Problem Set 2

CIVIL-425: Continuum Mechanics and Applications

6 March 2025

Exercise 1

Given the deformation mapping

$$\boldsymbol{\varphi}(\boldsymbol{X}) = [x_1,\, x_2,\, x_3]^\top = [X_1 + \alpha X_2^2 t,\, (1+\beta t)X_2,\, X_3]^\top \tag{1}$$

, where α, $\beta \in \mathbb{R}$, compute Lagrangian velocity and deformation fields.

Exercise 2

Consider a deformation mapping of the type:

$$x_1 = X_1\,, \tag{2}$$
$$x_2 = X_2\,, \tag{3}$$
$$x_3 = X_3 + w(X_1, X_2)\ \ , \tag{4}$$

where $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This mapping, when restricted to the plane $X_3 = 0$, represents moderate deformation of a membrane occupying a domain $\Omega \subset \mathbb{R}^2$.

1. Using the Piola transformation, write an expression for the element of oriented membrane area on the deformed configuration.
2. Write an integral expression for the area of the deformed membrane.
3. Show that, for small deflections, i.e., for $\omega \to 0$, the deformed area of the membrane may be approximated as

$$a = A + \int_\Omega \frac{1}{2}(w_{,1}^2 + w_{,2}^2) dX_1 dX_2\,. \tag{5}$$

Exercise 3

Consider a cylindrical solid referred to an orthonormal Cartesian reference frame $\{X_1,\, X_2,\, X_3\}$. Let the axis of the solid be aligned with the X_3-direction and let its normal cross-section occupy a region Ω in the $X_1 - X_2$ plane of boundary $\partial\Omega$. An *anti-plane shear* deformation of the solid can be defined as one for which the deformation mapping is of the form:

$$\boldsymbol{\varphi}(\boldsymbol{X}) = [x_1,\, x_2,\, x_3]^\top = [X_1,\, X_2,\, X_3 + w(X_1, X_2)]^\top\,, \tag{6}$$

where, in this definition, the spatial and material reference frames are taken to coincide, and the function w is defined over Ω.

i) Sketch the deformation of the region.

ii) Compute the deformation gradient field, the right Cauchy-Green deformation tensor field and the Jacobian of the deformation field in terms of w. Does the solid change volume during the deformation process? Are the local impenetrability conditions satisfied?

iii) Consider the unit vectors:

$$\boldsymbol{A} = \frac{w_{,1}\boldsymbol{G_1} + w_{,2}\boldsymbol{G_2}}{\sqrt{w_{,1}^2 + w_{,2}^2}}\,, \qquad \boldsymbol{B} = \frac{-w_{,2}\boldsymbol{G_1} + w_{,1}\boldsymbol{G_2}}{\sqrt{w_{,1}^2 + w_{,2}^2}}\,, \tag{7}$$

where $\{\boldsymbol{G_I}\}$, $I = 1, 2, 3$ are the (orthonormal) material basis vectors. How are $\boldsymbol{A}$ and $\boldsymbol{B}$ related to the level contours of $w(X_1, X_2)$? Compute, in terms of w, the change in length (measured by the corresponding strecth ratios) of $\boldsymbol{A}$ and $\boldsymbol{B}$, as well as the change in the angle subtended by $\boldsymbol{A}$ and $\boldsymbol{B}$. Interpret results.

iv) Using the Piola transformation, compute (in terms of w) the change of area of, and in the normal to, an infinitesimal material area contained in the X_1-X_2 plane.

v) Derive an integral expression for the deformed area of the domain Ω.

vi) Let the boundary $\partial\Omega$ of Ω be defined paramatrically by the equations

$$X_1 = X_1(S)\,, \qquad X_2 = X_2(S)\,, \tag{8}$$

where $0 \leq S \leq L$ is the arc-length measured along $\partial\Omega$. Derive an integral expression for the perimeter of the deformed boundary $\boldsymbol{\varphi}(\partial\Omega)$.

