Problem Set 9

CIVIL-425: Continuum Mechanics and Applications

08 May 2025

Exercise 1: Traction test of Elastomer Specimen

We are interested in studying the response of an incompressible elastomeric specimen subjected to homogeneous deformations. We consider a specimen whose reference underformed configuration is a cube, whose edges of length L are directed along the axes. The specimen is an elastomer whose constitutive law has the form:

$$\underline{\underline{\sigma}} = -p\underline{\underline{\mathbb{I}}} + f(I_1)\underline{\underline{F}} \cdot \underline{\underline{F}}^T$$

with

$$\begin{cases} f(I_1) = \mu = \rho c k_B T & \text{Neo-Hookean model} \\ f(I_1) = \mu \left(3\sqrt{\frac{I_1}{3N}}\right)^{-1} \mathcal{L}^{-1}\left(\sqrt{\frac{I_1}{3N}}\right) & \text{8-chain model} \end{cases}$$

where ρ is the mass density, $I_1 = (\boldsymbol{C} : \boldsymbol{I})$ is the first strain invariant and $\mathcal{L}$ is the Langevin function defined as $\mathcal{L}(x) = \coth x - 1/x$. It is assumed that the deformations are quasi-static, isothermal, and that the volumetric strains are negligible.

We consider that the specimen is subjected to an extension by a factor of α in the direction $\underline{e}_1$ while its lateral faces (with external normal $\underline{e}_2$ and $\underline{e}_3$) are stress-free. By denoting β as the contraction factor in the directions $\underline{e}_2$ and $\underline{e}_3$, we are interested in transformations of the form:

$$\underline{\phi}(\underline{X}) = \alpha X_1 \underline{e}_1 + \beta \left(X_2 \underline{e}_2 + X_3 \underline{e}_3\right)$$

1. Explicit the deformation gradient tensor $\underline{\underline{F}}$ and show that $\beta = \frac{1}{\sqrt{\alpha}}$. Calculate $I_1 = \underline{\underline{C}} : \underline{\underline{\mathbb{I}}}$.
2. Determine the Cauchy stress tensor using the local equilibrium and the boundary conditions. Is p free to vary?
3. Deduce the resultant $\underline{R}$ of the force exerted on the deformed face from the face $(X_1 = L)$ in the initial configuration as a function of α, then plot $\frac{R \cdot e_1}{L^2}$. Consider the change of area from reference to deformed configuration.

Solution

1. The uniaxial traction corresponds to a transformation of the form:

$$\underline{\underline{F}} = \alpha \underline{e}_1 \otimes \underline{e}_1 + \beta \left(\underline{e}_2 \otimes \underline{e}_2 + \underline{e}_3 \otimes \underline{e}_3\right); \det \underline{\underline{F}} = \alpha\beta^2 \,.$$

2. The incompressibility condition yields the relationship between α and β, $\det \underline{\underline{F}} = 1 \implies \beta = \frac{1}{\sqrt{\alpha}}$. We then obtain:

$$\underline{\underline{C}} = \underline{\underline{F}}^T \cdot \underline{\underline{F}} = \alpha^2 \underline{e}_1 \otimes \underline{e}_1 + \frac{1}{\alpha}\left(\underline{e}_2 \otimes \underline{e}_2 + \underline{e}_3 \otimes \underline{e}_3\right)$$
$$\underline{\underline{B}} = \underline{\underline{F}} \cdot \underline{\underline{F}}^T = \alpha^2 \underline{e}_1 \otimes \underline{e}_1 + \frac{1}{\alpha}\left(\underline{e}_2 \otimes \underline{e}_2 + \underline{e}_3 \otimes \underline{e}_3\right)$$
$$I_1 = \mathrm{Tr}\,\underline{\underline{C}} = \mathrm{Tr}\,\underline{\underline{B}} = \alpha^2 + \frac{2}{\alpha}$$
$$\underline{\underline{F}}^{-1} = \frac{1}{\alpha}\underline{e}_1 \otimes \underline{e}_1 + \sqrt{\alpha}\left(\underline{e}_2 \otimes \underline{e}_2 + \underline{e}_3 \otimes \underline{e}_3\right)$$

The constitutive law gives us:

$$\underline{\underline{\sigma}} = -p\underline{\underline{\mathbb{I}}} + f(I_1)\underline{\underline{B}} = \left(\alpha^2 f(I_1) - p\right)\underline{e}_1 \otimes \underline{e}_1 + \left(\frac{1}{\alpha} f(I_1) - p\right)\left(\underline{e}_2 \otimes \underline{e}_2 + \underline{e}_3 \otimes \underline{e}_3\right)$$

