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In-class Exercise Week #5: Iterative Techniques and Corotational Transformation

Exercise #1:

Consider the following column:

$$A = 1.27 \cdot 10^4 mm^2, I = 3.66 \cdot 10^7 mm^2, E = 200{,}000 MPa$$

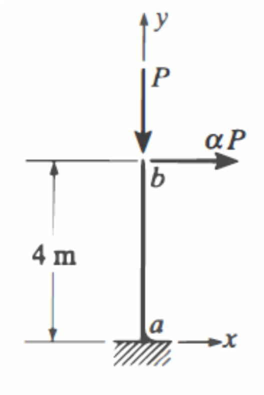

Figure 1.1. Column under axial and lateral loading

Analyze the behavior of the system for the following cases:

1. $\alpha = 0$ (determine the critical load)

Using load control and nonlinear geometry (with the corotational formulation), determine the secondary equilibrium path of the structure when:

2. $\alpha = 0.05$
3. $\alpha = 0.05$ and P directed upward

Hint: for 2d elastic beam elements, the element stiffness matrix in the basic reference frame $\hat{\mathbf{K}}$ is given by:

$$\hat{\mathbf{K}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{4EI}{L} & \frac{2EI}{L} \\ 0 & \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$$

Solution:
In this solution, the column is modeled using a single 2D beam-column element. Similar procedures can be applied when more beam-column elements are used to model the column. The key difference lies in assembling the resisting force vectors and stiffness matrices of all elements to form the global internal force vector and global stiffness matrix for the structure.

a) For $\alpha = 0$, to determine the critical load, the following steps are used:
1) Define the member properties (E, I, A and l)
2) Define the connectivity matrix and the mapping matrix between local and global degrees of freedom
3) For each member, determine the transformation matrix **T** between local and global coordinates. In this exercise, the local x-axis is defined in the axial direction of the element, and therefore corresponds to the global Y-axis
4) Assemble the initial structure stiffness matrix $\mathbf{K}_{structure}$. The following matrix is obtained:

$$\mathbf{K}_{structure} = \begin{bmatrix} 1.37\cdot10^3 & 0 & -2.75\cdot10^6 & -1.37\cdot10^3 & 0 & -2.75\cdot10^6 \\ 0 & 6.35\cdot10^5 & 0 & 0 & -6.35\cdot10^5 & 0 \\ -2.75\cdot10^6 & 0 & 7.32\cdot10^9 & 2.75\cdot10^6 & 0 & 3.66\cdot10^9 \\ -1.37\cdot10^3 & 0 & 2.75\cdot10^6 & 1.37\cdot10^3 & 0 & 2.75\cdot10^6 \\ 0 & -6.35\cdot10^5 & 0 & 0 & 6.35\cdot10^5 & 0 \\ -2.75\cdot10^6 & 0 & 3.66\cdot10^9 & 2.75\cdot10^6 & 0 & 7.32\cdot10^9 \end{bmatrix}$$

5) Define the boundary conditions, the external loads (i.e., apply the reference load $\mathbf{P}^{ref}$), the fixed and the free degrees of freedom of the problem
6) Compute the structure displacements **v** corresponding to the refence load $\mathbf{P}^{ref}$:

$$\mathbf{v}_f = \left(\mathbf{K}_{structure,f}\right)^{-1}\mathbf{P}^{ref}$$

Where the subscript f denotes the free degrees of freedom of the system
The following displacement vector is obtained:

$$\mathbf{v} = 10^{-6}\cdot\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1.575 \\ 0 \end{pmatrix}$$

7) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}$ and $\mathbf{K}_{g,structure}$. With a loop, go over all elements:

7.1) Determine the element displacement vector in the local reference frame **u**

$$\mathbf{u}_{elem} = \mathbf{T}_{elem}\mathbf{v}_{elem}$$

Where the subscript $elem$ denotes the degrees of freedom corresponding to element $elem$. For a single element, the following vector is obtained:

$$\mathbf{u} = 10^{-6}\cdot\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1.575 \\ 0 \\ 0 \end{pmatrix}$$

7.2) Using the corotational formulation, compute the element displacements in the basic reference frame $\bar{\mathbf{u}} = [\bar{u}_1, \bar{u}_2, \bar{u}_3]^T$. The following equations from Slide 32 of the lecture notes of Week #4 are used:

$$\bar{u}_1 = l_n - l$$
$$\bar{u}_2 = \mathbf{u}(3) - \beta$$

$$\bar{u}_3 = \mathbf{u}(6) - \beta$$

With

$$l_n = \sqrt{\left(l + \mathbf{u}(4) - \mathbf{u}(1)\right)^2 + \left(\mathbf{u}(5) - \mathbf{u}(2)\right)^2}$$

$$\beta = \arctan\left(\frac{\mathbf{u}(5) - \mathbf{u}(2)}{l + \mathbf{u}(4) - \mathbf{u}(1)}\right)$$

For a single element, the following vector is obtained:

$$\bar{\mathbf{u}} = 10^{-6} \cdot \begin{pmatrix} -1.575 \\ 0 \\ 0 \end{pmatrix}$$

7.3) Compute the element internal forces in the basic reference system, $\bar{\mathbf{q}}$:

$$\bar{\mathbf{q}} = \bar{\mathbf{K}}\bar{\mathbf{u}}$$

For a single element, the following value is obtained:

$$\bar{\mathbf{q}} = \begin{pmatrix} -1.0 \\ 0 \\ 0 \end{pmatrix}$$

7.4) Determine the transformation matrix $\mathbf{L}$ from the basic to the local reference system. The following equation from Slide 34 of the lecture notes of Week #4 are used:

$$\mathbf{L} = \begin{bmatrix} -c & -s & 0 & c & s & 0 \\ -s/L_n & c/L_n & 1 & s/L_n & -c/Ln & 0 \\ -s/L_n & c/L_n & 0 & s/L_n & -c/L_n & 1 \end{bmatrix}$$

Where

$c = \cos(\beta)$ and $s = \sin(\beta)$

For a single element, the following matrix is obtained:

$$\mathbf{L} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2.5 \cdot 10^{-4} & 1 & 0 & -2.5 \cdot 10^{-4} & 0 \\ 0 & 2.5 \cdot 10^{-4} & 0 & 0 & -2.5 \cdot 10^{-4} & 1 \end{bmatrix}$$

