Problem Set 4

CIVIL-425: Continuum Mechanics and Applications

27 March 2025

Exercise 1, Principal directions and stretches

Consider a local deformation gradient $\mathbf{F}$ and the corresponding left and right Cauchy-Green deformation tensors $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, respectively. Let the spectral decomposition of the right Cauchy-Green deformation tensor be:

$$\mathbf{C} = \sum_{\alpha=1}^{3} c_\alpha \mathbf{N}_\alpha \otimes \mathbf{N}_\alpha$$

here c_α and $\mathbf{N}_\alpha, \alpha = 1, 2, 3$, are the eigenvalues and eigenvectors of C, i. e.

$$(\mathbf{C} - c_\alpha \mathbf{I}) \cdot \mathbf{N}_\alpha = 0, \quad \|\mathbf{N}_\alpha\| = 1, \quad \alpha = 1, 2, 3$$

Let $\lambda_\alpha = \sqrt{c_\alpha}$ be the principal stretches; $\mathbf{F} = \mathbf{V}\mathbf{R} = \mathbf{R}\mathbf{U}$ the polar decompositions of $\mathbf{F}$, the $\mathbf{R} \in SO(3), \mathbf{U} = \mathbf{U}^T = \sqrt{\mathbf{C}}, \mathbf{V} = \mathbf{V}^T = \sqrt{\mathbf{B}}$; and set $\mathbf{n}_\alpha = \mathbf{R}\mathbf{N}_\alpha$.

i) Prove the identities:

i.1) $\mathbf{n}_\alpha = \lambda_\alpha^{-1}\mathbf{F}\mathbf{N}_\alpha$.

We have $\mathbf{U} = \sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{N}_\alpha \otimes \mathbf{N}_\alpha$. Hence

$$\mathbf{F}\mathbf{N}_\alpha = (\mathbf{R}\mathbf{U})\mathbf{N}_\alpha = \mathbf{R}\left(\sum_{\beta=1}^{3} \lambda_\beta \mathbf{N}_\beta \otimes \mathbf{N}_\beta\right)\mathbf{N}_\alpha = \lambda_\alpha \mathbf{R}\mathbf{N}_\alpha = \lambda_\alpha \mathbf{n}_\alpha\,,$$

where we used the identity $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{a} \otimes \mathbf{c})\mathbf{b}$, and the fact $\mathbf{N}_\alpha \otimes \mathbf{N}_\beta = \boldsymbol{\delta}_{\alpha\beta}$, since $||\mathbf{N}_i|| = 1$.

i.2) $\mathbf{F} = \sum_\alpha \lambda_\alpha \mathbf{n}_\alpha \otimes \mathbf{N}_\alpha$.

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{R}\left(\sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{N}_\alpha \otimes \mathbf{N}_\alpha\right) = \sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{R}\mathbf{N}_\alpha \otimes \mathbf{N}_\alpha = \sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{n}_\alpha \otimes \mathbf{N}_\alpha$$

i.3) $\mathbf{R} = \sum_\alpha \mathbf{n}_\alpha \otimes \mathbf{N}_\alpha$.

$$\mathbf{F} = \sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{n}_\alpha \otimes \mathbf{N}_\alpha = \left(\sum_{\alpha=1}^{3} \mathbf{n}_\alpha \otimes \mathbf{N}_\alpha\right)\left(\sum_{\beta=1}^{3} \lambda_\beta \mathbf{N}_\beta \otimes \mathbf{N}_\beta\right) = \mathrm{R}\mathbf{U}$$

Exercise 3, Linearization

Linearize the following kinematic relations with respect to a small displacement $\mathbf{u}$ field superposed on the spatial configuration:

ii) The volumetric-deviatoric decomposition of the deformation gradients is:

$$\mathbf{F} = \mathbf{F}^{\text{vol}}\mathbf{F}^{\text{dev}}, \quad \mathbf{F}^{\text{vol}} \equiv J^{1/3}\mathbf{I}, \quad \mathbf{F}^{\text{dev}} \equiv J^{-1/3}\mathbf{F}$$

where J is the Jacobian of the deformation and $\mathbf{I}$ is the identity tensor.

Keeping only the linear terms in $||\nabla\mathbf{u}||$ and using the Taylor's expansion for he volumetric part we obtain:

$$J = 1 + \text{tr}(\nabla\mathbf{u}) + \mathcal{O}(||\nabla\mathbf{u}||^2) \tag{1}$$

$$J^{1/3} \approx (1 + \text{tr}(\nabla\mathbf{u}))^{1/3} \approx 1 + \frac{1}{3}\text{tr}(\nabla\mathbf{u}) \tag{2}$$

$$F^{vol} = \mathbf{I} + \frac{1}{3}\text{tr}(\nabla\mathbf{u})\mathbf{I}\,. \tag{3}$$

Analogously for the deviatoric part:

$$J = 1 + \text{tr}(\nabla\mathbf{u}) + \mathcal{O}(||\nabla\mathbf{u}||^2) \tag{4}$$

$$J^{-1/3} \approx 1 - \frac{1}{3}\text{tr}(\nabla\mathbf{u}) \tag{5}$$

$$\begin{aligned} F^{dev} &= J^{-1/3}\mathbf{F} \approx \left(1 - \frac{1}{3}\text{tr}(\nabla\mathbf{u})\right)(\mathbf{I} + \nabla\mathbf{u}) = \\ &\mathbf{I} + \nabla\mathbf{u} - \frac{1}{3}\text{tr}(\nabla\mathbf{u})\mathbf{I} - \frac{1}{3}\text{tr}(\nabla\mathbf{u})\nabla\mathbf{u} \approx \mathbf{I} + \nabla\mathbf{u} - \frac{1}{3}\text{tr}(\nabla\mathbf{u})\mathbf{I} \end{aligned} \tag{6}$$