Problem Set 3

CIVIL-425: Continuum Mechanics and Applications

14 March 2024

Exercise 1, compatibility conditions

Consider a cylindrical solid referred to an orthonormal Cartesian reference frame $\{X_1,\, X_2,\, X_3\}$. Let the axis of the solid be aligned with the X_3-direction and let its normal cross-section occupy a region Ω in the $X_1 - X_2$ plane of boundary $\partial\Omega$. Consider deformations which result in the deformation gradients of the form:

$$[\mathbf{F}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma_1 & \gamma_2 & 1 \end{pmatrix}, \tag{1}$$

where γ_1 and γ_2 are functions of (X_1, X_2).

1. What condition do (γ_1, γ_2) need to satifisfy to verify compatibility?

   We know the compatibility condition is:

   $$\mathbf{F}_{iI,J}\epsilon_{JIK} = 0$$

   This condition is identically satisfied for $i = 1, 2$. For $i = 3$:

   $$\begin{aligned} K = 1 &\Rightarrow F_{\cdot 32,3} - F_{\cdot 33,2} = 0 \\ K = 2 &\Rightarrow F_{\cdot 31,3} - F_{\cdot 33,1} = 0 \\ K = 3 &\Rightarrow F_{\cdot 31,2} - F_{\cdot 32,1} = 0 \end{aligned}$$

   The first two are identically satisfied, the third says:

   $$\gamma_{1,2} - \gamma_{2,1} = 0$$

2. Assuming that the compatibility conditions derived in the previously are satisfied, express the deformation mapping in terms of (γ_1, γ_2) and show that it is of the same form as of 1 for some $w(X_1, X_2)$.

   Assume $\mathbf{0} \in B_0$ and $\phi(\mathbf{0}) = \mathbf{0}$. Then:

   $$\begin{aligned} \phi_1(\mathbf{P}) &= \phi_1(\mathbf{0}) + \int_\Gamma dX_1 = X_1 \\ \phi_2(\mathbf{P}) &= \phi_2(\mathbf{0}) + \int_\Gamma dX_2 = X_2 \\ \phi_3(\mathbf{P}) &= \phi_3(\mathbf{0}) + \int_\Gamma (\gamma_1 dX_1 + \gamma_2 dX_2 + dX_3) = X_3 + w\,(X_1, X_2) \end{aligned}$$

   For $\phi(\mathrm{P})$ to be independent of the choice of Γ must have:

   $$\oint \mathrm{F}_{iJ} dX_J = 0 \text{ for all closed } \Gamma$$

   It suffices to consider $i = 3 \Rightarrow$

$$\oint_\Gamma (\gamma_1 dX_1 + \gamma_2 dX_2 + dX_3) = 0$$

By Stokes' theorem, this is equivalent to:

$$\int_\Sigma (\gamma_{2,1} - \gamma_{1,2}) N_3 dA = \int_\Sigma' (\gamma_{2,1} - \gamma_{1,2}) dA' = 0$$

which is satisfied if $\gamma_{2,1} - \gamma_{1,2} = 0$

3. Consider the deformation defined by:

$$\gamma_1 = -\frac{\sin\theta}{r}, \gamma_2 = \frac{\cos\theta}{r} \tag{2}$$

where (r, θ) are polar coordinates centered at the origin, with θ measured from X_1, so that:

$$X_1 = r\cos\theta, X_2 = r\sin\theta. \tag{3}$$

Verify that the deformation field in eq. 2 satisfies the compatibility condition derived in the first point everywhere in the $X_1 - X_2$ plane excluding the origin.

Recall that

$$\frac{\partial r}{\partial X_1} = \cos\theta$$
$$\frac{\partial r}{\partial X_2} = \sin\theta$$
$$\frac{\partial \theta}{\partial X_1} = \frac{-\sin\theta}{r}$$
$$\frac{\partial \theta}{\partial X_2} = \frac{\cos\theta}{r}$$

$$\gamma_1 = -\frac{\sin\theta}{r}, \gamma_2 = \frac{\cos\theta}{r} \Rightarrow$$
$$\gamma_{1,2} = \frac{\partial \gamma_1}{\partial r}\frac{\partial r}{\partial X_2} + \frac{\partial \gamma_1}{\partial \theta}\frac{\partial \theta}{\partial X_2} = \frac{\sin\theta}{r^2}\sin\theta - \frac{\cos\theta\cos\theta}{r^2}$$
$$\gamma_{2,1} = \frac{\partial \gamma_2}{\partial r}\frac{\partial r}{\partial X_1} + \frac{\partial \gamma_2}{\partial \theta}\frac{\partial \theta}{\partial X_1} = -\frac{\cos\theta}{r^2}\cos\theta + \frac{\sin\theta\sin\theta}{r^2}$$

$$\Rightarrow \gamma_{2,1} - \gamma_{1,2} = 0 \text{ except at } r = 0, \text{ where } \gamma_{2,1}, \gamma_{1,2} \text{ are undefined.}$$

4. Compute the displacement field $w(X_1, X_2)$ which gives rise to the deformation 2, assuming that $w = 0$ on the positive half-line $X_2 = 0, X_1 > 0$. Sketch the deformed shape of an initially circular region centered at the origin and contained in the $X_1 - X_2$ plane.