This stress field balances the external forces if:

$$\mathrm{div}\,\underline{\underline{\sigma}} = \underline{0}(\text{in }\Omega); \quad \underline{\underline{\sigma}} \cdot \underline{e}_2 = \underline{0}\left(\text{on } x_2 = \pm\frac{L}{2\sqrt{\alpha}}\right); \quad \underline{\underline{\sigma}} \cdot \underline{e}_3 = \underline{0}\left(\text{on } x_3 = \pm\frac{L}{2\sqrt{\alpha}}\right)$$

so that :

$$\begin{cases} \mathrm{grad}\, p = \alpha^2 \frac{\partial f(I_1)}{\partial x}\underline{e}_1 + \frac{1}{\alpha}\frac{\partial f(I_1)}{\partial y}\underline{e}_2 + \frac{1}{\alpha}\frac{\partial f(I_1)}{\partial z}\underline{e}_3 \\ \frac{1}{\alpha} f(I_1) - p = 0 \end{cases}$$

This imposes: $p = \frac{1}{\alpha} f(I_1)$. Therefore, the pressure is not a free variable but is fixed due to the incompressibility constraint of the material. If $f(I_1)$ does not depend on $\underline{x}$, which will notably be satisfied if $\underline{\underline{C}}$ is constant, then

$$\underline{\underline{\sigma}} = \left(\alpha^2 - \frac{1}{\alpha}\right) f\left(\alpha^2 + \frac{2}{\alpha}\right)\underline{e}_1 \otimes \underline{e}_1$$

3. The sum of the forces exerted on the deformed face from the face ($X_1 = L$) in the initial configuration is given by:

$$\underline{R} = \int_{x_1 = \alpha L/2} \underline{\underline{\sigma}} \cdot \underline{n} da = \int_{X_1 = L/2} \underline{\underline{\sigma}} \cdot (\det \underline{\underline{F}})\underline{\underline{F}}^{-T} \cdot \underline{e}_1 dA = \left(\alpha - \frac{1}{\alpha^2}\right) f\left(\alpha^2 + \frac{2}{\alpha}\right) S_0 \underline{e}_1$$

We can then plot $\frac{R \cdot e_1}{S_0} = \left(\alpha - \frac{1}{\alpha^2}\right) f\left(\alpha^2 + \frac{2}{\alpha}\right)$ as a function of α, see Figure 1.

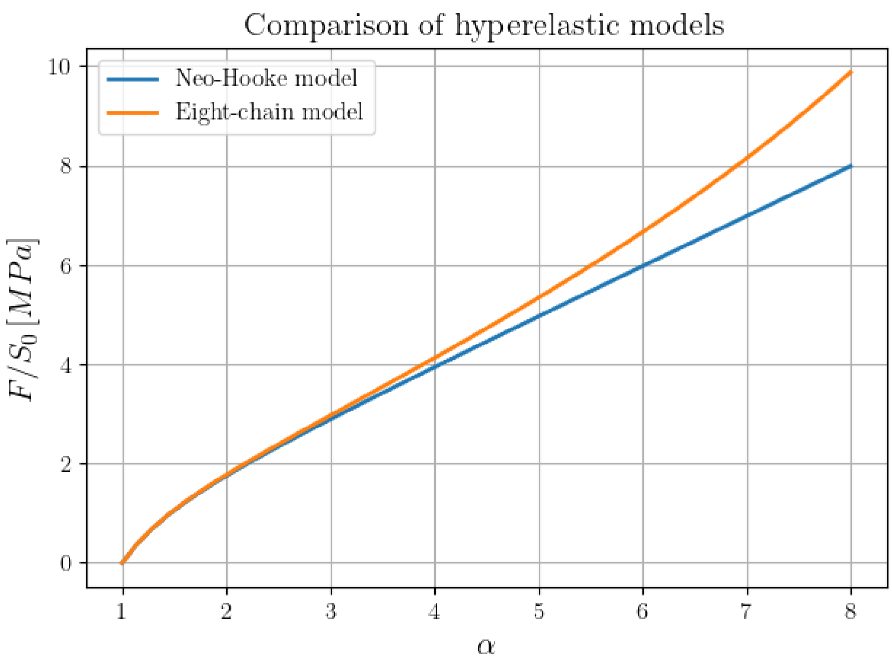

Figure 1: Comparison of the Neo-Hookean and Arruda-Boyce model for a uniaxial tension test ($N = 75$, $\mu = 1\,[MPa]$). The two curves show a divergent at values of stretch higher than 3