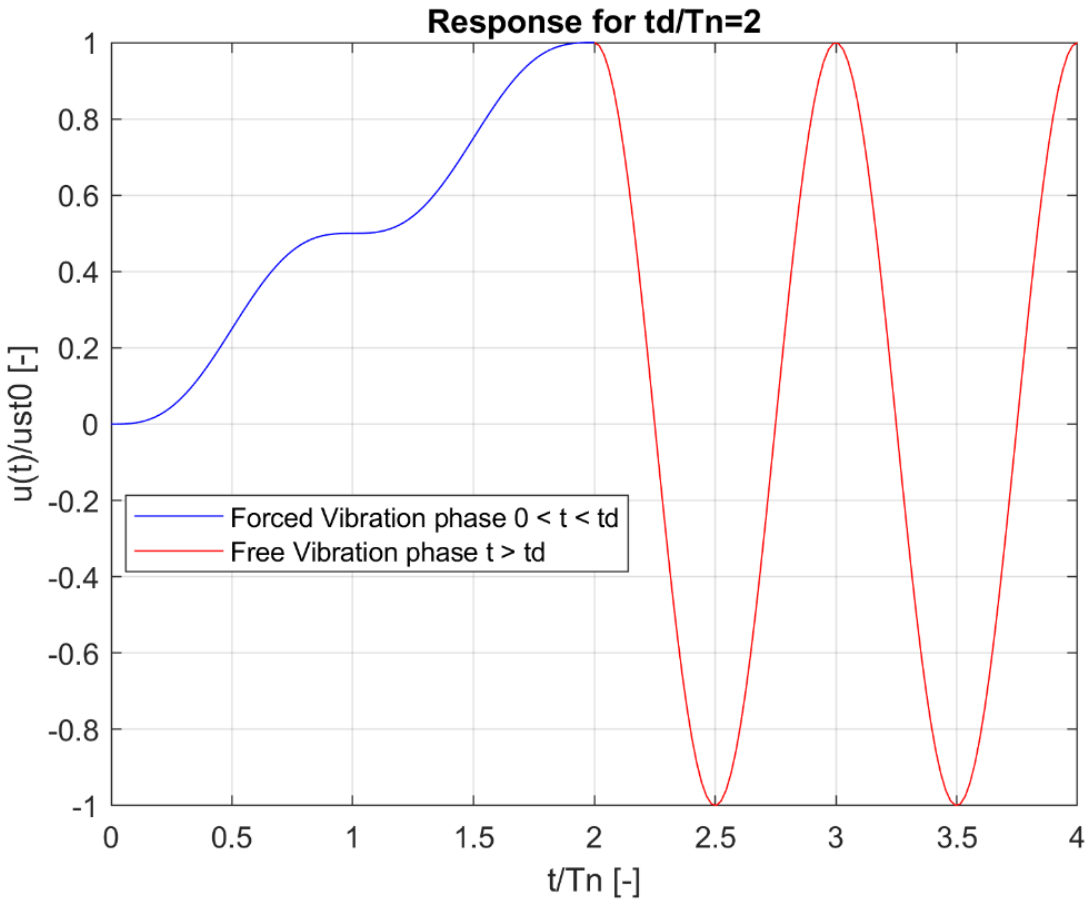
Rewriting in terms of $\frac{t_d}{T_n}$ gives:

$$\frac{u(t)}{(u_{st})_0} = \cos\left[\frac{2\pi}{T_n}(t - t_d)\right] + \frac{1}{2\pi}\left(\frac{T_n}{t_d}\right)\sin\left(\frac{2\pi}{T_n}(t - t_d)\right) - \frac{1}{2\pi}\left(\frac{T_n}{t_d}\right)\sin\left(\frac{2\pi}{T_n}t\right); \quad t \geq t_d \tag{8}$$

(1/4)

iii- Plotting the responses: (2 points)





4.2. **(5 points)**

During the forced vibration phase, u is a non-decreasing function of t. Thus, the maximum value of u during this phase can be found by evaluating equation (3) at $t = t_d$:

$$R_d = \frac{u(t)}{(u_{st})_0} = 1 - \frac{T_n}{2\pi t_d} \sin\left(\frac{2\pi}{T_n} t_d\right); \tag{9}$$

(2/5)

In the free vibration phase, the response of the system is given by equation (4) with the amplitude:

$$u_0 = \sqrt{u(t_d)^2 + \left[\frac{\dot{u}(t_d)}{\omega_n}\right]^2} \; ; \tag{10}$$

(2/5)