7.5) Determine the element geometric stiffness matrix in the local reference frame $\mathbf{K}_{g,elem}$. The following equations from Slide 34 of the lecture notes of Week #4 are used:

$$\mathbf{K}_{g,elem} = \frac{\bar{q}_1}{l_n}\begin{bmatrix} s^2 & -cs & 0 & -s^2 & cs & 0 \\ -cs & c^2 & 0 & cs & -c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & cs & 0 & s^2 & -cs & 0 \\ cs & -c^2 & 0 & -cs & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\bar{q}_2 + \bar{q}_3}{l_n^2}\begin{bmatrix} -2sc & c^2 - s^2 & 0 & 2sc & -c^2 + s^2 & 0 \\ c^2 - s^2 & 2cs & 0 & -c^2 + s^2 & -2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2sc & -c^2 + s^2 & 0 & -2sc & c^2 - s^2 & 0 \\ -c^2 + s^2 & -2cs & 0 & c^2 - s^2 & 2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For a single element, the following value is obtained:

$$\mathbf{K}_{g,elem} = 10^{-4} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.50 & 0 & 0 & 2.50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.50 & 0 & 0 & -2.50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.6) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}$ and $\mathbf{K}_{g,structure}$ with the element quantities

8) Determine the critical load multiplier λ_{crit}:

8.1) Solve the following eigenvalue problem

$$[\mathbf{K}_{e,structure,f} + \lambda \mathbf{K}_{g,structure,f}]\boldsymbol{\Delta}_f = \mathbf{0}$$

8.2) The critical load multiplier λ_{crit} corresponds to the minimum of all eigenvectors λ

8.3) The critical load $\mathbf{P}_{crit}$ is then given by, $\mathbf{P}_{crit} = \lambda_{crit}\mathbf{P}^{ref}$

For a single element, the following value is obtained, $\mathbf{P}_{crit} = 1372\ kN$

b) To determine the secondary equilibrium path for $\alpha = 0.05$, the following steps are used:

1) Define the member properties (E, I, A and l)

2) Define the connectivity matrix and the mapping matrix between local and global degrees of freedom

3) For each member, determine the transformation matrix **T** between local and global coordinates. In this exercise, the local x-axis is defined in the axial direction of the element, and therefore corresponds to the global Y-axis

4) Assemble the initial structure stiffness matrix $\mathbf{K}_{structure}$. The following matrix is obtained:

$$\mathbf{K}_{structure} = \begin{bmatrix} 1.37\cdot 10^3 & 0 & -2.75\cdot 10^6 & -1.37\cdot 10^3 & 0 & -2.75\cdot 10^6 \\ 0 & 6.35\cdot 10^5 & 0 & 0 & -6.35\cdot 10^5 & 0 \\ -2.75\cdot 10^6 & 0 & 7.32\cdot 10^9 & 2.75\cdot 10^6 & 0 & 3.66\cdot 10^9 \\ -1.37\cdot 10^3 & 0 & 2.75\cdot 10^6 & 1.37\cdot 10^3 & 0 & 2.75\cdot 10^6 \\ 0 & -6.35\cdot 10^5 & 0 & 0 & 6.35\cdot 10^5 & 0 \\ -2.75\cdot 10^6 & 0 & 3.66\cdot 10^9 & 2.75\cdot 10^6 & 0 & 7.32\cdot 10^9 \end{bmatrix}$$

5) Define the boundary conditions, the external loads (i.e., apply the reference load $\mathbf{F}^{ref}$), the fixed and the free degrees of freedom of the problem

6) Initialize the variables used within the Newton-Raphson scheme: $\lambda = 0, \mathbf{v} = \mathbf{0}$
Where λ denotes the load multiplier (i.e. $\mathbf{F}_{ext} = \lambda\mathbf{F}^{ref}$)

7) Define the parameters for the load-control algorithm:

- The final external load $\mathbf{F}_{ext}^{\text{tot}} = \lambda^{tot}\mathbf{F}^{ref}$
- The number of steps to apply the final external load: n_{tot}
- At each load step, the increment in external force is given by $\Delta\bar{\lambda} = \lambda^{tot}/n_{tot}$
- The tolerance tol
- The maximum number of iterations per iteration of the Newton-Raphson scheme $i_{\max}$

For the solution procedure described herein, the following values are used: $\lambda^{tot} = 1100000$, $\mathbf{F}^{ref} = [0\ 0\ 0\ \alpha - 1\ 0]^T$, $n_{tot} = 100$ and $tol = 1e-4$

8) For load increment n, perform the Newton-Raphson iterations:

8.1) For $i = 1$, set all vectors to the previously converged step: $\Delta\mathbf{F}_{ext}^{n,i=1} = \Delta\bar{\lambda}\mathbf{F}^{ref}$, $\mathbf{F}_{int}^{n,i=1} = \mathbf{F}_{int}^{n-1}$, $\mathbf{K}_{\text{structure}}^{n,1} = \mathbf{K}_{\text{structure}}^{n-1}$ and $\mathbf{v}^{n,1} = \mathbf{v}^{n-1}$

8.2) Compute the incremental structural displacement vector, $\Delta\mathbf{v}^{n,i}$:

$$\Delta\mathbf{v}_f^{n,i} = \left(\mathbf{K}_{structure,f}^{n,i}\right)^{-1}\Delta\mathbf{F}_{ext}^{n}$$

Where the subscript f denotes the free degrees of freedom of the system. The following displacement vector is obtained for the first step and first iteration of the Newton-Raphson scheme:

$$\Delta\mathbf{v}^{1,1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1.60 \\ -0.017 \\ -6.00 \cdot 10^{-4} \end{pmatrix}$$

8.3) Update the structure displacements for the current step and current iteration:

$$\mathbf{v}^{n,i} = \mathbf{v}^{n,i-1} + \Delta\mathbf{v}^{n,i}$$

The following displacement vector is obtained:

$$\mathbf{v}^{1,1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1.60 \\ -0.017 \\ -6.00 \cdot 10^{-4} \end{pmatrix}$$

8.4) Assemble the material and geometric stiffness matrices of the entire structure (one element in this case) $\mathbf{K}_{e,structure}^{n,i}$ and $\mathbf{K}_{g,structure}^{n,i}$, respectively, as well as the structure resisting force vector $\mathbf{F}_{int}^{n,i}$. With a loop, go over all elements (if more):

8.4.1) Determine the element displacement vector in the local reference frame $\boldsymbol{u}^{n,i}$

$$\mathbf{u}_{elem}^{n,i} = \mathbf{T}_{elem}\mathbf{v}_{elem}^{n,i}$$

Where the subscript $elem$ denotes the degrees of freedom corresponding to element $elem$. For a single element, the following vector is obtained:

$$\mathbf{u}^{n,i} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -0.017 \\ 1.60 \\ -6.00 \cdot 10^{-4} \end{pmatrix}$$

8.4.2) Using the corotational geometric formulation (or the linear geometric formulation), compute the element displacements in the basic reference system $\bar{\mathbf{u}} = [\bar{u}_1, \bar{u}_2, \bar{u}_3]^T$. For the corotational formulation, the following equations from Slide 32 of the lecture notes of Week #4 are used:

$$\bar{u}_1^{n,i} = l_n^{n,i} - l$$
$$\bar{u}_2^{n,i} = \mathbf{u}^{n,i}(3) - \beta^{n,i}$$
$$\bar{u}_3^{n,i} = \mathbf{u}^{n,i}(6) - \beta^{n,i}$$

