Lecture 7: Externalities
Exercises and solutions

April 21, 2025

In this scenario, we consider two separate firms (Firm 1 and Firm 2) operating in perfectly competitive markets. Each firm produces a different good, and there is no substitution between the goods—meaning each firm serves its own independent market. Because the markets are perfectly competitive, each firm is a price taker. This implies that each firm treats the market price of its product as given and beyond its control. The objective of each firm is to choose the quantity of output that maximizes its profit. given its cost structure and the market price. Costs are linear quadratic

$$C_i(q_i) = a_i q_i + b_i q_i^2 \tag{1}$$

Exercise 1.
Compute the competitive equilibrium.

The setting is now slightly changed: Firm 1 produces now an externality on Firm 2, by causing an increase on its marginal production cost, directly proportional to the quantity produced q_1.

The cost function for Firm 1 will thus be unchanged, while the new cost for Firm 2 will be defined by

$$C_2(q_1, q_2) = a_2 q_2 + b_2 q_2^2 + e q_1 q_2 \tag{2}$$

Exercise 2.
For this new setting, compute the competitive equilibrium.

If the externality is produced, a priori the competitive equilibrium does not coincide with the socially optimal quantity produced (namely, the quantities that, if produced, would maximise the combined profits of the two firms).

Exercise 3.
Compute the socially optimal quantities produced by the two firms.

Suppose now that there exists a policy maker who has the objective of making the companies produce the socially optimal quantities.
The policy maker can only set a tax by increasing the marginal production cost for Firm 1, yielding a new cost function

$$C_1(q_1) = a_1 q_1 + b_1 q_1^2 + k q_1 \tag{3}$$

Exercise 4.
Compute the value of the optimal tax k, that is the tax that makes the Firms produce, at competitive equilibrium, the socially optimal quantities.
Suppose then that, except for the tax and externalities, the Firms are symmetrical, that is

$$a_i = a$$
$$b_i = b$$
$$p_i = p$$

What does the tax reduce to? What is the difference between the optimal quantities produced by the two Firms? How do they differ with respect to the non-taxed equilibrium?

Solution 1.
The equilibrium will be defined by the quantities that maximise the profit of both firms simultaneously. Since the markets are assumed to be independent, the profit of each firm can be modelled independently:

$$\begin{aligned}\Pi_i(q_i) &= p_i q_i - C_i(q_i)\\ &= p_i q_i - a_i q_i - b_i q_i^2\\ &= (p_i - a_i) q_i - b_i q_i^2\end{aligned}$$

where p_i is the price of good produced by firm i.

The profit is thus a parabola, and can thus be minimised if and only if the coefficient of the quadratic term is positive, namely $b_i > 0$. We will assume this to be true from now on.

To find the quantity that maximises the profit, we solve the first order condition:

$$\begin{aligned}\frac{\partial \Pi_i}{\partial q_i}(q_i^*) &= 0\\ p_i - a_i - 2b_i q_i^* &= 0\\ q_i^* &= \frac{p_i - a_i}{2b_i}\end{aligned}$$

The optimal quantity will thus be directly proportional to the difference $p_i - a_i$, and inversely proportional to the quadratic coefficient b_0.

Solution 2.
The profit of Firm 1 is unchanged, and will thus be maximised for the same quantity computed in Exercise 1:

$$q_1^* = \frac{p_1 - a_1}{2b_1}$$

According to this behaviour by Firm 1, the profit of Firm 2 will be equal to

$$\begin{aligned}\Pi_2(q_1^*, q_2) &= p_2 q_2 - C_2(q_1^*, q_2)\\ &= p_2 q_2 - a_2 q_2 - b_2 q_2^2 - e q_1^* q_2\\ &= (p_2 - a_2 - q_1^* e)\, q_2 - b_2 q_2^2\end{aligned}$$

The quantity which maximises the profit will solve the first order conditions (again, if assuming $b_2 > 0$:

$$\begin{aligned}\frac{\partial \Pi_2}{\partial q_2}(q_1^*, q_2^*) &= 0\\ q_2^* &= \frac{p_2 - a_2 - q_1^* e}{2b_2}\\ &= \frac{p_2 - a_2 - \frac{p_1 - a_1}{2b_1} e}{2b_2}\end{aligned}$$

As expected, the externality acts in this case exactly as an increased marginal cost of production, decreasing the optimal quantity by increasing both the quantity produced by the Firm 1 and the coefficient e.

Solution 3.
Computing the social optimum, all the Firms (that are, in this case, only 2) have to be treated as a single one: the parameter e doesn't determine anymore an externality (since no externality can ever be present if only one firm is present), but is internalized and determines an added internal cost.

The profit to maximise will be the sum of profits of the two firms:

$$\begin{aligned}\Pi(q_1, q_2) &= \Pi_1(q_1, q_2) + \Pi_2(q_1, q_2) \\ &= (p_1 - a_1)q_1 - b_1 q_1^2 + (p_2 - a_2)q_2 - b_2 q_2^2 - e q_1 q_2\end{aligned}$$

Computing the first order conditions for this combined profit yields two different equations:

$$\begin{cases} \dfrac{\partial \Pi}{\partial q_1}(q_1^o, q_2^o) = 0 \\ \dfrac{\partial \Pi}{\partial q_2}(q_1^o, q_2^o) = 0 \end{cases}$$

that is

$$\begin{cases} 2b_1 q_1 + e q_2 = p_1 - a_1 \\ e q_1 + 2b_2 q_2 = p_2 - a_2 \end{cases} \tag{4}$$

Solving this system (by, for instance, inverting the coefficient matrix) gives the solution

$$\begin{cases} q_1^o = \frac{2b_2(p_1-a_1)-e(p_2-a_2)}{4b_1b_2-e^2} \\ \\ q_2^o = \frac{2b_1(p_2-a_2)-e(p_1-a_1)}{4b_1b_2-e^2} \end{cases} \tag{5}$$