Choose path: $X_1 = a\cos\alpha; X_2 = a\sin\alpha; 0 \le \alpha \le \theta$. Integrate:

$$w(\mathbf{P}) = \int_\Gamma (\gamma_1 dX_1 + \gamma_2 dX_2) = \int_0^\theta \left(\frac{\sin\alpha}{a} a\sin\alpha + \frac{\cos\alpha}{a} a\cos\alpha\right) d\alpha = \int_0^\theta d\alpha$$
$$\Rightarrow w = \theta$$

5. Consider the Burgers circuit $r = const, 0 < \theta < 2\pi$ centered at the origin. Does the same circuit close in the deformed configuration ? Interpret the results in term of dislocations.

Clearly, Γ does not close in deformed configuration:

$$\oint_\Gamma (\gamma_1 dX_1 + \gamma_2 dX_2) = w(2\pi) - w(0).$$

Also:

$$\oint_\Gamma \mathbf{F}_{iJ} dX_J = 2\pi\delta_{i3} \equiv b_i,$$

any circuit encircling the X_3-axis fails to close by $b_i \implies$ The X_3-axis contains a (screw) dislocation with Burgers vector $\mathbf{b} = 2\pi\mathbf{e}_3$ (see figure 1 for a depiction).

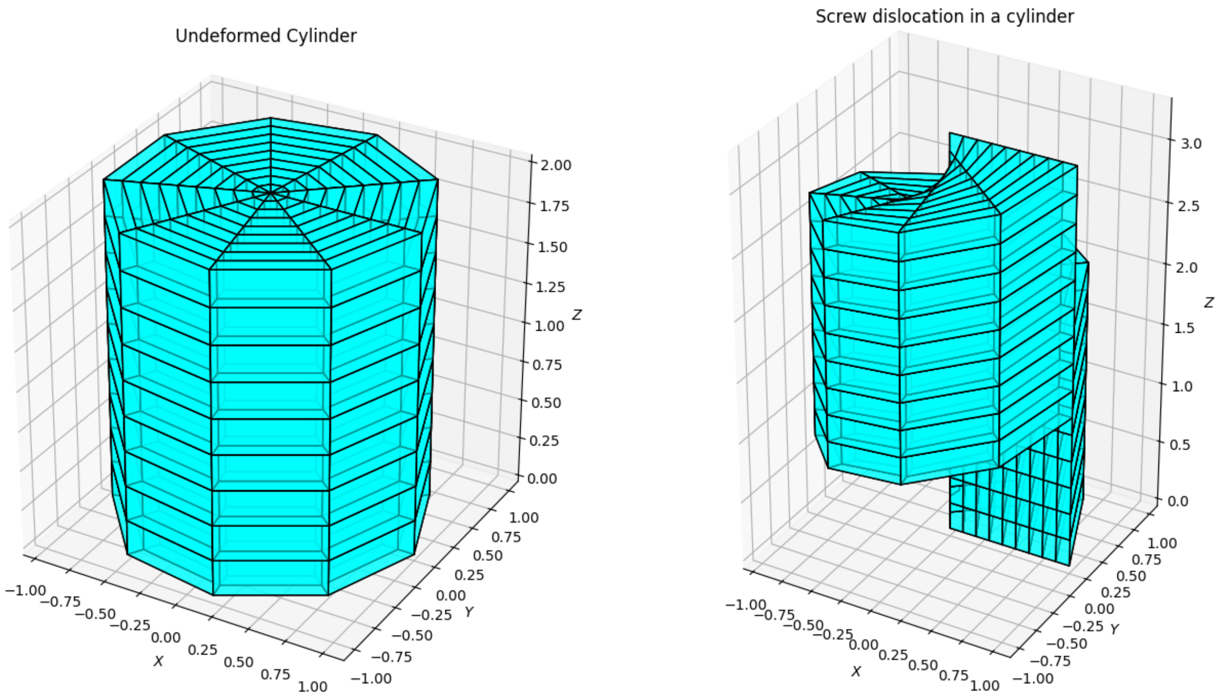

Figure 1: Undeformed cylinder (left) and after deformation(right) skew dislocation of a cylinder

Exercise 2, Orthogonal cutting

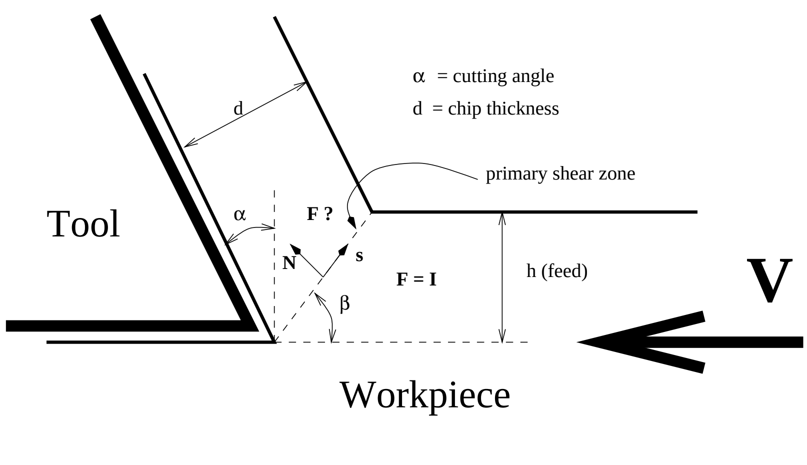

Figure 2: Geometry of orthogonal cutting.

