School of Architecture, Civil and Environmental Engineering
CIVIL ENGINEERING INSTITUTE
EPFL – RESILIENT STEEL STRUCTURES LABORATORY

Course: CIVIL 449 - Nonlinear analysis of structures

Instructors:
Prof. Dr. Dimitrios Lignos, Dr. Savvas Saloustros

Assignment #2: Nonlinear geometric analysis of structures

Q1 (65 points): Extend the program you wrote for Assignment #1 to conduct nonlinear analysis of frame structures by considering nonlinear geometric effects. The program can be written at any programming language of your preference. Your program should consider the following:

- Linear and corotational geometric transformation
- Load and displacement control for load application

Your program should be able to determine the nodal displacements, member forces and support reactions for planar frames by nonlinear analysis. Assume that the members are all prismatic, i.e., the axial and flexural rigidities of the members are constant along their length.

Q2 (35 points): Use your program from Q1 to compute the following:

1. The critical load, P_{cr} and the buckled shape of the frame shown in Figure 1.
2. What is the horizontal displacement, δ_{cr}, at which the frame reaches P_{cr}?
3. Compare the moment diagram for linear and nonlinear analyses for $P = 0.5P_{cr}$.
4. Compare the total base shear versus lateral displacement equilibrium path based on the linear and the corotational transformation.
5. Compare the total base shear versus lateral displacement equilibrium paths when considering load and displacement control. Comment on your results based on the choice of the analysis used.

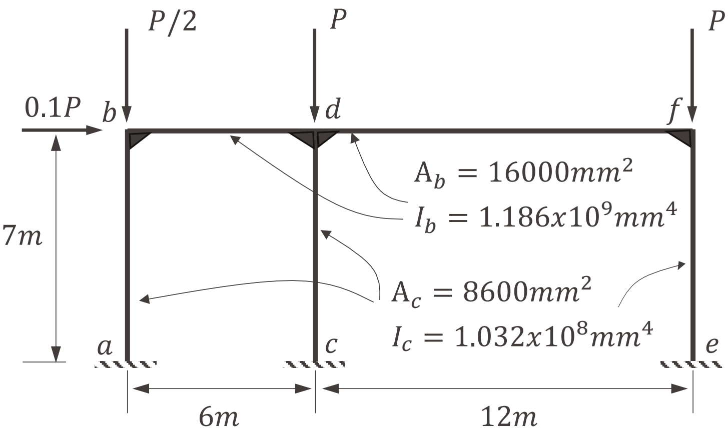

Figure 1. Planar frame

Solution:
To determine the critical load, the following steps are used:
1) Define the member properties (E, I, A and l)
2) Define the connectivity matrix and the mapping matrix between local and global degrees of freedom. In this case, each column is divided into 50 2D elastic beam-column elements, while the beams are modeled with a single 2D elastic beam-column element
3) Assemble the initial structure stiffness matrix $\boldsymbol{K}_{structure}$
4) Define the boundary conditions, the external loads (i.e., apply the reference load $\boldsymbol{P}^{ref}$), the fixed and the free degrees of freedom of the problem
5) Compute the structure displacements $\boldsymbol{v}$ corresponding to the refence load $\boldsymbol{P}^{ref}$:

$$\boldsymbol{v}_f = \left(\boldsymbol{K}_{structure,f}\right)^{-1}\boldsymbol{P}^{ref}$$

Where the subscript f denotes the free degrees of freedom of the system
6) Assemble the structure material and geometric stiffness matrices $\boldsymbol{K}_{e,structure}$ and $\boldsymbol{K}_{g,structure}$. With a loop, go over all elements:

6.1) Determine the element displacement vector in the local reference frame $\boldsymbol{u}$

$$\boldsymbol{u}_{elem} = \boldsymbol{T}_{elem}\boldsymbol{v}_{elem}$$

Where $\boldsymbol{T}_{elem}$ is the transformation matrix between local and global degrees of freedom and the subscript $elem$ denotes the degrees of freedom corresponding to element $elem$.

