Problem Set 7

CIVIL-425: Continuum Mechanics and Applications

17 April 2025

Problem 1: Finite element formulation

Consider a two-dimensional simplex element which, in its deformed configuration, occupies a triangular region Ω whose corners, or 'nodes', are labelled 1,2 and 3 consecutively and counterclockwise. The element lies in the $x_1 - x_2$ plane. Let $\mathbf{x}_a$, $a = 1, 2, 3$, be the coordinates of the nodes and $\mathbf{x} \in \Omega$ a point in the element. Denote $\mathbf{r}_a = \mathbf{x}_a - \mathbf{x}, a = 1, 2, 3$. Define 'shape functions' of the form:

$$N_1(\mathbf{x}) = [\mathbf{e}_3, \mathbf{r}_2, \mathbf{r}_3] / (2A)$$
$$N_2(\mathbf{x}) = [\mathbf{e}_3, \mathbf{r}_3, \mathbf{r}_1] / (2A)$$
$$N_3(\mathbf{x}) = [\mathbf{e}_3, \mathbf{r}_1, \mathbf{r}_2] / (2A)$$

where A is the area of the element and $\mathbf{e}_3$ is the unit vector normal to the plane of the element.

i) Show that $N_1 + N_2 + N_3 = 1$.

Incremental displacements $\mathbf{u}(\mathbf{x})$ and accelerations $\ddot{\mathbf{u}}(\mathbf{x})$ over the element are defined by interpolation,

$$\mathbf{u}(\mathbf{x}) = \sum_{a=1}^{3} \mathbf{u}_a N_a(\mathbf{x})$$
$$\ddot{\mathbf{u}}(\mathbf{x}) = \sum_{a=1}^{3} \ddot{\mathbf{u}}_a N_a(\mathbf{x})$$

where $\mathbf{u}_a$ and $\ddot{\mathbf{u}}_a, a = 1, 2, 3$, are the nodal displacements and accelerations. The Cauchy stresses $\boldsymbol{\sigma}$ are taken to be constant over the element. The element is under the action of body forces $\mathbf{b}$ defined over Ω, and boundary tractions $\bar{\mathbf{t}}$ applied over a part $\partial\Omega_\tau$ of the boundary of the element.

ii) Using the principle of virtual work and restricting virtual displacements to be of the form (2), show that the equation of linear momentum balance for the element reduces to:

$$\mathbf{M}\ddot{\mathbf{u}} + A\mathbf{B}^T\boldsymbol{\sigma} = \mathbf{f}$$

where $\ddot{\mathbf{u}} = \{\ddot{u}_1, \ddot{u}_2, \ddot{u}_3\}$ is an array which collects all nodal accelerations, $\mathbf{M}$ is a 6×6 'mass' matrix, $\boldsymbol{\sigma}$ is redefined as the array $\{\sigma_{11}, \sigma_{22}, \sigma_{12}\}$, $\mathbf{B}$ is a 3×6 matrix, and $\mathbf{f}$ is a 6-dimensional 'nodal force' array. Compute the arrays $\mathbf{M}, \mathbf{B}$ and $\mathbf{f}$ in terms of the nodal coordinates, the mass density $\rho, \mathbf{b}$ and $\bar{\mathbf{t}}$.

bonus: Entropy and concentration equilibrium

Consider two perfectly miscible solid solutions placed in contact through an adiabatic but permeable boundary. The combined system is otherwise isolated from the rest of the world. Suppose that the subsystems have N_1 and N_2 intersticial locations, and, initially, n_1 and n_2 atoms, respectively. We proceed to compute the equilibrium concentrations in the two subsystems.

i) Show that $c_1^{eq} = c_2^{eq} = c^{eq}$ at equilibrium.

ii) Compute the internal entropy $S_{int} \geq 0$ produced in going from the initial conditions to equilibrium and show that $\triangle S_{int} \geq 0$.