Problem Set 6 Solutions

CIVIL-425 : Continuum Mechanics and Applications

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Exercise 1: Conservation of angular momentum

Let $\boldsymbol{G}(E) \in \mathbb{R}^3$ be the total angular momentum contained in subset $E \subset B$, and let $\boldsymbol{M}(E) \in \mathbb{R}^3$ be the resultant moment of all forces acting on E. Then, the principle of conservation of angular momentum states that

$$\frac{d\boldsymbol{G}}{dt}(E) = \boldsymbol{M}(E), \quad \forall E \subset B \tag{1}$$

For simple bodies, the angular momentum contained in a subbody E is

$$\boldsymbol{G}(E) = \int_{\varphi(E)} \boldsymbol{x} \times \rho \boldsymbol{v} dV \tag{2}$$

where $\boldsymbol{x}$ is the spatial position vector. In addition, the resultant torque is

$$\boldsymbol{M}(E) = \int_{\varphi(E)} \boldsymbol{x} \times \rho \boldsymbol{b} dV + \int_{\partial\varphi(E)} \boldsymbol{x} \times t(\boldsymbol{n}) ds \tag{3}$$

Prove that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$:

Solution

Combining Equations 1, 2, 3 we obtain:

$$\frac{d}{dt} \int_{\varphi(E)} \boldsymbol{x} \times (\rho \boldsymbol{v}) dV = \int_{\varphi(E)} \boldsymbol{x} \times \rho \boldsymbol{b} dV + \int_{\partial\varphi(E)} \boldsymbol{x} \times \boldsymbol{t}(\boldsymbol{n}) ds$$

which is an integral statement of angular momentum balance in spatial form. In order to obtain the corresponding local form we assume that mass is conserved, so that

$$\frac{d}{dt} \int_{\varphi(E,t)} \rho f dV = \int_{\varphi(E,t)} \rho \frac{Df}{Dt} dV = \tag{4}$$

applies, and make use of identity $\boldsymbol{t}(\boldsymbol{n}) = \boldsymbol{\sigma}\boldsymbol{n}$ to obtain

$$\int_{\varphi(E)} \rho \frac{D}{Dt}(\boldsymbol{x} \times \boldsymbol{v}) dV = \int_{\varphi(E)} \boldsymbol{x} \times \rho \boldsymbol{b} dV + \int_{\partial\varphi(E)} \boldsymbol{x} \times (\boldsymbol{\sigma}\boldsymbol{n}) ds$$

The left hand side is

$$\int_{\varphi(E)} \rho \frac{D}{Dt}(\boldsymbol{x} \times \boldsymbol{v}) dV = \int_{\varphi(E)} \left(\rho\Big(\frac{D\boldsymbol{x}}{Dt} \times \boldsymbol{v} + \boldsymbol{x} \times \frac{D\boldsymbol{v}}{Dt}\Big) \right) dV = \int_{\varphi(E)} \rho \left(\cancel{\boldsymbol{v} \times \boldsymbol{v}} + \boldsymbol{x} \times \boldsymbol{a}) \right) dV \tag{5}$$

For the second term on the right hand side, an application of the divergence theorem gives:

$$\int_{\partial\varphi(E)} \epsilon_{ijk} x_j \sigma_{kl} n_l ds = \int_{\varphi(E)} \epsilon_{ijk} (x_j \sigma_{kl})_{,l} dV = \int_{\varphi(E)} \epsilon_{ijk} \sigma_{kj} dV + \int_{\varphi(E)} \epsilon_{ijk} x_j (\sigma_{kl})_{,l} dV\,. \tag{6}$$

Using the previously derived expressions the angular momentum balance reads

$$\int_{\varphi(E)} \epsilon_{ijk} \left\{ x_j \left[\rho \left(a_k - b_k \right) - \sigma_{kl,l} \right] + \sigma_{jk} \right\} dv = 0$$

Assuming in addition that linear momentum is conserved simplifies this expression to

$$\int_{\varphi(E)} \epsilon_{ijk} \sigma_{jk} dv = 0$$

But since E is an arbitrary subbody we must have

$$\epsilon_{ijk} \sigma_{jk} = 0, \quad \text{in } \varphi(B)$$

This identity is satisfied if and only if

$$\sigma_{ji} = \sigma_{ij}$$

i. e., if the Cauchy stress tensor is symmetric. We have thus proved the following: For a mass, linear and angular-momentum conserving motion of a simple body the Cauchy stress tensor is symmetric.

Exercise 2: Soap Bubble

A spherical soap bubble contains a certain mass of an ideal monatomic gas. The soap has surface tension $\sigma\,[J/m^2]$ both with the inside and outside fluid, and the outside atmospheric pressure is p_0. The soap shell has mass m_s. In terms of the radius $a(t)$ of the bubble and its time derivative $\dot{a}(t)$, find:

i) The kinetic energy $K(t)$ of the bubble.

ii) The external power $P^E(t)$.

iii) The internal energy rate $\dot{E}(t)$ of the bubble.

iv) Assuming isothermal conditions, from the principle of conservation of energy, derive the amount of heat exchange needed to sustain the bubble.

v) Assuming adiabatic conditions, derive an ordinary differential equation governing the evolution of $a(t)$. Solve that equation for $a(t)$ and plot the result. What kind of motion does the bubble undergo?