6.2) Using the corotational formulation, compute the element displacements in the basic reference frame $\bar{\boldsymbol{u}} = [\bar{u}_1, \bar{u}_2, \bar{u}_3]^T$. The following equations from Slide 32 of the lecture notes of Week #4 are used:

$$\bar{u}_1 = l_n - l$$
$$\bar{u}_2 = \boldsymbol{u}(3) - \beta$$
$$\bar{u}_3 = \boldsymbol{u}(6) - \beta$$

With

$$l_n = \sqrt{\left(l + \boldsymbol{u}(4) - \boldsymbol{u}(1)\right)^2 + \left(\boldsymbol{u}(5) - \boldsymbol{u}(2)\right)^2}$$
$$\beta = \arctan\left(\frac{\boldsymbol{u}(5) - \boldsymbol{u}(2)}{l + \boldsymbol{u}(4) - \boldsymbol{u}(1)}\right)$$

6.3) Compute the element internal forces in the basic reference frame $\bar{\boldsymbol{q}}$:

$$\bar{\boldsymbol{q}} = \bar{\boldsymbol{K}}\bar{\boldsymbol{u}}$$

6.4) Determine the transformation matrix $\boldsymbol{L}$ from the basic to the local reference frame. The following equation from Slide 34 of the lecture notes of Week #4 are used:

$$\boldsymbol{L} = \begin{bmatrix} -c & -s & 0 & c & s & 0 \\ -s/L_n & c/L_n & 1 & s/L_n & -c/L_n & 0 \\ -s/L_n & c/L_n & 0 & s/L_n & -c/L_n & 1 \end{bmatrix}$$

Where

$$c = \cos(\beta) \text{ and } s = \sin(\beta)$$

6.5) Determine the element geometric stiffness matrix in the local reference frame $\boldsymbol{K}_{g,elem}$. The following equations from Slide 34 of the lecture notes of Week #4 are used:

$$\boldsymbol{K}_{g,elem} = \frac{\bar{q}_1}{l_n}\begin{bmatrix} s^2 & -cs & 0 & -s^2 & cs & 0 \\ -cs & c^2 & 0 & cs & -c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & cs & 0 & s^2 & -cs & 0 \\ cs & -c^2 & 0 & -cs & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\bar{q}_2 + \bar{q}_3}{l_n^2}\begin{bmatrix} -2sc & c^2 - s^2 & 0 & 2sc & -c^2 + s^2 & 0 \\ c^2 - s^2 & 2cs & 0 & -c^2 + s^2 & -2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2sc & -c^2 + s^2 & 0 & -2sc & c^2 - s^2 & 0 \\ -c^2 + s^2 & -2cs & 0 & c^2 - s^2 & 2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6.6) Assemble the structure material and geometric stiffness matrices $\boldsymbol{K}_{e,structure}$ and $\boldsymbol{K}_{g,structure}$ with the element quantities

7) Determine the critical load multiplier λ_{crit}:

7.1) Solve the following eigenvalue problem

$$\left[\boldsymbol{K}_{e,structure,f} + \lambda \boldsymbol{K}_{g,structure,f}\right]\boldsymbol{\Delta}_f = \boldsymbol{0}$$

7.2) The critical load multiplier λ_{crit} corresponds to the minimum of all eigenvectors λ

7.3) The critical load $\boldsymbol{P}_{crit}$ is then given by

$$\boldsymbol{P}_{crit} = \lambda_{crit}\boldsymbol{P}^{ref}$$

For the number of elements discussed above, the following value is obtained:

$$\boldsymbol{P}_{crit} = 4803.4\ kN$$

8) The eigen vectors $\boldsymbol{\Delta}_f$ associated with the critical load multiplier λ_{crit} are of the nodal displacements of the buckled structure. The following buckled shape is obtained:

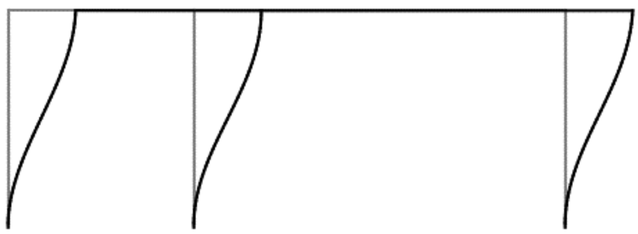

Figure 2. Buckled shape

To compute the base shear versus lateral displacement equilibrium path, the following steps are used:

