Problem Set 6

CIVIL-425 : Continuum Mechanics and Applications

10 April 2025

Exercise 1: Conservation of angular momentum

Let $\boldsymbol{G}(E) \in \mathbb{R}^3$ be the total angular momentum contained in subset $E \subset B$, and let $\boldsymbol{M}(E) \in \mathbb{R}^3$ be the resultant moment of all forces acting on E. Then, the principle of conservation of angular momentum states that

$$\frac{d\boldsymbol{G}}{dt}(E) = \boldsymbol{M}(E), \quad \forall E \subset B \tag{1}$$

For simple bodies, the angular momentum contained in a subbody E is

$$\boldsymbol{G}(E) = \int_{\varphi(E)} \boldsymbol{x} \times \rho \boldsymbol{v} dV \tag{2}$$

where $\boldsymbol{x}$ is the spatial position vector. In addition, the resultant torque is

$$\boldsymbol{M}(E) = \int_{\varphi(E)} \boldsymbol{x} \times \rho \boldsymbol{b} dV + \int_{\partial\varphi(E)} \boldsymbol{x} \times t(\boldsymbol{n}) ds \tag{3}$$

Prove that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$:

Exercise 2: Soap Bubble

A spherical soap bubble contains a certain mass of an ideal monatomic gas. The soap has surface tension $\sigma\,[J/m^2]$ both with the inside and outside fluid, and the outside atmospheric pressure is p_0. The soap shell has mass m_s. In terms of the radius $a(t)$ of the bubble and its time derivative $\dot{a}(t)$, find:

i) The kinetic energy $K(t)$ of the bubble.

ii) The external power $P^E(t)$.

iii) The internal energy rate $\dot{E}(t)$ of the bubble.

iv) Assuming isothermal conditions, from the principle of conservation of energy, derive the amount of heat exchange needed to sustain the bubble.

v) Assuming adiabatic conditions, derive an ordinary differential equation governing the evolution of $a(t)$. Solve that equation for $a(t)$ and plot the result.

Bonus: Exercise 3

A cylindrical bar of radius R collides head on with a rigid surface. The material in the bar is incompressible. To obtain approximate solutions, it is assumed that thin slices normal to the axis of the bar at time t and at a distance z from the wall remain flat and circular after the deformation. The deformation of the axis is described by the deformation mapping $z = \phi(Z, t), Z \geq 0$, where Z is the distance to the wall along the axis of the bar in the undeformed configuration. Let $\lambda(Z, t)$ denote the stretch ratio in the axial direction, S_0 the undeformed cross-sectional area, $S(z, t)$ the deformed cross-sectional area, $V(Z, t)$, $v(z, t)$, $A(Z, t)$ and $a(z, t)$ the material and spatial velocity and acceleration fields over the axis, respectively.

- Knowing that the bar is free of applied loads, write the virtual work expression in material and spatial forms. Consider states of uniaxial stress, and let $\sigma \equiv \sigma_{33}$ and $P \equiv P_{33}$ denote the axial components of the Cauchy and Piola-Kirchhoff stress tensors, respectively. How are σ and P related? To obtain an axial equation of motion, consider virtual displacements of the form $\delta\phi_1 = \delta\phi_2 = 0$, $\delta\phi_3 = \delta\phi$. If the bar is of infinite length and its velocity is prescribed at infinity, what essential boundary conditions must $\delta\phi$ satisfy? Enforcing the virtual work principle for all variations of this type, obtain axial equations of motion in material and spatial form.

- Obtain the same equations of motions directly by establishing the dynamic equilibrium of thin slices of the bar in its undeformed and deformed configurations.