Problem Set 2

CIVIL-425: Continuum Mechanics and Applications

6 March 2025

Exercise 1

Given the deformation mapping

$$\boldsymbol{\varphi}(\boldsymbol{X}) = [x_1,\, x_2,\, x_3]^\top = [X_1 + \alpha X_2^2 t,\, (1+\beta t)X_2,\, X_3]^\top \tag{1}$$

, where α, $\beta \in \mathbb{R}$, compute Lagrangian velocity and deformation fields.

Exercise 2

Consider a deformation mapping of the type:

$$x_1 = X_1\,, \tag{2}$$
$$x_2 = X_2\,, \tag{3}$$
$$x_3 = X_3 + w(X_1, X_2)\ \ , \tag{4}$$

where $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This mapping, when restricted to the plane $X_3 = 0$, represents moderate deformation of a membrane occupying a domain $\Omega \subset \mathbb{R}^2$.

1. Using the Piola transformation, write an expression for the element of oriented membrane area on the deformed configuration.
2. Write an integral expression for the area of the deformed membrane.
3. Show that, for small deflections, i.e., for $\omega \to 0$, the deformed area of the membrane may be approximated as

$$a = A + \int_\Omega \frac{1}{2}(w_{,1}^2 + w_{,2}^2)dX_1 dX_2\,. \tag{5}$$

Exercise 3

Consider a cylindrical solid referred to an orthonormal Cartesian reference frame $\{X_1,\, X_2,\, X_3\}$. Let the axis of the solid be aligned with the X_3-direction and let its normal cross-section occupy a region Ω in the $X_1 - X_2$ plane of boundary $\partial\Omega$. An *anti-plane shear* deformation of the solid can be defined as one for which the deformation mapping is of the form:

$$\boldsymbol{\varphi}(\boldsymbol{X}) = [x_1,\, x_2,\, x_3]^\top = [X_1,\, X_2,\, X_3 + w(X_1, X_2)]^\top\,, \tag{6}$$

where, in this definition, the spatial and material reference frames are taken to coincide, and the function w is defined over Ω.

i) Sketch the deformation of the region.

ii) Compute the deformation gradient field, the right Cauchy-Green deformation tensor field and the Jacobian of the deformation field in terms of w. Does the solid change volume during the deformation process? Are the local impenetrability conditions satisfied?

iii) Consider the unit vectors:

$$\boldsymbol{A} = \frac{w_{,1}\boldsymbol{G_1} + w_{,2}\boldsymbol{G_2}}{\sqrt{w_{,1}^2 + w_{,2}^2}}\,, \qquad \boldsymbol{B} = \frac{-w_{,2}\boldsymbol{G_1} + w_{,1}\boldsymbol{G_2}}{\sqrt{w_{,1}^2 + w_{,2}^2}}\,, \tag{7}$$

where $\{\boldsymbol{G_I}\}$, $I = 1, 2, 3$ are the (orthonormal) material basis vectors. How are $\boldsymbol{A}$ and $\boldsymbol{B}$ related to the level contours of $w(X_1, X_2)$? Compute, in terms of w, the change in length (measured by the corresponding strecth ratios) of $\boldsymbol{A}$ and $\boldsymbol{B}$, as well as the change in the angle subtended by $\boldsymbol{A}$ and $\boldsymbol{B}$. Interpret results.

iv) Using the Piola transformation, compute (in terms of w) the change of area of, and in the normal to, an infinitesimal material area contained in the X_1-X_2 plane.

v) Derive an integral expression for the deformed area of the domain Ω.

vi) Let the boundary $\partial\Omega$ of Ω be defined paramatrically by the equations

$$X_1 = X_1(S)\,, \qquad X_2 = X_2(S)\,, \tag{8}$$

where $0 \leq S \leq L$ is the arc-length measured along $\partial\Omega$. Derive an integral expression for the perimeter of the deformed boundary $\boldsymbol{\varphi}(\partial\Omega)$.