It may be interesting to compare this with the competitive equilibrium. For comparing them, we set an inequality:

$$\begin{aligned} q_1^o &< q_1^* \\ \frac{2b_2(p_1 - a_1) - e(p_2 - a_2)}{4b_1 b_2 - e^2} &< \frac{p_1 - a_1}{2b_1} \\ 4b_1 b_2(p_1 - a_1) - 2eb_1(p_2 - a_2) &< 4b_1 b_2(p_1 - a_1) - e^2(p_1 - a_1) \\ e\left(e(p_1 - a_1) - 2b_1(p_2 - a_2)\right) &< 0 \\ 0 < e &< 2b_1 \frac{p_2 - a_2}{p_1 - a_2} \end{aligned}$$

and similarly,

$$\begin{aligned}
q_2^o &< q_2^* \\
\frac{2b_1(p_2-a_2)-e(p_1-a_1)}{4b_1b_2-e^2} &< \frac{p_2-a_2-\frac{p_1-a_1}{2b_1}e}{2b_2} \\
4b_1b_2(p_2-a_2)-2b_2(p_1-a_1)e &< 4b_1b_2(p_2-a_2)-2b_2(p_1-a_1)e-(p_2-a_2)e^2+\frac{p_1-a_1}{2b_1}e^3 \\
0 &< e^2\left(-(p_2-a_2)+\frac{p_1-a_1}{2b_1}e\right) \\
e &> 2b_1\frac{p_2-a_2}{p_1-a_1}
\end{aligned}$$

Supposing the externality to be positive, for really high values of the parameter e we see that the socially optimal quantity produced by Firm 1 becomes higher than the quantity produced at equilibrium, while for smaller values of e it is lower.

This is because, for low values of e, it is convenient to mitigate the total externality produced by Firm 1 by decreasing its production.

When, on the other hand, the coefficient e becomes too high, the externality becomes so big that it is convenient to increase the quantity produced by Firm 1, since its cost of production is not affected by the increase of cost determined by the externality it produces.

Solution 4.

Recall that the policy maker is now imposing a tax on Firm 1, in the form of the increase of its marginal production cost by a constant k: the new cost for Firm 1 is

$$C_1(q_1) = a_1q_1 + b_1q_1^2 + kq_1$$

At the competitive equilibrium, Firm 1 maximises its profit, defined by

$$\Pi_1(q_1) = (p_1 - a_1 - k)q_1 - b_1q_1^2$$

The first order condition yields

$$\begin{aligned}
\frac{\partial \Pi_1}{\partial q_1}(q_1^*) &= 0 \\
p_1 - a_1 - k - 2b_1q_1^* &= 0 \\
q_1^* &= \frac{p_1-a_1-k}{2b_1}
\end{aligned}$$

As you can see, the value k of the tax behaves exactly as the externality eq_1 caused by Firm 1 on Firm 2.

This illustrates what is known as the Pigovian Tax, or the *Polluter Pays* principle: the Firm that causes an externality is taxed for the externality caused to the other firms, and this tax will yield a competitive equilibrium which is closer to the social optimum.

The goal of the policy maker is now having this equal to the quantity q_1^o found in (5). By simply considering both the equations, we get

$$
\begin{aligned}
q_1^* &= q_1^o \\
\frac{p_1 - a_1 - k}{2b_1} &= \frac{2b_2(p_1 - a_1) - e(p_2 - a_2)}{4b_1b_2 - e^2} \\
k &= p_1 - a_1 - 2b_1\frac{2b_2(p_1 - a_1) - e(p_2 - a_2)}{4b_1b_2 - e^2}
\end{aligned}
$$

In the symmetric case, this reduces to

$$
\begin{aligned}
k &= p - a - 2b\frac{2b(p-a) - e(p-a)}{4b^2 - e^2} \\
&= (p-a)\left(1 - 2b\frac{2b - e}{(2b-e)(2b+e)}\right) \\
&= (p-a)\left(1 - \frac{2b}{2b+e}\right) \\
&= (p-a)\frac{e}{2b+e}
\end{aligned}
$$

It is easy to verify that, for this value of the tax k, both Firms will produce the same quantity:

$$
\begin{aligned}
q_1^* &= \frac{p - a - k}{2b} \\
&= \frac{p - a - (p-a)\frac{e}{2b+e}}{2b} \\
&= \frac{p-a}{2b}\frac{2b}{2b+e} \\
&= \frac{p-a}{2b+e}
\end{aligned}
$$

while, for Firm 2,

$$
\begin{aligned}
q_2^* &= \frac{p - a - q_1^* e}{2b} \\
&= \frac{p - a - e\frac{p-a}{2b+e}}{2b} \\
&= \frac{p - a\left(1 - \frac{e}{2b+e}\right)}{2b} \\
&= \frac{p-a}{2b}\frac{2b}{2b+e} \\
&= \frac{p-a}{2b+e}
\end{aligned}
$$

As you can see (and as expected), with the optimal value of the tax k the Firms (when symmetrical) produce the same quantities.

On top of this, the produced quantities are the same as the optimal quantities computed in (5): q_1^o, q_2^o reduce indeed in the symmetric case to

$$\begin{aligned} q_1^o &= \frac{2b_2(p_1 - a_1) - e(p_2 - a_2)}{4b_1b_2 - e^2} \\ &= \frac{(2b - e)(p - a)}{(2b + e)(2b - e)} \\ &= \frac{p - a}{2b + e} \\ &= q_1^o = q_1^* = q_2^* \end{aligned}$$

This ends the solutions. If anything is not clear, you're welcome to contact one of the TA by email at any time!