With

$$l_n^{n,i} = \sqrt{\left(l + \mathbf{u}^{n,i}(4) - \mathbf{u}^{n,i}(1)\right)^2 + \left(\mathbf{u}^{n,i}(5) - \mathbf{u}^{n,i}(2)\right)^2}$$

$$\beta^{n,i} = \arctan\left(\frac{\mathbf{u}^{n,i}(5) - \mathbf{u}^{n,i}(2)}{l + \mathbf{u}^{n,i}(4) - \mathbf{u}^{n,i}(1)}\right)$$

For a single element, the following vector is obtained using the corotational formulation:

$$\overline{\mathbf{u}}^{n,i} = \begin{pmatrix} -0.017 \\ 4.00 \cdot 10^{-4} \\ 2.00 \cdot 10^{-4} \end{pmatrix}$$

8.4.3) Compute the element internal forces in the basic reference frame $\overline{\mathbf{q}}^{n,i}$:

$$\overline{\mathbf{q}}^{n,i} = \overline{\mathbf{K}}^{n,i}\overline{\mathbf{u}}^{n,i}$$

For a single element, the following value is obtained:

$$\overline{\mathbf{q}}^{1,1} = \begin{pmatrix} -1.08 \cdot 10^{4} \\ 2.20 \cdot 10^{6} \\ 18.82 \end{pmatrix}$$

8.4.4) Determine the transformation matrix $\mathbf{L}^{n-1}$ from the basic to the local reference system. The following equation from Slide 34 of the lecture notes of Week #4 are used for the corotational formulation:

$$\mathbf{L}^{n,i} = \begin{bmatrix} -c & -s & 0 & c & s & 0 \\ -s/L_n & c/L_n & 1 & s/L_n & -c/Ln & 0 \\ -s/L_n & c/L_n & 0 & s/L_n & -c/L_n & 1 \end{bmatrix}$$

Where

$$L_n = L_n^{n,i},\ c = \cos(\beta^{n,i}) \text{ and } s = \sin(\beta^{n,i})$$

For a single element, the following matrix is obtained:

$$\mathbf{L}^{n,i} = \begin{bmatrix} -1 & 4.01 \cdot 10^{-4} & 0 & 1 & -4.01 \cdot 10^{-4} & 0 \\ 1.00 \cdot 10^{-7} & 2.50 \cdot 10^{-4} & 1 & -1.00 \cdot 10^{-7} & -2.50 \cdot 10^{-4} & 0 \\ 1.00 \cdot 10^{-7} & 2.50 \cdot 10^{-4} & 0 & -1.00 \cdot 10^{-7} & -2.50 \cdot 10^{-4} & 1 \end{bmatrix}$$

8.4.5) Compute the element internal force vector in the local reference system:

$$\mathbf{Q}_{elem}^{n,i} = \left(\mathbf{L}^{n,i}\right)^{T}\overline{\mathbf{q}}^{n,i}$$

For a single element, the following vector is obtained:

$$\mathbf{Q}_{elem}^{n,i} = \begin{pmatrix} 1.08 \cdot 10^{4} \\ 545.69 \\ 2.20 \cdot 10^{6} \\ -1.08 \cdot 10^{4} \\ -545.7 \\ 18.82 \end{pmatrix}$$

8.4.6) Determine the element geometric stiffness matrix in the local reference frame $\mathbf{K}_{g,elem}^{n,i}$. The following equations from Slide 34 of the lecture notes of Week #4 are used:

$$\mathbf{K}_{g,elem} = \frac{\overline{q}_1}{l_n}\begin{bmatrix} s^2 & -cs & 0 & -s^2 & cs & 0 \\ -cs & c^2 & 0 & cs & -c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & cs & 0 & s^2 & -cs & 0 \\ cs & -c^2 & 0 & -cs & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\overline{q}_2 + \overline{q}_3}{l_n^2}\begin{bmatrix} -2sc & c^2 - s^2 & 0 & 2sc & -c^2 + s^2 & 0 \\ c^2 - s^2 & 2cs & 0 & -c^2 + s^2 & -2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2sc & -c^2 + s^2 & 0 & -2sc & c^2 - s^2 & 0 \\ -c^2 + s^2 & -2cs & 0 & c^2 - s^2 & 2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For a single element, the following value is obtained:

$$\mathbf{K}_{g,elem}^{n,i} = \begin{bmatrix} 1.10 \cdot 10^{-4} & 0.14 & 0 & -1.10 \cdot 10^{-4} & -0.14 & 0 \\ 0.14 & -2.70 & 0 & -0.14 & 2.70 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1.10 \cdot 10^{-4} & -0.14 & 0 & 1.10 \cdot 10^{-4} & 0.14 & 0 \\ -0.14 & 2.70 & 0 & 0.14 & -2.70 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8.4.7) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}^{n,i}$ and $\mathbf{K}_{g,structure}^{n,i}$ as well as the structure internal force vector $\mathbf{F}_{int}^{n,i}$ with the element quantities

8.5) Compute the unbalanced load vector $\mathbf{F}_{unb}^{n,i} = \mathbf{F}_{int}^{n,i} - \mathbf{F}_{ext}^{n}$
For a single element, the following value is obtained:

$$\mathbf{F}_{unb}^{n,i} = \begin{pmatrix} -545.69 \\ 1.08 \cdot 10^{4} \\ 2.20 \cdot 10^{6} \\ -4.31 \\ 203.72 \\ 18.82 \end{pmatrix}$$

8.6) Check if the Newton-Raphson scheme has converged. In source code provided in the solution with MATLAB, convergence is achieved once:

$$\left\|\mathbf{F}_{unb,f}^{n,i}\right\| < tol$$

8.7) If iteration i has converged, go to next load step n, else set $i = i + 1$ and $\Delta\mathbf{F}_{ext}^{n,i} = -\mathbf{F}_{unb,f}^{n,i-1}$ and go to the next step

Figure 1.2 compares results obtained by neglecting the second-order geometric effects (termed ‘Linear’) with those considering geometric nonlinearities using the corotational formulation (termed ‘Corotational’). Notice that the results are practically identical only when the lateral displacement is less than about 50mm.