Solution

i) We assume that the ideal gas inside the bubble does not contribute to the kinetic energy, but rather to the internal energy of the system. Statistically the particles move in random direction with a velocity which depends on the thermodynamic conditions, but that is generally isotropic. Then, the kinetic energy $K(t)$ of the bubble only includes the contribution of the thin film of soap which is expanding. As the soap film has mass m_s this means that $K(t) = \frac{1}{2} m_s \dot{a}(t)^2$.

ii) The external power is given by the work of the atmospheric pressure to contrast the expansion of the bubble:

$$P^E(t) = F_{atm} \dot{a}(t) = -p_0 \cdot 4\pi a(t)^2 \cdot \dot{a}(t)$$

iii) We know that since there are two free surfaces (the interior and the exterior of the soap bubble) the energy due to the spherical soap bubble is

$$E_b(t) = 8\pi\sigma a^2(t)\,.$$

For an ideal monoatomic gas at constant temperature, the internal energy is simply

$$E_g(t) = \frac{3}{2} N k_b \theta(t) = \frac{3}{2} p_{in} V(t) = const$$

The internal energy rate is thus equal to zero $\dot{E}_g(t) = 0$

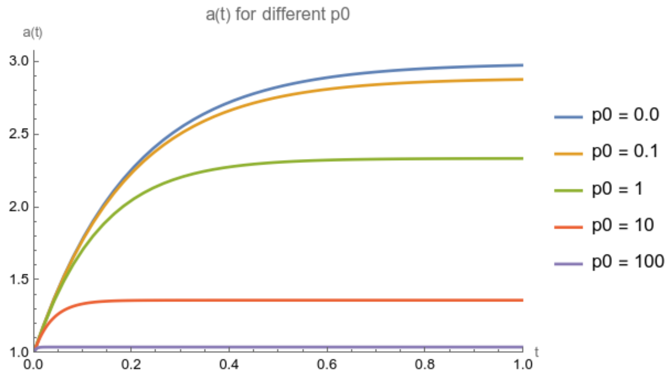

Figure 1: Soap radius dynamics $a(t)$, as a function of external pressure. $m_s = 10$, $a_0 = 1.0$, $\gamma = 1.4$, $\sigma = 1.0$

iv) The principle of conservation of energy is:

$$\dot{E}(t) + \dot{K}(t) = P^E(t) + \dot{Q}(t)\,.$$

Therefore

$$\dot{Q}(t) = m_b a(t)\ddot{a}(t) + 16\pi\sigma a(t)\dot{a}(t) + 4\pi p_0 a^2(t)\dot{a}(t)$$

If the soap expands at a constant speed $\ddot{a} = 0$ heat needs to moves from the outside to the inside to keep the temperature constant.

v) The previously derived formulas up to (iii) still hold, except the one for E_g. Since now the gas is adiabatic we have:

$$\dot{E}_g(t) = p\dot{V} = p_0\left(\frac{M_g}{\rho_0 V}\right)^{\gamma}\dot{V} = p_0\left(\frac{a_0^3}{a^3(t)}\right)^{\gamma} 4\pi a^2(t)\dot{a}(t)$$

Assuming adiabatic conditions we can neglect the term $\dot{Q}(t)$. We are left with the following differential equation:

$$m_s\ddot{a} + p_0\left(\frac{a_0^3}{a^3(t)}\right)^{\gamma} 4\pi a(t)\dot{a}(t) + 16\pi\sigma\dot{a}(t) + 4\pi p_0 a(t)\dot{a}(t) = 0\,.$$

The solution with some specific initial conditions is in plotted in Figure 1.

Bonus: Exercise 3

A cylindrical bar of radius R collides head on with a rigid surface. The material in the bar is incompressible. To obtain approximate solutions, it is assumed that thin slices normal to the axis of the bar at time t and at a distance z from the wall remain flat and circular after the deformation. The deformation of the axis is described by the deformation mapping $z = \phi(Z,t), Z \geq 0$, where Z is the distance to the wall along the axis of the bar in the undeformed configuration. Let $\lambda(Z,t)$ denote the stretch ratio in the axial direction, S_0 the undeformed cross-sectional area, $S(z,t)$ the deformed cross-sectional area, $V(Z,t)$, $v(z,t)$, $A(Z,t)$ and $a(z,t)$ the material and spatial velocity and acceleration fields over the axis, respectively.

- Knowing that the bar is free of applied loads, write the virtual work expression in material and spatial forms. Consider states of uniaxial stress, and let $\sigma \equiv \sigma_{33}$ and $P \equiv P_{33}$ denote the axial components of the Cauchy and Piola-Kirchhoff stress tensors, respectively. How are σ and P related? To obtain an axial equation of motion, consider virtual displacements of the form $\delta\phi_1 = \delta\phi_2 = 0$, $\delta\phi_3 = \delta\phi$. If the bar is of infinite length and its velocity is prescribed at infinity, what essential boundary conditions must $\delta\phi$ satisfy? Enforcing the virtual work principle for all variations of this type, obtain axial equations of motion in material and spatial form.