1) Define the member properties (E, A, I and l)

2) Define the connectivity matrix and the mapping matrix between local and global degrees of freedom. In this case, each column is divided into two 2D elastic beam-column elements, while the beams are modeled with a single 2D elastic beam-column element

3) Assemble the initial structure stiffness matrix $\mathbf{K}_{structure}$

4) Define the boundary conditions, the fixed and the free degrees of freedom of the problem, the external loads (i.e., apply the reference load $\overline{\boldsymbol{F}}_{ext}$)

5) Initialize the variables used within the displacement-control and load-control procedures

$$\lambda = 0, \boldsymbol{v} = \boldsymbol{0}$$

6) Define the parameters defining the displacement-control and load-control algorithms:

For displacement-control:

- The dof controlling the displacement q_{dof}
- The final displacement at q_{dof}: $v_{c,max}$
- The number of steps n_{tot}
- At each load step, the increment in displacement is given by $\Delta\bar{v} = v_{c,max}/n_{tot}$
- The tolerance tol
- The maximum number of iterations for each iterations of the displacement control loop $i_{\max}$

For load-control:

- The maximum value of the load multiplier: λ_{max}
- The number of steps n_{tot}
- At each load step, the increment in external force is given by $\Delta\bar{\lambda} = \lambda_{max}/n_{tot}$
- The tolerance tol
- The maximum number of iterations for each iterations of the displacement control loop $i_{\max}$

7) For the displacement or load increment n, perform the Newton-Raphson iterations

7.1) For $i = 1$, set:
$\lambda^{n,i=1} = \lambda^{n-1}$, $\mathbf{F}_{int}^{n,i=1} = \mathbf{F}_{int}^{n-1}$, $\mathbf{K}_{structure}^{n,1} = \mathbf{K}_{structure}^{n-1}$ and $\mathbf{v}^{n,1} = \mathbf{v}^{n-1}$

7.2) Determine $\delta\mathbf{v}_r^{n,i}$and $\delta\mathbf{v}_p^{n,i}$:

$$\delta\mathbf{v}_{p,f}^{n,i} = \left(\mathbf{K}_{structure}^{n,i-1}\right)^{-1}\bar{\mathbf{F}}_{ext}$$
$$\delta\mathbf{v}_{r,f}^{n,i} = -\left(\mathbf{K}_{structure}^{n,i-1}\right)^{-1}\mathbf{F}_{unb}^{n,i-1}$$

Where the subscript f denotes the free degrees of freedom of the system

7.3) Compute the increment in the load multiplier $\delta\lambda^{n,i}$:

For load control:

$$\delta\lambda^{n,i} = \begin{cases} \Delta\bar{\lambda}^n \; if \; i = 1 \\ 0 \; else \end{cases}$$

For displacement control:

$$\delta\lambda^{n,i} = \begin{cases} \dfrac{\Delta\bar{v}^n}{\delta\mathbf{v}_p^{n,i}} \text{ if } i = 1 \; (note, \; \delta\mathbf{v}_r^{n,1} = \mathbf{0}) \\ -\dfrac{\delta\mathbf{v}_r^{n,i}}{\delta\mathbf{v}_p^{n,i}} \text{ else} \end{cases}$$

7.4) Compute the increment in structure displacements $\Delta\mathbf{v}^{n,i}$:

$$\delta\mathbf{v}^{n,i} = \delta\lambda^{n,i}\delta\mathbf{v}_p^{n,i} + \delta\mathbf{v}_r^{n,i}$$

7.5) Update the structure displacements and the load multiplier:

$$\mathbf{v}^{n,i} = \mathbf{v}^{n,i-1} + \delta\mathbf{v}^{n,i}$$
$$\lambda^{n,i} = \lambda^{n,i-1} + \Delta\lambda^{n,i}$$

7.6) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}^{n,i}$ and $\mathbf{K}_{g,structure}^{n,i}$, as well as the structure resisting force vector $\mathbf{F}_{int}^{n,i}$. With a loop, go over all elements:

7.6.1) Determine the element displacement vector in the local reference frame $\boldsymbol{u}^{n,i}$

$$\mathbf{u}_{elem}^{n,i} = \mathbf{T}_{elem}\mathbf{v}_{elem}^{n,i}$$

Where the subscript $elem$ denotes the degrees of freedom corresponding to element $elem$