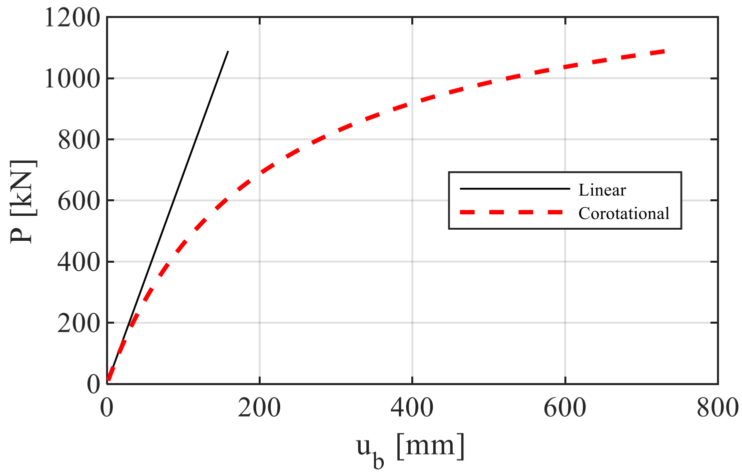

Figure 1.2. Effects of geometric transformation on member response under compressive load

c) The same approach discussed above is used to determine the secondary equilibrium path for $\alpha = 0.05$ and a tensile axial load. In this case, tension is anticipated to create a stable equilibrium path as the column is in traction. Figure 1.3 compares the results by neglecting the second-order geometric effects (termed 'Linear') with those considering geometric nonlinearities using the corotational formulation (termed 'Corotational'):

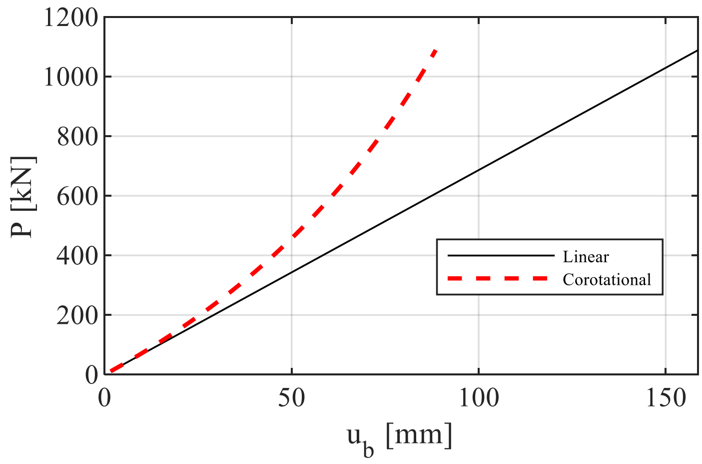

Figure 1.3. Effects of geometric transformation on member response under tensile load

Exercise #2:

Consider the following frame:

- Members ab and bc: $A = 2.50 \cdot 10^4 mm^2, I = 6.36 \cdot 10^8 mm^2, E = 200{,}000 MPa$
- Members bd and ce: $A = 1.76 \cdot 10^4 mm^2, I = 8.61 \cdot 10^8 mm^2, E = 200{,}000 MPa$

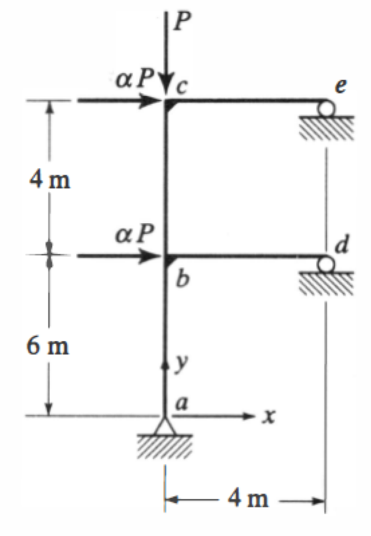

Figure 2.1. Planar frame under axial and lateral loading

Analyze the behavior of the system for the following cases:

1. $\alpha = 0$ (determine the critical load)

Using load control and the corotational formulation, determine the secondary equilibrium path of the structure when:

2. $\alpha = 0.01$

Solution:
In this solution, each member is modeled using a single 2D elastic beam-column element.

a) For $\alpha = 0$, to determine the critical load, the following steps are used:
1) Define the member properties (E, I, A and l)
2) Define the connectivity matrix and the mapping matrix between local and global degrees of freedom. The figure below shows the global degrees of freedom used for the structure

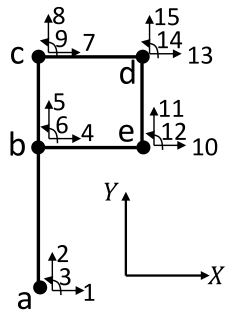

Figure 2.2. Global degrees of freedom

The mapping matrix $\boldsymbol{numEq}$ is as follows:

$$\boldsymbol{numEq} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 10 & 11 & 12 \\ 7 & 8 & 9 & 13 & 14 & 15 \end{bmatrix}$$

3) For each member, determine the transformation matrix **T** between local and global coordinates. In this exercise, the local x-axis is defined in the axial direction of the element

4) Assemble the initial structure stiffness matrix $\mathbf{K}_{structure}$

5) Define the boundary conditions, the fixed and free degrees of freedom, and the external loads i.e., apply the reference load:

$$\mathbf{P}^{ref} = [0\ 0\ 0\ 0\ 0\ 0\ 0 - 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^T$$

6) Compute the structure displacements **v** corresponding to the refence load $\mathbf{P}^{ref}$:

$$\mathbf{v}_f = \left(\mathbf{K}_{structure,f}\right)^{-1}\mathbf{P}^{ref}$$

where the subscript f denotes the free degrees of freedom of the system

The following displacement vector is obtained:

$$
\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 3.33 \cdot 10^{-4} \\ -2.00 \cdot 10^{-6} \\ -1.20 \cdot 10^{-6} \\ 3.33 \cdot 10^{-10} \\ -3.60 \cdot 10^{-6} \\ -2.00 \cdot 10^{-6} \\ 4.67 \cdot 10^{-10} \\ -2.00 \cdot 10^{-6} \\ 0 \\ 2.84 \cdot 10^{-10} \\ -3.60 \cdot 10^{-6} \\ 0 \\ 5.16 \cdot 10^{-10} \end{pmatrix}
$$

7) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}$ and $\mathbf{K}_{g,structure}$. With a loop, go over all elements:

7.1) Determine the element displacement vector in the local reference frame $\mathbf{u}$

$$\mathbf{u}_{elem} = \mathbf{T}_{elem}\mathbf{v}_{elem}$$

where the subscript $elem$ denotes the degrees of freedom corresponding to the element $elem$.