- Obtain the same equations of motions directly by establishing the dynamic equilibrium of thin slices of the bar in its undeformed and deformed configurations.

Solution

We start with writing the principle of virtual work in material and spatial forms. Neglecting the body forces and using the notation from above

$$\int_{B_o} P_{ij}\delta\phi_{i,j}dV_o + \int_{B_o} \rho_o A_i \delta\phi_i dV_o = 0\,, \tag{7}$$

in material form and

$$\int_{B} \sigma_{ij}\delta\phi_{i,j}dV + \int_{B} \rho a_i \delta\phi_i dV = 0\,, \tag{8}$$

in spatial.

Now consider uniaxial stress loading, such that the only non-zero components of Piola-Kirchoff and Cauchy stress tensors are P_{33} and σ_{33}, respectively. The connection between initial and current configurations for stresses is given as $\boldsymbol{P} = J\boldsymbol{\sigma}F^{-1}$, where F is deformation gradient and J is the Jacobian of the deformation.

To derive both F and J, we recall the assumption that thin slices normal to the axis of the bar remain flat and circular after the deformation. Deformation is thus pure expansion

$$x = \lambda_L X \tag{9}$$

$$y = \lambda_L Y \tag{10}$$

$$z = \lambda Z \tag{11}$$

recall that here $\lambda(Z,t)$ is stretch ratio in the axial direction whereas $\lambda_L(Z,t)$ is stretch ratio in the cross-section plane.

This allows to write deformation gradient

$$F = \begin{pmatrix} \lambda_L & 0 & 0 \\ 0 & \lambda_L & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

The material of the bar is incompressible, which essentially means that the bar can change the shape but its overall volume remains constant. This requires $J \equiv 1$ or $\lambda_L = \lambda^{-1/2}$. We can rewrite F as

$$F = \begin{pmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

which gives the following relation between the only non-zero components of stress $P_{33} = \dfrac{\sigma_{33}}{\lambda}$.

The next step is to obtain equation of motion using the principle of virtual work in both initial and current configurations for an infinite bar with virtual displacements $\delta\phi_1 = \delta\phi_2 = 0$ and $\delta\phi_3 = \delta\phi(z)$.

In initial configuration we write

$$\int_{B_o} P_{33}\frac{d\delta\phi}{dz}dV_o + \int_{B_o} \rho_o A_3 \delta\phi_3 dV_o = \tag{12}$$

$$= \int_0^{L_o} P_{33}\frac{d\delta\phi}{dz}S_o dz + \int_0^{L_o} \rho_o A_3 S_o \delta\phi dz = 0 \tag{13}$$

where S_o is bar cross-section in the initial configuration.

Remember that virtual work principle is valid for all admissible displacement fields. This requires that at the part of the boundary where the tractions are not prescribed, the virtual displacements $\delta\phi$ must be zero. In our problem of infinite bar $L_o \to \infty$, the essential boundary conditions for $\phi(z)$ are defined as $\delta\phi(0) = 0$ and $\delta\phi(\infty) = 0$.

$$\int_0^{+\infty} P_{33}\frac{d\delta\phi}{dz}S_o dz + \int_0^{+\infty} \rho_o A_3 S_o \delta\phi dz = 0\,, \tag{14}$$

Integrating previous equation by parts and taking boundary conditions into account we obtain

$$\int_0^{+\infty} \left(-(P_{33}S_o)' + \rho_o A_3 S_o\right) \delta\phi dz = 0\,, \tag{15}$$

since it is valid for any admissible $\delta\phi$, the following must hold (equation of motion in material form)

$$-P'_{33} + \rho_o A_3 = 0 \tag{16}$$

The same is applied to obtain equation of motion in spatial form

$$\int_0^{+\infty} \sigma_{33} \frac{d\delta\phi}{dz} S dz + \int_0^{+\infty} \rho A_3 S \delta\phi dz = 0 \tag{17}$$

where $S = S(z,t)$ is bar cross-section which now depends on vertical coordinate.
Integrating by parts yields

$$\int_0^{+\infty} \left(-(\sigma_{33}S)' + \rho a_3 S\right) \delta\phi dz = 0 \tag{18}$$

Finally, equation of motion in spatial form is

$$-(\sigma_{33}S)' + \rho a_3 S = 0 \tag{19}$$

The same can be obtained considering thin slices of the bar. The forces acting on a thin slice in deformed configuration are $(\sigma_{33} + d\sigma_{33})(S + dS)$ and $-\sigma_{33}S$ which gives

$$(\sigma_{33} + d\sigma_{33})(S + dS) - \sigma_{33}S = \rho S dz a_3 \;\; => \sigma_{33}S' + \sigma'_{33}S = \rho S a_3 \tag{20}$$

Now in undeformed configuration

$$(P_{33} + dP_{33})S_o - P_{33}S_o = \rho_o S_o dZ A_3 \;\; => P'_{33} = \rho_o A_3 \tag{21}$$