7.6.2) Using the linear or the corotational formulation for the 2d elastic beam-column element, compute the element displacements in the basic reference frame $\bar{\mathbf{u}}$

7.6.3) Compute the element internal forces in the basic reference frame $\bar{\mathbf{q}}^{n,i}$:

$$\bar{\mathbf{q}}^{n,i} = \bar{\mathbf{K}}^{n,i}\bar{\mathbf{u}}^{n,i}$$

7.6.4) Determine the transformation matrix $\mathbf{L}^{n-1}$ from the basic to the local reference system

7.6.5) Compute the element internal force vector in the local reference frame:

$$\mathbf{Q}_{elem}^{n,i} = \left(\mathbf{L}^{n,i}\right)^T \overline{\mathbf{q}}^{n,i}$$

7.6.6) Determine the element geometric stiffness matrix in the local reference frame $\mathbf{K}_{g,elem}^{n,i}$

7.6.7) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}^{n,i}$ and $\mathbf{K}_{g,structure}^{n,i}$ as well as the structure internal force vector $\mathbf{F}_{int}^{n,i}$ with the element quantities

7.7) Compute the unbalanced load vector $\mathbf{F}_{unb}^{n,i} = \mathbf{F}_{int}^{n,i} - \mathbf{F}_{ext}^{n}$

7.8) Check if the Newton-Raphson procedure has converged. In the source code, convergence is achieved once

$$\left\|\boldsymbol{F}_{unb,f}^{n,i}\right\| < tol$$

7.9) If iteration i has converged, go to next load step n, else set $i = i + 1$ and go back to step **(7.2)**

The following figure compares the total base shear versus lateral displacement equilibrium path based on the linear and the corotational transformation:

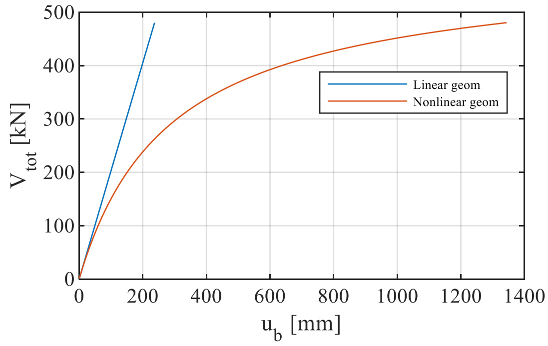

Figure 3. Total base shear versus lateral displacement equilibrium paths

From this figure, it can be seen that since the equilibrium path does not exhibit a snap-through response, both displacement-control and load-control will give the same equilibrium path.

When considering linear geometry, the frame reaches P_{cr} at a horizontal displacement $\delta_{cr} = 237.1$mm. When considering nonlinear geometry with the corotational formulation, the frame reaches P_{cr} at a horizontal displacement $\delta_{cr} = 1344.0$mm.

The internal forces (i.e., normal and shear forces and the bending moment) are obtained by reading the element resisting force vector in the local reference frame $\mathbf{Q}_{elem}^{n,i}$.
Figures 4 and 5 below compare the bending moment diagrams for linear and nonlinear analysis for $P = 0.5P_{cr}$. Figure 4 shows the diagram for linear geometry, while Figure 5 shows the diagram for nonlinear geometry.

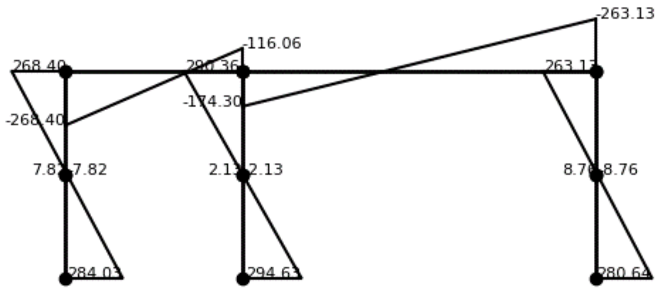

Figure 4. Bending moment diagram linear geometry (units: kNm)

Figure 5. Bending moment diagram nonlinear geometry (units: kNm)