7.2) Using the corotational formulation, compute the element displacements in the basic reference frame $\bar{\mathbf{u}} = [\bar{u}_1, \bar{u}_2, \bar{u}_3]^T$. The following equations from Slide 32 of the lecture notes of Week #4 are used:

$$\bar{u}_1 = l_n - l$$
$$\bar{u}_2 = \mathbf{u}(3) - \beta$$
$$\bar{u}_3 = \mathbf{u}(6) - \beta$$

With

$$l_n = \sqrt{\left(l + \mathbf{u}(4) - \mathbf{u}(1)\right)^2 + \left(\mathbf{u}(5) - \mathbf{u}(2)\right)^2}$$
$$\beta = \arctan\left(\frac{\mathbf{u}(5) - \mathbf{u}(2)}{l + \mathbf{u}(4) - \mathbf{u}(1)}\right)$$

7.3) Compute the element internal forces in the basic reference frame $\bar{\mathbf{q}}$:

$$\bar{\mathbf{q}} = \bar{\mathbf{K}}\bar{\mathbf{u}}$$

7.4) Determine the transformation matrix $\mathbf{L}$ from the basic to the local reference system. The equation from Slide 34 of the lecture notes of Week #4 is used:

$$\mathbf{L} = \begin{bmatrix} -c & -s & 0 & c & s & 0 \\ -s/L_n & c/L_n & 1 & s/L_n & -c/Ln & 0 \\ -s/L_n & c/L_n & 0 & s/L_n & -c/L_n & 1 \end{bmatrix}$$

where

$$c = \cos(\beta) \text{ and } s = \sin(\beta)$$

7.5) Determine the element geometric stiffness matrix in the local reference frame $\mathbf{K}_{g,elem}$. The following equations from Slide 34 of the lecture notes of Week #4 are used:

$$\mathbf{K}_{g,elem} = \frac{\bar{q}_1}{l_n}\begin{bmatrix} s^2 & -cs & 0 & -s^2 & cs & 0 \\ -cs & c^2 & 0 & cs & -c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & cs & 0 & s^2 & -cs & 0 \\ cs & -c^2 & 0 & -cs & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\bar{q}_2 + \bar{q}_3}{l_n^2}\begin{bmatrix} -2sc & c^2 - s^2 & 0 & 2sc & -c^2 + s^2 & 0 \\ c^2 - s^2 & 2cs & 0 & -c^2 + s^2 & -2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2sc & -c^2 + s^2 & 0 & -2sc & c^2 - s^2 & 0 \\ -c^2 + s^2 & -2cs & 0 & c^2 - s^2 & 2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following quantities are obtained for the four elements of the planar structure:
-Element ab:

$$\mathbf{u}_{ab} = \begin{pmatrix} 0 \\ 0 \\ 3.33 \cdot 10^{-10} \\ -1.20 \cdot 10^{-6} \\ 2.00 \cdot 10^{-6} \\ 3.33 \cdot 10^{-10} \end{pmatrix}$$

$$\bar{\mathbf{u}}_{ab} = 10^{-6} \cdot \begin{pmatrix} -1.20 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{\mathbf{q}}_{ab} = \begin{pmatrix} -1.00 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{L}_{ab} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1.67 \cdot 10^{-4} & 1 & 0 & -1.67 \cdot 10^{-4} & 0 \\ 0 & 1.67 \cdot 10^{-4} & 0 & 0 & -1.67 \cdot 10^{-4} & 1 \end{bmatrix}$$

$$\mathbf{K}_{g,ab} = 10^{-4} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.67 & 0 & 0 & 1.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.67 & 0 & 0 & -1.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

-Element bc:

$$\mathbf{u}_{bc} = 10^{-6} \cdot \begin{pmatrix} -1.20 \\ 2.00 \\ 0 \\ -2.00 \\ 3.60 \\ 0 \end{pmatrix}$$

$$\bar{\mathbf{u}}_{bc} = 10^{-6} \cdot \begin{pmatrix} -0.80 \\ 0.00 \\ 0.00 \end{pmatrix}$$

$$\bar{\mathbf{q}}_{bc} = \begin{pmatrix} -1.00 \\ -4.26 \\ 4.26 \end{pmatrix}$$

$$\mathbf{L}_{bc} = 10^{-4} \cdot \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2.50 & 1 & 0 & -2.50 & 0 \\ 0 & 2.50 & 0 & 0 & -2.50 & 1 \end{bmatrix}$$

$$\mathbf{K}_{g,bc} = 10^{-4} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.50 & 0 & 0 & 2.50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.50 & 0 & 0 & -2.50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

-Element bd:

$$\mathbf{u}_{bd} = \begin{pmatrix} -2.00 \cdot 10^{-6} \\ -1.20 \cdot 10^{-6} \\ 3.33 \cdot 10^{-10} \\ -2.00 \cdot 10^{-6} \\ 0 \\ 2.84 \cdot 10^{-10} \end{pmatrix}$$

$$\overline{\mathbf{u}}_{bd} = 10^{-11} \cdot \begin{pmatrix} 0 \\ 3.30 \\ -1.65 \end{pmatrix}$$

$$\overline{\mathbf{q}}_{bd} = \begin{pmatrix} 0 \\ 4.26 \\ 0 \end{pmatrix}$$

$$\mathbf{L}_{bd} = 10^{-4} \cdot \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2.50 & 1 & 0 & -2.50 & 0 \\ 0 & 2.50 & 0 & 0 & -2.50 & 1 \end{bmatrix}$$

$$\mathbf{K}_{g,bd} = 10^{-7} \cdot \begin{bmatrix} 0 & 2.66 & 0 & 0 & -2.66 & 0 \\ 2.66 & 0 & 0 & -2.66 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.66 & 0 & 0 & 2.66 & 0 \\ -2.66 & 0 & 0 & 2.66 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

-Element ce:

$$\mathbf{u}_{ce} = \begin{pmatrix} -3.60 \cdot 10^{-6} \\ -2.00 \cdot 10^{-6} \\ 4.67 \cdot 10^{-10} \\ -3.60 \cdot 10^{-6} \\ 0 \\ 5.16 \cdot 10^{-10} \end{pmatrix}$$

$$\overline{\mathbf{u}}_{ce} = 10^{-11} \cdot \begin{pmatrix} 0 \\ -3.30 \\ 1.65 \end{pmatrix}$$

$$\overline{\mathbf{q}}_{ce} = \begin{pmatrix} 0 \\ -4.26 \\ 0 \end{pmatrix}$$

$$\mathbf{L}_{ce} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2.50 \cdot 10^{-4} & 1 & 0 & -2.50 \cdot 10^{-4} & 0 \\ 0 & 2.50 \cdot 10^{-4} & 0 & 0 & -2.50 \cdot 10^{-4} & 1 \end{bmatrix}$$

$$\mathbf{K}_{g,ce} = 10^{-7} \cdot \begin{bmatrix} 0 & -2.66 & 0 & 0 & 2.66 & 0 \\ -2.66 & 0 & 0 & 2.66 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.66 & 0 & 0 & -2.66 & 0 \\ 2.66 & 0 & 0 & -2.66 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.6) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}$ and $\mathbf{K}_{g,structure}$ with the element quantities

8) Determine the critical load multiplier λ_{crit}:
8.1) Solve the following eigenvalue problem

$$\left[\mathbf{K}_{e,structure,f} + \lambda \mathbf{K}_{g,structure,f}\right]\boldsymbol{\Delta}_f = \mathbf{0}$$