A rigid tool cuts a layer from a metal workpiece with its edge perpendicular to the direction of motion (orthogonal machining). The feed is h, the rake angle is α and the angle of the primary shear zone to the surface of the workpiece is β. The thickness of the resulting chip is d. The material is assumed to be incompressible and undeformed prior to cutting.

1. Calculate the thickness of the chip as a function of h, α and β, by considering conservation of mass at the primary shear zone.
   Taking into account conservation of mass at the primary shear zone we get

$$d \sin\beta = h\cos(\alpha-\beta) d = \frac{\cos(\alpha-\beta)}{\sin\beta} h$$

2. Calculate the deformation $\boldsymbol{F}$ in the chip as a function of α and β. In particular, calculate the shear $\gamma = \boldsymbol{s}\cdot(\boldsymbol{F}\boldsymbol{N})$ through the primary shear zone, where $\boldsymbol{s}$ and $\boldsymbol{N}$ are the unit vectors tangent and normal to the primary shear zone, respectively. *Hint: consider there is a discontinuous jump in the deformation field normal to the primary shear zone*
   To calculate $\mathbf{F}$ in the chip we use weak (Hadamard) compatibility, considering a shock at the primary shear zone:
   $$\mathrm{F} - \mathbf{I} = \mathbf{a} \otimes \mathrm{N}$$
   for some a. Assume incompressibility: $\det(\mathbf{F}) = 1$
   $$\begin{aligned} 1 = \det(\mathbf{F}) &= \det(\mathbf{I} + \mathbf{a} \otimes \mathbf{N}) \\ &= \det(\mathbf{I}) + \det(\mathbf{a} \otimes \mathbf{N}) + \mathrm{tr}(\mathbf{a} \otimes \mathbf{N}) \\ &= 1 + \mathbf{a} \cdot \mathbf{N} \end{aligned}$$
   Since:
   $$\mathbf{N} = \left\{ \begin{array}{c} -\sin\beta \\ \cos\beta \end{array} \right\} \Rightarrow \mathbf{a} = \gamma \left\{ \begin{array}{c} \cos\beta \\ \sin\beta \end{array} \right\}$$
   for some γ. To determine γ, note that direction $(-1, 0)$ deforms into direction $-(\sin\alpha, \cos\alpha)$ tangent to tool-chip interface. This requires:
   $$F \left\{ \begin{array}{c} -1 \\ 0 \end{array} \right\} = \lambda \left\{ \begin{array}{c} -\sin\alpha \\ \cos\alpha \end{array} \right\}$$
   for some λ.
   $$\left[I + \gamma \left\{ \begin{array}{c} \cos\beta \\ \sin\beta \end{array} \right\} (-\sin\beta, \cos\beta) \right] \left\{ \begin{array}{c} -1 \\ 0 \end{array} \right\} = \lambda \left\{ \begin{array}{c} -\sin\alpha \\ \cos\alpha \end{array} \right\}$$
   $$\begin{gathered} -1 + \gamma \sin\beta \cos\beta = -\lambda \sin\alpha \\ \gamma \sin^2\beta = \lambda \cos\alpha \\ \Rightarrow \frac{-1 + \gamma \sin\beta\cos\beta}{\gamma \sin^2\beta} = -\tan\alpha \\ \Rightarrow \gamma = \frac{1}{\sin\beta\cos\beta + \sin^2\beta\tan\alpha} = \frac{\cos\alpha}{\sin\beta\cos(\alpha - \beta)} \end{gathered}$$

3. Suppose that the material deforms so as to minimize the shear deformation γ. Compute β from this condition. Calculate the corresponding γ and chip thickness.
   Suppose β is such that γ (and hence plastic dissipation) is minimized. Then:
   $$\begin{gathered} \frac{\partial\gamma}{\partial\beta} = -\frac{\cos\alpha\cos(\alpha - 2\beta)}{\sin^2\beta\cos^2(\alpha - \beta)} = 0 \\ \Rightarrow 2\beta - \alpha = \frac{\pi}{2} \Rightarrow \beta_{\min} = \frac{\pi}{4} + \frac{\alpha}{2}; \gamma_{\min} = \frac{2\cos\alpha}{1 + \sin\alpha} \end{gathered}$$
   For $\alpha = 0 \Rightarrow \gamma_{\min} = 2$
   $$\lambda = \frac{\sin^2\beta}{\cos\alpha}\gamma = \frac{\sin\beta}{\cos(\alpha - \beta)} = \frac{h}{d}$$