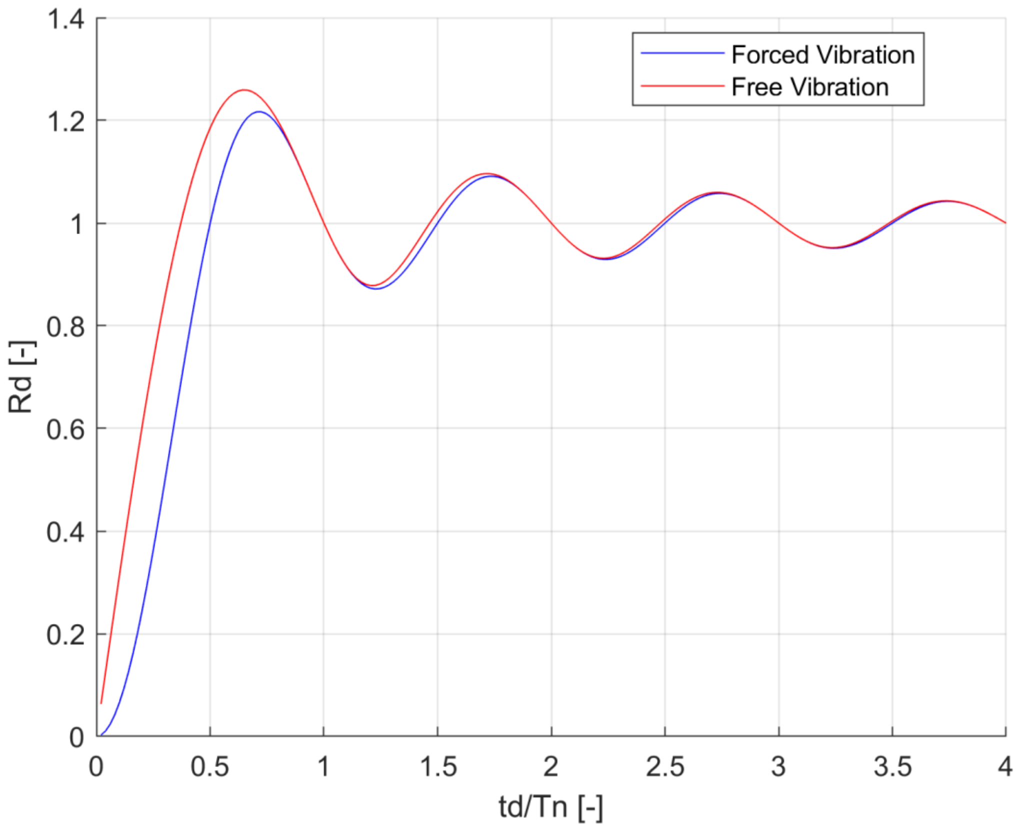
Substituting equations (5) and (6) and manipulating gives:

$$R_d = \sqrt{\left[1 - \frac{T_n}{2\pi t_d} \cdot \sin\left(\frac{2\pi \cdot t_d}{T_n}\right)\right]^2 + \frac{T_n^2}{\pi^2 t_d^2} . \sin^4\left(\frac{\pi \cdot t_d}{T_n}\right)} \tag{11}$$

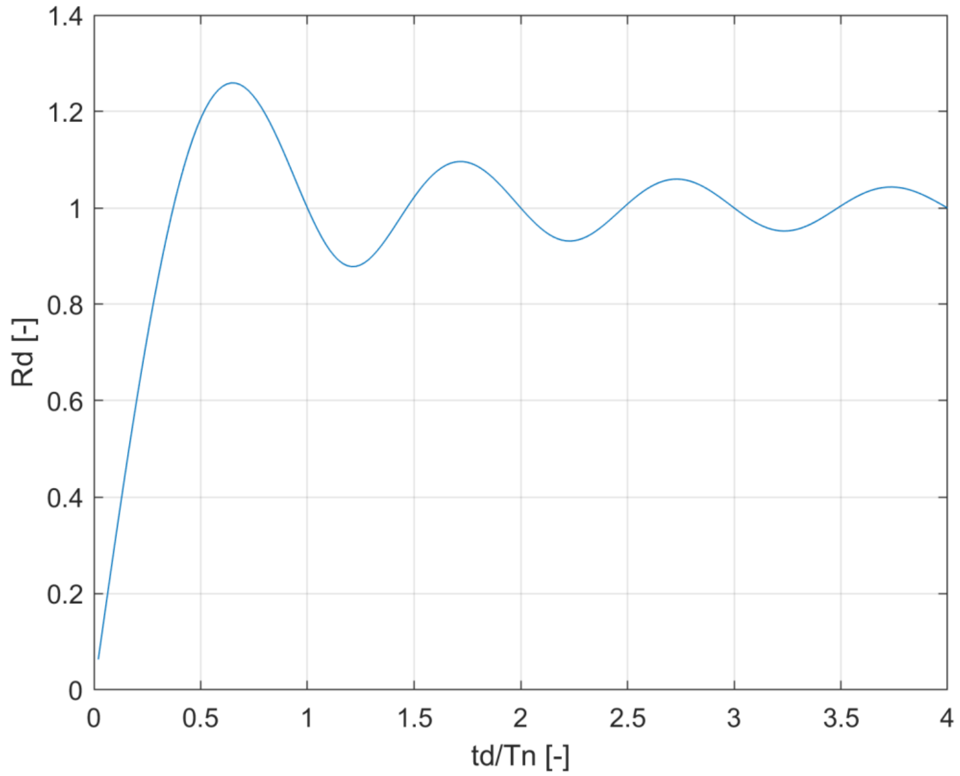
(1/5)