8.2) The critical load multiplier λ_{crit} corresponds to the minimum of all eigenvectors λ
8.3) The critical load $\mathbf{P}_{crit}$ is then given by: $\mathbf{P}_{crit} = \lambda_{crit}\mathbf{P}^{ref}$

For a single element, the following value is obtained: $\mathbf{P}_{crit} = 7300\ kN$

b) To determine the secondary equilibrium path for $\alpha = 0.01$, the following steps are used:
1) Define the member properties (E, I, A and l)
2) Define the connectivity matrix and the mapping matrix between local and global degrees of freedom. The figure below shows the global degrees of freedom used for the structure

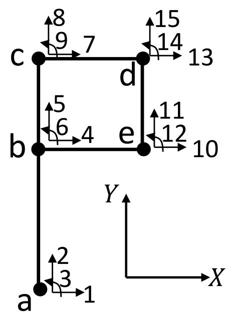

Figure 2.3. Global degrees of freedom

The mapping matrix $\boldsymbol{numEq}$ is therefore given by

$$\boldsymbol{numEq} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 10 & 11 & 12 \\ 7 & 8 & 9 & 13 & 14 & 15 \end{bmatrix}$$

3) For each member, determine the transformation matrix $\mathbf{T}$ between local and global coordinates. In this exercise, the local x-axis is defined in the axial direction of the element, and therefore corresponds to the global Y-axis
4) Assemble the initial structure stiffness matrix $\mathbf{K}_{structure}$.

5) Define the boundary conditions, the fixed and the free degrees of freedom of the problem, and the external loads i.e., apply the reference load:

$$\mathbf{F}^{ref} = [0\ 0\ 0\ 0.01\ 0\ 0\ 0.01 - 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^T$$

6) Initialize the variables used within the Newton-Raphson scheme

$$\lambda = 0, \mathbf{v} = \mathbf{0}$$

where λ denotes the load multiplier (i.e. $\mathbf{F}_{ext} = \lambda\mathbf{F}^{ref}$)

7) Define the parameters defining the load-control algorithm:

- The final external load $\mathbf{F}_{ext}^{\text{tot}} = \lambda^{tot}\mathbf{F}^{ref}$
- The number of steps n_{tot}
- At each load step, the increment in external force is given by $\Delta\bar{\lambda} = \lambda^{tot}/n_{tot}$
- The tolerance tol
- The maximum number of iterations for each iterations of the Newton-Raphson scheme, $i_{\max}$

For the solution procedure described herein, the following values are used: $\lambda^{tot} = 4000000$, $n_{tot} = 100$ and $tol = 1e - 4$

8) For load increment n, perform the Newton-Raphson iterations

8.1) For $i = 1$, set, $\Delta\mathbf{F}_{ext}^{n,i=1} = \Delta\bar{\lambda}\mathbf{F}^{ref}$, $\mathbf{F}_{int}^{n,i=1} = \mathbf{F}_{int}^{n-1}$, $\mathbf{K}_{structure}^{n,1} = \mathbf{K}_{structure}^{n-1}$ and $\mathbf{v}^{n,1} = \mathbf{v}^{n-1}$

8.2) Compute the increment in structure displacements $\Delta\mathbf{v}^{n,i}$:

$$\Delta\mathbf{v}_f^{n,i} = \left(\mathbf{K}_{structure,f}^{n,i}\right)^{-1}\Delta\mathbf{F}_{ext}^{n}$$

Where the subscript f denotes the free degrees of freedom of the system. The following displacement vector is obtained:

$$\Delta\mathbf{v}^{1,1} = \begin{pmatrix} 0 \\ 0 \\ -1.38 \cdot 10^{-4} \\ 0.60 \\ -0.046 \\ -2.44 \cdot 10^{-5} \\ 0.65 \\ -0.078 \\ 5.80 \cdot 10^{-6} \\ 0.60 \\ 0 \\ 2.95 \cdot 10^{-5} \\ 0.65 \\ 0 \\ 2.62 \cdot 10^{-5} \end{pmatrix}$$

8.3) Update the structure displacements:

$$\mathbf{v}^{n,i} = \mathbf{v}^{n,i-1} + \Delta\mathbf{v}^{n,i}$$

The following displacement vector is obtained:

$$\mathbf{v}^{1,1} = \begin{pmatrix} 0 \\ 0 \\ -1.38 \cdot 10^{-4} \\ 0.60 \\ -0.046 \\ -2.44 \cdot 10^{-5} \\ 0.65 \\ -0.078 \\ 5.80 \cdot 10^{-6} \\ 0.60 \\ 0 \\ 2.95 \cdot 10^{-5} \\ 0.65 \\ 0 \\ 2.62 \cdot 10^{-5} \end{pmatrix}$$

8.4) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}^{n,i}$ and $\mathbf{K}_{g,structure}^{n,i}$, as well as the structure resisting force vector $\mathbf{F}_{int}^{n,i}$. With a loop, go over all elements:

8.4.1) Determine the element displacement vector in the local reference system, $\mathbf{u}^{n,i}$

$$\mathbf{u}_{elem}^{n,i} = \mathbf{T}_{elem}\mathbf{v}_{elem}^{n,i}$$

where the subscript $elem$ denotes the degrees of freedom corresponding to element $elem$.

8.4.2) Using the corotational or the linear formulation, compute the element displacements in the basic reference frame $\bar{\mathbf{u}} = [\bar{u}_1, \bar{u}_2, \bar{u}_3]^T$. For the corotational formulation, the following equations from Slide 32 of the lecture notes of Week #4 are used:

$$\bar{u}_1^{n,i} = l_n^{n,i} - l$$
$$\bar{u}_2^{n,i} = \mathbf{u}^{n,i}(3) - \beta^{n,i}$$
$$\bar{u}_3^{n,i} = \mathbf{u}^{n,i}(6) - \beta^{n,i}$$

With

$$l_n^{n,i} = \sqrt{\left(l + \mathbf{u}^{n,i}(4) - \mathbf{u}^{n,i}(1)\right)^2 + \left(\mathbf{u}^{n,i}(5) - \mathbf{u}^{n,i}(2)\right)^2}$$
$$\beta^{n,i} = \arctan\left(\frac{\mathbf{u}^{n,i}(5) - \mathbf{u}^{n,i}(2)}{l + \mathbf{u}^{n,i}(4) - \mathbf{u}^{n,i}(1)}\right)$$

8.4.3) Compute the element internal forces in the basic reference system, $\bar{\mathbf{q}}^{n,i}$:

$$\bar{\mathbf{q}}^{n,i} = \bar{\mathbf{K}}^{n,i}\bar{\mathbf{u}}^{n,i}$$

8.4.4) Determine the transformation matrix $\mathbf{L}^{n-1}$ from the basic to the local reference frame. The equation from Slide 34 of the lecture notes of Week #4 is used for the corotational formulation:

$$\mathbf{L}^{n,i} = \begin{bmatrix} -c & -s & 0 & c & s & 0 \\ -s/L_n & c/L_n & 1 & s/L_n & -c/Ln & 0 \\ -s/L_n & c/L_n & 0 & s/L_n & -c/L_n & 1 \end{bmatrix}$$

