Problem Set 5 Solutions

CIVIL-425: Continuum Mechanics and Applications

3 April 2025

Exercise 1

Using the principle of conservation of linear momentum, solve the following problems:

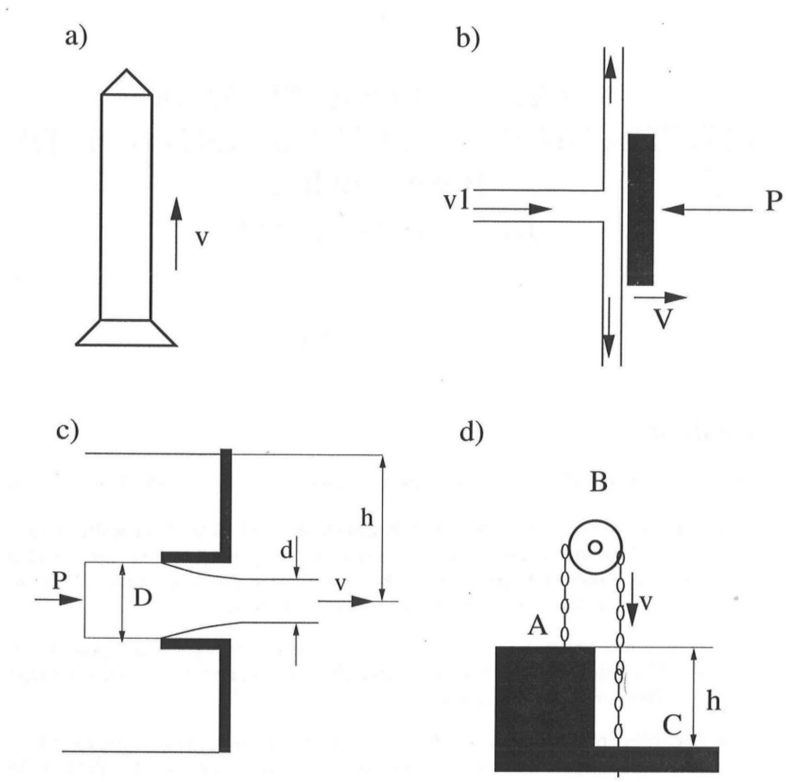

Figure 1: Illustration of the different configurations

a. A rocket of initial mass m_0 (including shell and fuel) is fired vertically at time $t = 0$. The fuel is consumed at a constant rate $q = \frac{dm}{dt}$ and is expelled at a constant speed u relative to the rocket. Derive an expression for the velocity of the rocket at time t neglecting the resistance of air.

Solution: Since we are not interested in the details of fluid motion inside the rocket, nor on elastic deformation of the latter, we can set our control volume Ω around the rocket and we highlight the portion of the contour S' where the nozzle is. Ambient pressure is to be ignored. Since the rocket is moving, we would like out control volume to follow its motion. For a moving control volume, with velocity $\mathbf{w} = (0, 0, \mathrm{v})$, mass conservation yields:

$$\frac{dm}{dt}(\Omega(t), t) = \int_{\partial\Omega(t)} \rho(\mathbf{w} - \mathbf{u}) \cdot \mathbf{n} dS = \int_{S'} -\rho u dS = q \qquad \Longrightarrow m(t) = m_0 - qt \tag{1}$$

where we used the fact that the fluid only goes out through the nozzle and that u is already the relative velocity of the fluid with respect to the moving control volume, whereas $\mathbf{u}$ is the fluid velocity from a fixed observer. Next, the balance of linear momentum reads:

$$\frac{d\mathbf{L}}{dt}(\Omega(t),t) = -\int_{\partial\Omega(t)} \rho\mathbf{u}(\mathbf{u}-\mathbf{w})\cdot\mathbf{n}dS + \int_{\partial\Omega(t)} \boldsymbol{\sigma}\mathbf{n}dS + \int_{\Omega(t)} \rho\mathbf{b}dV \tag{2}$$

$$m(t)\frac{d\mathrm{v}}{dt} + \mathrm{v}(t)\frac{dm}{dt} = \int_{S'} \rho(u-\mathrm{v})udS + 0 + \int_{\Omega(t)} -\rho g dV = \tag{3}$$

$$m(t)\frac{d\mathrm{v}}{dt} + \mathrm{v}(t)\frac{dm}{dt} = (u-\mathrm{v})\underbrace{\int_{S'} \rho u dS}_{-q} - m(t)g \tag{4}$$

$$m(t)\frac{d\mathrm{v}}{dt} = -u\frac{dm}{dt} - m(t)g\,. \tag{5}$$

We assumed that $(u-\mathrm{v})$ is constant over S'.