4.3. **(3 points)**

The equations (11) and (9) are plotted: (2/3)



The overall maximum response is the larger of the maxima shown in the figure above, it is plotted in the following figure to obtain the shock spectrum: (1/3)



4.4. **(7 points)**

For small enough values of the ratio t_d/T_n, the force acting on the system is considered as an impulse as:

$$I = \int_0 t_d . p(t)\, dt = \frac{p_0 \cdot t_d}{2}; \quad (12)$$

(2/7)

Hence:

$$u_0 = \frac{p_0}{k} . \frac{t_d \pi}{T_n};$$

(1/7)

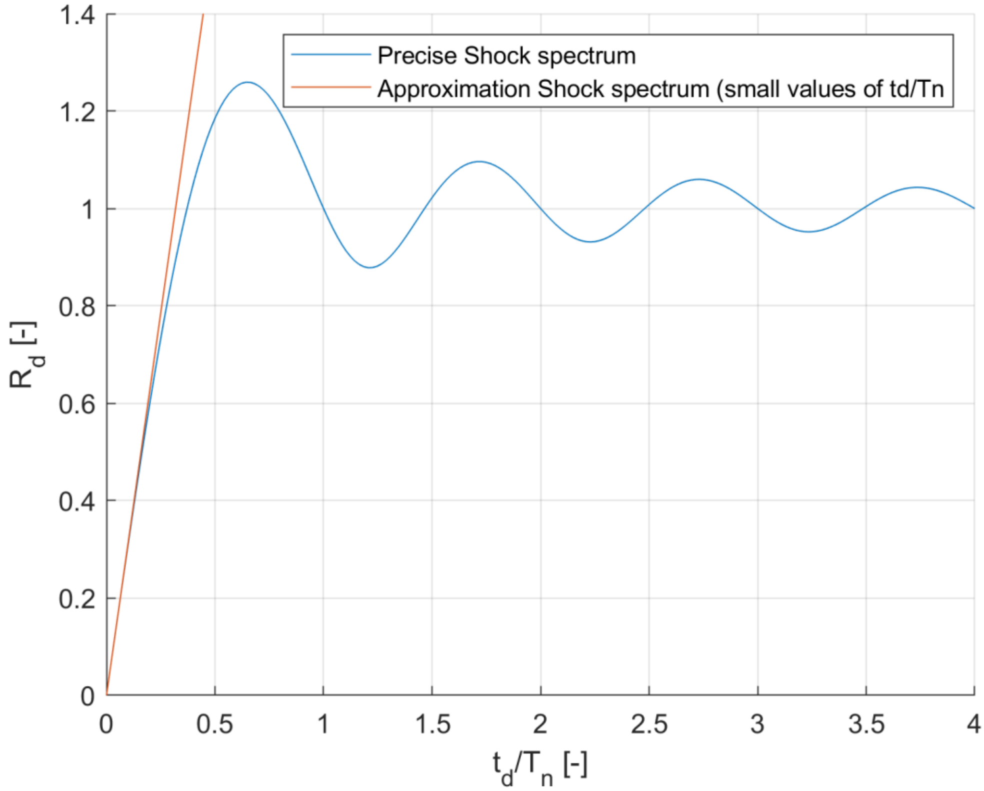
With :

$$u_{st} = \frac{p_0}{k};$$

$$R_d = \frac{u_0}{u_{st}} = \frac{t_d \pi}{T_n}; \quad (13)$$

(2/7)

The shock spectrum by this approach superimposed on the plot from Question 4.3 gives: (1/7)



The error for the value of $\frac{t_d}{T_n} = \frac{1}{4}$ is given as:

$$error = \frac{\left|R_{d,precise}\left(\frac{t_d}{T_n}=\frac{1}{4}\right) - R_{d,approximate}\left(\frac{t_d}{T_n}=\frac{1}{4}\right)\right|}{R_{d,precise}\left(\frac{t_d}{T_n}=\frac{1}{4}\right)} = 7.09\ \% \qquad (1/7)$$

$$R_{d,precise}\left(\frac{t_d}{T_n} = \frac{1}{4}\right) = 0.733\ [-]$$

$$R_{d,approximate}\left(\frac{t_d}{T_n} = \frac{1}{4}\right) = \ 0.785\ [-]$$