Where

$$L_n = L_n^{n,i},\ c = \cos(\beta^{n,i}) \text{ and } s = \sin(\beta^{n,i})$$

8.4.5) Compute the element internal force vector in the local reference frame:

$$\mathbf{Q}_{elem}^{n,i} = \left(\mathbf{L}^{n,i}\right)^T \overline{\mathbf{q}}^{n,i}$$

8.4.6) Determine the element geometric stiffness matrix in the local reference frame $\mathbf{K}_{g,elem}^{n,i}$. The following equations from Slide 34 of the lecture notes of Week #4 are used:

$$\mathbf{K}_{g,elem} = \frac{\bar{q}_1}{l_n}\begin{bmatrix} s^2 & -cs & 0 & -s^2 & cs & 0 \\ -cs & c^2 & 0 & cs & -c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & cs & 0 & s^2 & -cs & 0 \\ cs & -c^2 & 0 & -cs & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\bar{q}_2 + \bar{q}_3}{l_n^2}\begin{bmatrix} -2sc & c^2 - s^2 & 0 & 2sc & -c^2 + s^2 & 0 \\ c^2 - s^2 & 2cs & 0 & -c^2 + s^2 & -2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2sc & -c^2 + s^2 & 0 & -2sc & c^2 - s^2 & 0 \\ -c^2 + s^2 & -2cs & 0 & c^2 - s^2 & 2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following quantities are obtained for the four elements:

-Element ab:

$$\mathbf{u}_{ab}^{1,1} = \begin{pmatrix} 0 \\ 0 \\ -1.38 \cdot 10^{-4} \\ -0.046 \\ -0.60 \\ -2.44 \cdot 10^{-5} \end{pmatrix}$$

$$\overline{\mathbf{u}}_{ab}^{1,1} = \begin{pmatrix} -0.0461 \\ -3.77 \cdot 10^{-5} \\ 7.55 \cdot 10^{-5} \end{pmatrix}$$

$$\overline{\mathbf{q}}_{ab}^{1,1} = \begin{pmatrix} -3.84 \cdot 10^{4} \\ 97.53 \\ 4.80 \cdot 10^{6} \end{pmatrix}$$

$$\mathbf{L}_{ab}^{1,1} = \begin{bmatrix} -1 & 9.99 \cdot 10^{-5} & 0 & 1 & -9.99 \cdot 10^{-5} & 0 \\ 1.66 \cdot 10^{-8} & 1.67 \cdot 10^{-4} & 1 & -1.66 \cdot 10^{-8} & -1.67 \cdot 10^{-4} & 0 \\ 1.66 \cdot 10^{-8} & 1.67 \cdot 10^{-4} & 0 & -1.66 \cdot 10^{-8} & -1.67 \cdot 10^{-4} & 1 \end{bmatrix}$$

$$\mathbf{K}_{g,ab}^{1,1} = \begin{bmatrix} 2.66 \cdot 10^{-5} & 1.33 \cdot 10^{-1} & 0 & -2.66 \cdot 10^{-5} & -1.33 \cdot 10^{-1} & 0 \\ 1.33 \cdot 10^{-1} & -6.40 & 0 & -1.33 \cdot 10^{-1} & 6.40 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2.66 \cdot 10^{-5} & -1.33 \cdot 10^{-1} & 0 & 2.66 \cdot 10^{-5} & 1.33 \cdot 10^{-1} & 0 \\ -1.33 \cdot 10^{-1} & 6.40 & 0 & 1.33 \cdot 10^{-1} & -6.40 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

-Element bc:

$$\mathbf{u}_{bc}^{1,1} = \begin{pmatrix} -0.046 \\ -0.60 \\ -2.44 \cdot 10^{-5} \\ -0.078 \\ -0.65 \\ 5.80 \cdot 10^{-6} \end{pmatrix}$$

$$\overline{\mathbf{u}}_{bc}^{1,1} = \begin{pmatrix} -0.0316 \\ -1.09 \cdot 10^{-5} \\ 1.09 \cdot 10^{-5} \end{pmatrix}$$

$$\bar{\mathbf{q}}_{bc}^{1,1} = \begin{pmatrix} -3.96 \cdot 10^{4} \\ -1.60 \cdot 10^{5} \\ 1.76 \cdot 10^{6} \end{pmatrix}$$

$$\mathbf{L}_{bc}^{1,1} = \begin{bmatrix} -1 & 1.35 \cdot 10^{-5} & 0 & 1 & -1.35 \cdot 10^{-5} & 0 \\ 3.37 \cdot 10^{-9} & 2.50 \cdot 10^{-4} & 1 & -3.37 \cdot 10^{-9} & -2.50 \cdot 10^{-4} & 0 \\ 3.37 \cdot 10^{-9} & 2.50 \cdot 10^{-4} & 0 & -3.37 \cdot 10^{-9} & -2.50 \cdot 10^{-4} & 1 \end{bmatrix}$$

$$\mathbf{K}_{g,bc}^{1,1} = \begin{bmatrix} 2.70 \cdot 10^{-6} & 9.99 \cdot 10^{-2} & 0 & -2.70 \cdot 10^{-6} & -9.99 \cdot 10^{-2} & 0 \\ 9.99 \cdot 10^{-2} & -9.89 & 0 & -9.99 \cdot 10^{-2} & 9.89 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2.70 \cdot 10^{-6} & -9.99 \cdot 10^{-2} & 0 & 2.70 \cdot 10^{-6} & 9.99 \cdot 10^{-2} & 0 \\ -9.99 \cdot 10^{-2} & 9.89 & 0 & 9.99 \cdot 10^{-2} & -9.89 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