Integrating over time and applying the initial condition $\mathrm{v}(0) = \mathrm{v}_0$ we obtain the rocket equation:

$$\mathrm{v}(t) = -gt + u\ln\left(\frac{m_0}{m_0 - qt}\right) \tag{6}$$

The previous equation is valid only for times $t < m_0/q$.

b. A stream of water of cross-sectional area A and velocity v_1 strikes a plate which is held motionless by a force P. Determine the magnitude of P knowing that $A = 500mm^2$ and $v_1 = 25m/s$.

Solution: Considering that the stream flow is in steady-state and assuming (a) a prismatic control volume enclosing the arriving flow is defined from a section far away from the wall up to the point where the wall starts, (b) that the flow changes directions (from horizontal, towards the wall, to vertical, over the wall) out of this control volume, we only need to apply balance of linear momentum in integral form:

$$\cancel{\frac{d\mathbf{L}}{dt}} = -\int_{\partial\Omega} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dS + \int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n}dS + \cancel{\int_{\Omega} \rho\mathbf{b}dV} \tag{7}$$

the first term is zero since the system has reached steady state, and the body force is also assumed to be zero. Projecting along the direction of incoming flow, we obtain:

$$\int_{\partial\Omega} \rho v_1^2(-1)dS = -\int_{\partial\Omega} pdS = \int_A pdS \tag{8}$$

the velocity at the right end of the control volume (termed S_2) is zero by boundary condition while the pressure at the left end (termed S_1) is neglected. Assume that the left section of the jet has an area A and that the density and velocities are uniform (the density assumption is a classic one when working with liquids) there (so that the velocity can be taken out of the integral) to get

$$\rho v^2 A = \int_{\partial\Omega} pdS = P$$

from the assumptions that we made at the beginning (regarding the direction of the flow becoming purely vertical out of the control volume), the second integral is the reaction force exerted by the wall over the fluid. Notice that we could have reached the same result with the Bernoulli principle.

c. A circular reentrant orifice (also called Borda's mouthpiece) of diameter D is placed at a depth h below the surface of a tank. Knowing that the speed of the issuing stream is $v = \sqrt{2gh}$ and assuming that the speed of approach v_1 is zero, show that the diameter of the stream is $d = D\sqrt{2}$.

Problem 3

A circular reentrant orifice (also called Borda's mouthpiece) of diameter D is placed at a depth h below the surface of a tank. Knowing that the speed of the issuing stream is $v = \sqrt{2gh}$ and assuming that the speed of approach v_1

is zero, show that the diameter of the stream is $d = D\sqrt{2}$. (Hint: Consider the section of water indicated, and note that P is equal to the pressure at a depth h multiplied by the area of the orifice.)

In this case as well, we need to level assumption of steady-state, along with a few others: we assume that the diameter of the intake, D, is much smaller than the vertical distance between its center and the free surface ($D/h \ll 1$), we also take the velocity at the intake to be negligible.

Now, let us work with balance of linear momentum in integral form once again (the time derivatives are omitted as we assumed steady-state regime):

$$\int_{\partial\Omega} \rho \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n})dS = -\int_{\partial\Omega} p\boldsymbol{n}dS + \int_{\Omega} \rho\boldsymbol{b}dV$$

Let us take Ω to be a control volume enclosing the output jet as well as extending into the deposit deep enough as for the intake section to satisfy the negligible-velocities condition mentioned earlier. It is also assumed that the velocities increase in magnitude within the volume yet they always remain horizontal over its side contours (this assumption will come in handy later) while being uniform at the output section (where $v = \sqrt{2gh}$ is known).

Under these assumptions, the horizontal projection of the prior equation yields

$$\int_{S_{\text{out}}} \rho v^2 \cdot \boldsymbol{n}dS = \underbrace{\rho v^2}_{=2\rho g h} \frac{\pi d^2}{4} = \int_{S_{\text{in}}} pdS$$

assume the pressure being uniform at the intake (since changes over distance D are tiny compared to $\rho g h$ due to the assumption $D/h \ll 1$) and neglecting the ambiance pressure

$$= (\rho g h)\frac{\pi D^2}{4}$$

from this equality the relation between diameters is readily obtained: $d = D/\sqrt{2}$.