Solutions

Exercise 1

(a) In order to obtain the velocities we have to compute regular time derivatives (Lagrangian velocities):

$$V_i = \frac{\partial \varphi_i}{\partial t} \to V_1 = \frac{\partial \varphi_1}{\partial t} = \alpha X_2^2\,, \tag{9a}$$

$$V_2 = \frac{\partial \varphi_2}{\partial t} = \beta X_2\,, \tag{9b}$$

$$V_3 = \frac{\partial \varphi_3}{\partial t} = 0\,. \tag{9c}$$

(b)
Likewise,

$$A_1 = A_2 = A_3 = 0\,, \tag{10a}$$

Exercise 2

(1) Begin by computing the deformation gradient and its inverse, which will be necessary later.

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} \Rightarrow \boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ w_{,1} & w_{,2} & 1 \end{bmatrix}\,, \tag{11}$$

$$F_{Ji}^{-1} = \frac{\partial X_J}{\partial x_i} \Rightarrow \boldsymbol{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -w_{,1} & -w_{,2} & 1 \end{bmatrix}\,, \tag{12}$$

$$\tag{13}$$

moreover, see that $J = \det \boldsymbol{F} = 1$ (hence no volume changes).

To compute area changes, we use the relation (Nanson's formula a.k.a. Piola transformations)

$$n_i da = J F_{Ii}^{-1} N_I dA\,, \tag{14}$$

where dA is a differential element of area in the deformed configuration and dA idem in deformed configuration.

Since we are working in the $X_3 = 0$ plane, the normal vector will be $N_I = [0,\, 0,\, 1]^\top$, we have that

$$n_i da = 1 \cdot [0,\, 0,\, 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -w_{,1}- & w_{,2} & 1 \end{bmatrix} dA = [-w_{,1},\, -w_{,2},\, 1]^\top dA\,. \tag{15}$$

The prior result represents the components of the oriented-area vector in deformed configuration. The infinitesimal area element is therefore the modulus of this vector:

$$da = ||d\boldsymbol{a}|| = ||\boldsymbol{n} da|| = \sqrt{1 + w_{,1}^2 + w_{,2}^2}\, dA\,. \tag{16}$$

To find the total area, integrate the differential element of area over the domain of definition Ω (which has not changed as the deformation does not involve the X_1 and X_2 components, only X_3):

$$a = \int_\Omega da = \int_\Omega \sqrt{1 + w_{,1}^2 + w_{,2}^2}\, dX_1 dX_2\,. \tag{17}$$

Assuming that $|w_{,1}^2 + w_{,2}^2| \ll 1$ holds all over the integration domain, we can write the square root as

$$\sqrt{1 + w_{,1}^2 + w_{,2}^2} = 1 + \frac{1}{2}|\nabla w|^2 + \mathcal{O}(|\nabla w|^4)\,, \tag{18}$$

where of course $\nabla w = [w_{,1},\, w_{,2}]^\top$ is the gradient of the scalar function $w(X_1,\, X_2)$, the latter means that the square root can be approximated with the quadratic function up to terms that decrease as $|\nabla w|^4$ when $|\nabla w| \to 0$, so they can arguably be neglected in favor or 1 and lower-order terms that scale as $|\nabla w|^2$.

Making this approximation, one finally reaches

$$a = \int_\Omega da = \int_\Omega \sqrt{1 + w_{,1}^2 + w_{,2}^2} dX_1 dX_2 \tag{19a}$$

$$\approx \int_\Omega \left(1 + \frac{w_{,1}^2 + w_{,2}^2}{2}\right) dX_1 dX_2 \tag{19b}$$

$$= A_0 + \frac{1}{2}\int_\Omega |\nabla w|^2 dX_1 dX_2 \tag{19c}$$

Exercise 3

(i) Sketch
(ii) Start with the deformation gradient tensor

$$F_{iJ} = \frac{\partial x_i}{\partial X_j} \to \boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ w_{,1} & w_{,2} & 1 \end{bmatrix}, \tag{20}$$

and thus the Jacobian det $\boldsymbol{F} = 1 \to$ no volume change during deformation. Next, the Cauchy-Green tensor:

$$C_{IJ} = F_{Ii}F_{iJ} \to \boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F} = \begin{bmatrix} 1 + w_{,1}^2 & w_{,1}w_{,2} & w_{,1} \\ w_{,2}w_{,1} & 1 + w_{,2}^2 & w_{,2} \\ w_{,1} & w_{,2} & 1 \end{bmatrix}. \tag{21}$$