-Element bd:

$$\mathbf{u}_{bd}^{1,1} = \begin{pmatrix} 0.60 \\ -0.046 \\ -2.44 \cdot 10^{-5} \\ 0.60 \\ 0 \\ 2.95 \cdot 10^{-5} \end{pmatrix}$$

$$\bar{\mathbf{u}}_{bd}^{1,1} = \begin{pmatrix} 2.65 \cdot 10^{-7} \\ -3.59 \cdot 10^{-5} \\ 1.80 \cdot 10^{-5} \end{pmatrix}$$

$$\bar{\mathbf{q}}_{bd}^{1,1} = \begin{pmatrix} 0.23 \\ -4.64 \cdot 10^{6} \\ 1.32 \cdot 10^{-4} \end{pmatrix}$$

$$\mathbf{L}_{bd}^{1,1} = \begin{bmatrix} -1 & -1.15 \cdot 10^{-5} & 0 & 1 & 1.15 \cdot 10^{-5} & 0 \\ -\ 2.88 \cdot 10^{-9} & 2.50 \cdot 10^{-4} & 1 & 2.88 \cdot 10^{-9} & -2.50 \cdot 10^{-4} & 0 \\ -2.88 \cdot 10^{-9} & 2.50 \cdot 10^{-4} & 0 & 2.88 \cdot 10^{-9} & -2.50 \cdot 10^{-4} & 1 \end{bmatrix}$$

$$\mathbf{K}_{g,bd}^{1,1} = \begin{bmatrix} 6.68 \cdot 10^{-6} & -2.90 \cdot 10^{-1} & 0 & -6.68 \cdot 10^{-6} & 2.90 \cdot 10^{-1} & 0 \\ -2.90 \cdot 10^{-1} & 5.17 \cdot 10^{-5} & 0 & 2.90 \cdot 10^{-1} & -5.17 \cdot 10^{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -6.68 \cdot 10^{-6} & 2.90 \cdot 10^{-1} & 0 & 6.68 \cdot 10^{-6} & -2.90 \cdot 10^{-1} & 0 \\ 2.90 \cdot 10^{-1} & -5.17 \cdot 10^{-5} & 0 & -2.90 \cdot 10^{-1} & 5.17 \cdot 10^{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

-Element ce:

$$\mathbf{u}_{ce}^{1,1} = \begin{pmatrix} 0.65 \\ -0.078 \\ 5.80 \cdot 10^{-6} \\ 0.65 \\ 0 \\ 2.62 \cdot 10^{-5} \end{pmatrix}$$

$$\bar{\mathbf{u}}_{ce}^{1,1} = \begin{pmatrix} 7.55 \cdot 10^{-7} \\ -1.36 \cdot 10^{-5} \\ 6.82 \cdot 10^{-5} \end{pmatrix}$$

$$\bar{\mathbf{q}}_{ce}^{1,1} = \begin{pmatrix} 0.66 \\ -1.76 \cdot 10^{6} \\ 6.32 \cdot 10^{-4} \end{pmatrix}$$

$$\mathbf{L}_{ce}^{1,1} = \begin{bmatrix} -1 & -1.94 \cdot 10^{-5} & 0 & 1 & 1.94 \cdot 10^{-5} & 0 \\ -\ 4.86 \cdot 10^{-9} & 2.50 \cdot 10^{-4} & 1 & 4.86 \cdot 10^{-9} & -2.50 \cdot 10^{-4} & 0 \\ -4.86 \cdot 10^{-9} & 2.50 \cdot 10^{-4} & 0 & 4.86 \cdot 10^{-9} & -2.50 \cdot 10^{-4} & 1 \end{bmatrix}$$

$$\mathbf{K}_{g,ce}^{1,1} = \begin{bmatrix} 4.28 \cdot 10^{-6} & -1.10 \cdot 10^{-1} & 0 & -4.28 \cdot 10^{-6} & 1.10 \cdot 10^{-1} & 0 \\ -1.10 \cdot 10^{-1} & 1.62 \cdot 10^{-4} & 0 & 1.10 \cdot 10^{-1} & -1.62 \cdot 10^{-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4.28 \cdot 10^{-6} & 1.10 \cdot 10^{-1} & 0 & 4.28 \cdot 10^{-6} & -1.10 \cdot 10^{-1} & 0 \\ 1.10 \cdot 10^{-1} & -1.62 \cdot 10^{-4} & 0 & -1.10 \cdot 10^{-1} & 1.62 \cdot 10^{-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8.4.7) Assemble the structure material and geometric stiffness matrices $\mathbf{K}_{e,structure}^{n,i}$ and $\mathbf{K}_{g,structure}^{n,i}$ as well as the structure internal force vector $\mathbf{F}_{int}^{n,i}$ with the element quantities. The following value of internal force vector is obtained:

$$\mathbf{F}_{int}^{1,1} = \begin{pmatrix} -796.21 \\ 38375.14 \\ 97.53 \\ 396.51 \\ 24.41 \\ 117.90 \\ 398.82 \\ -39999.55 \\ 20.37 \\ 0.22 \\ 1159.88 \\ 0 \\ 0.66 \\ 440.12 \\ 0 \end{pmatrix}$$

8.5) Compute the unbalanced load vector $\mathbf{F}_{unb}^{n,i} = \mathbf{F}_{int}^{n,i} - \mathbf{F}_{ext}^{n}$

The following value is obtained after doing the matrix operations with the two vectors:

$$\mathbf{F}_{unb}^{n,i} = \begin{pmatrix} -796.21 \\ 38375.14 \\ 97.53 \\ -3.49 \\ 24.41 \\ 117.90 \\ -1.18 \\ 0.45 \\ 20.37 \\ 0.22 \\ 1159.88 \\ 0 \\ 0.66 \\ 440.12 \\ 0 \end{pmatrix}$$

8.6) Check if the Newton-Raphson scheme has converged. In the source code that is provided with the solution in MATLAB, convergence is achieved once

$$\left\|\mathbf{F}_{unb,f}^{n,i}\right\| < tol$$

8.7) If iteration i has converged, go to next load step n, else set $i = i + 1$ and $\Delta\mathbf{F}_{ext}^{n,i} = -\mathbf{F}_{unb,f}^{n,i-1}$ and go to step

Figure 2.4 compares results obtained by using the linear transformation (termed 'Linear') with those considering geometric nonlinearities using the corotational formulation (termed 'Corotational'):

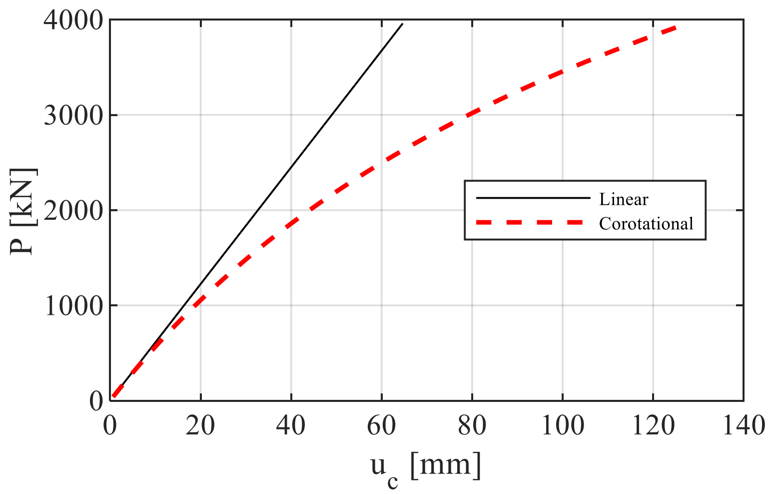

Figure 2.4. Effects of geometric transformation on structural response under the external load