Exercises for CIVIL-455 Transportation economics: Congestion Game

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May 23, 2025

Q1 Consider a four-link network as shown in Figure 1.

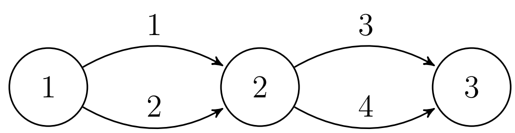

Figure 1: A four-link network

(a) Find the user equilibrium link flow and travel time for the network, where

$$
\begin{aligned}
t_1 &= 2 + x_1^2 \\
t_2 &= 3 + x_2 \\
t_3 &= 1 + 2x_3^2 \\
t_4 &= 2 + 4x_4
\end{aligned}
$$

The OD demand is 4 units of flow between nodes 1 and 3.

(b) Find the SO flow pattern for the network. Compare it to the UE flow pattern.

(c) Does the SO have a unique solution in terms of path flows? Explain your answer.

Solution:
(a) The objective function of Beckmann's formulation is

$$
\begin{aligned}
&\sum_e \int_0^{x_e} t_e(x)dx \\
&= 2x_1 + \frac{1}{3}x_1^3 + 3x_2 + \frac{1}{2}x_2^2 + x_3 + \frac{2}{3}x_3^3 + 2x_4 + 2x_4^2 \\
&= 2x_1 + \frac{1}{3}x_1^3 + 3(4 - x_1) + \frac{1}{2}(4 - x_1)^2 + x_3 + \frac{2}{3}x_3^3 + 2(4 - x_3) + 2(4 - x_3)^2
\end{aligned}
$$

By the first-order optimality condition, we have

$$\begin{aligned} x_1^2 + x_1 - 5 &= 0 \\ 2x_3^2 + 4x_3 - 17 &= 0 \end{aligned}$$

By solving the equations, we have $x_1 = 1.791$ and $x_3 = 2.082$. Given that $x_1+x_2 = x_3+x_4 = 4$, then we have $x_2 = 2.209$ and $x_4 = 1.918$.
(b)

$$\begin{aligned} &\sum_{e\in E} x_e t_e(x_e) \\ &= x_1(2+x_1^2) + x_2(3+x_2) + x_3(1+2x_3^2) + x_4(2+4x_4) \\ &= x_1(2+x_1^2) + (4-x_1)(3+4-x_1) + x_3(1+2x_3)^2 + (4-x_3)(2+4(4-x_3)) \end{aligned}$$

Consider the local minimum, we have

$$\begin{aligned} 3x_1^2 + 2x_1 - 9 &= 0 \\ 6x_3^2 + 8x_3 - 33 &= 0 \end{aligned}$$

Solve the equation set we have
$x_1 = 1.431, x_2 = 2.570, x_3 = 1.772, x_4 = 2.229$,
$t_1 = 4.046, t_2 = 5.570, t_3 = 7.276, t_4 = 10.914$.
The total travel time is 57.3 compared to 59.5 for UE. The travel times of the paths are 11.322, 14.960, 12.846, and 16.484. Some path travel times are larger than those in UE, while some are not.
(c) It is not unique. Recall that a path flow represents the number of travelers choosing a path, while a link flow is the sum of path flows of all paths traversing through a link. Given the solution x regarding the link flow, we have

$$\begin{aligned} f_1 + f_2 &= x_1 \\ f_1 + f_3 &= x_3 \\ f_3 + f_4 &= x_2 \\ f_2 + f_4 &= x_4 \end{aligned}$$

where f_1, f_2, f_3, and f_4 are respectively the flows on path 1 (link 1 and link 3), path 2 (link 1 and link 4), path 3 (link 2 and link 3), and path 4 (link 2 and link 4).

The matrix of the left-hand side parameters of the above equation system is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

which is not an invertible matrix (the determinant is zero). Therefore, the solution of f is not unique.

Q2 Recall the Cournot competition example with Uber and Lyft. Consider the following scenario: Uber decreases marginal cost by 33.3%

(a) Answer the following questions: What is the effect of the decrease in marginal cost of Uber on:

- Market share of both firms
- Equilibrium price
- Total supply
- Profit of either firm

(b) How would your answer change if competition is according to Bertrand?

Solution:
Demand:

$$P = 5 - 0.01Q, \quad Q = q_1 + q_2$$

Marginal Costs:

$$MC_1 = 3, \quad MC_2 = 2$$

Price as function of q_1, q_2:

$$P = 5 - 0.01(q_1 + q_2)$$

Total Revenue for firm 1:

$$TR_1 = P \cdot q_1 = (5 - 0.01(q_1 + q_2))\, q_1 = 5q_1 - 0.01q_1^2 - 0.01q_1q_2$$

Marginal Revenue:

$$MR_1 = \frac{dTR_1}{dq_1} = 5 - 0.02q_1 - 0.01q_2$$

$$MR_2 = 5 - 0.02q_2 - 0.01q_1$$

Best Response Functions:
Firm 1:

$$MR_1 = MC_1 \Rightarrow 5 - 0.02q_1 - 0.01q_2 = 3 \Rightarrow q_1 = 100 - 0.5q_2$$

Firm 2:

$$MR_2 = MC_2 \Rightarrow 5 - 0.02q_2 - 0.01q_1 = 2 \Rightarrow 3 = 0.02q_2 + 0.01q_1 \Rightarrow q_2 = 150 - 0.5q_1$$

Solving the system:
Substitute into each other:

$$q_1 = 100 - 0.5(150 - 0.5q_1) = 100 - 75 + 0.25q_1 \Rightarrow 0.75q_1 = 25 \Rightarrow q_1^* = \frac{100}{3} = 33\frac{1}{3}$$

$$q_2^* = 150 - 0.5 \cdot 33\frac{1}{3} = 133\frac{1}{3}$$

Total quantity:

$$Q = q_1 + q_2 = 33\frac{1}{3} + 133\frac{1}{3} = 166\frac{2}{3}$$

Price:

$$P = 5 - 0.01Q = 5 - 0.01 \cdot 166.67 \approx 3\frac{1}{3}$$

Profits:

$$\pi_1 = q_1(P - MC_1) = 33\frac{1}{3} \cdot (3\frac{1}{3} - 3) = 33\frac{1}{3} \cdot \frac{1}{3} \approx 11.11$$

$$\pi_2 = q_2(P - MC_2) = 133\frac{1}{3} \cdot (3\frac{1}{3} - 2) = 133\frac{1}{3} \cdot \frac{4}{3} \approx 177.77$$

(b) Uber would undercut Lyft by setting its price slightly lower and capture the entire market.