(iii) See that the vectors can also be written as

$$\boldsymbol{A} = \frac{1}{|\nabla w|}[w_{,1},\, w_{,2},\, 0]^\top, \qquad \boldsymbol{B} = \frac{1}{|\nabla w|}[-w_{,2},\, w_{,1},\, 0]^\top. \tag{22}$$

See that the vector $\boldsymbol{A}$ is aligned with the gradient, i.e., along the direction of maximum descent/ascent of w, thus it must be orthogonal to the contour curves. On the other hand, $\boldsymbol{B}$ is orthogonal to $\boldsymbol{A}$, hence it "follows" the level set, this it indicates the direction along which w does not change at all.

Recall that the change in length along the direction of a vector, say, $\boldsymbol{A}$ is $\lambda_A^2 = \boldsymbol{A}^\top \boldsymbol{C}\boldsymbol{A}$, hence

$$\lambda_A^2 = \frac{1}{|\nabla w|^2}[w_{,1},\, w_{,2},\, 0]\boldsymbol{C}\begin{bmatrix} w_{,1} \\ w_{,2} \\ 0 \end{bmatrix} = 1 + |\nabla w|, \tag{23a}$$

$$\lambda_B^2 = \frac{1}{|\nabla w|^2}[-w_{,2},\, w_{,1},\, 0]\boldsymbol{C}\begin{bmatrix} -w_{,2} \\ w_{,1} \\ 0 \end{bmatrix} = 1. \tag{23b}$$

Notice that the two vectors are clearly orthogonal in the undeformed configuration, $A_I B_I = 0$, but how the deformed one remain orthogonal as well:

$$\cos(\widehat{AB}) = \boldsymbol{A}^\top \boldsymbol{C}\boldsymbol{B} = 0. \tag{24}$$

What kind of deformation is this?
(iv) and (v) Just like in exercise 2.
(vi) Let us compute the tangent vector over the contour: each point is defined by $[X_1(s),\, X_2(s)]$ for some value of X_3, so the tangent vector $\boldsymbol{T}$ is

$$\boldsymbol{T} = \frac{\partial \boldsymbol{X}/\partial S}{|\partial \boldsymbol{X}/\partial S|} = \frac{1}{\sqrt{\left(\frac{\partial X_1}{\partial S}\right)^2 + \left(\frac{\partial X_2}{\partial S}\right)^2}}\left[\frac{\partial X_1}{\partial S},\, \frac{\partial X_2}{\partial S},\, 0\right]. \tag{25}$$

In order to compute the new perimeter we have to integrate over the arc-length of the contour in the deformed configuration, we can do so because, thanks to knowing $\boldsymbol{C}$, we can account for the differential stretch of each and every fiber making the contour up. Mathematically,

$$\ell = \int_{\varphi(\partial\Omega)} ds = \int_{\partial\Omega} \lambda_T dS, \tag{26}$$

where

$$\lambda_T^2 = \boldsymbol{T}^\top \boldsymbol{C}\boldsymbol{T} = (1+w_1^2)\left(\frac{\partial X_1}{\partial S}\right)^2 + 2w_1 w_2 \frac{\partial X_1}{\partial S}\frac{\partial X_2}{\partial S} + (1+w_2^2)\left(\frac{\partial X_2}{\partial S}\right)^2 , \tag{27}$$

thus, plugging the expression for λ_T in the integral formula,

$$\ell = \int_{\partial\Omega} \lambda_T dS \tag{28a}$$

$$= \int_{\Omega} \sqrt{(1+w_1^2)\left(\frac{\partial X_1}{\partial S}\right)^2 + 2w_1 w_2 \frac{\partial X_1}{\partial S}\frac{\partial X_2}{\partial S} + (1+w_2^2)\left(\frac{\partial X_2}{\partial S}\right)^2} dS \,. \tag{28b}$$

A good sanity check for the formula just obtained is making sure that in the absence of deformation, i.e. $\boldsymbol{\varphi}(\partial\Omega) = \partial\Omega$, $\ell = L$ (the undeformed perimeter length):

$$\ell|_{w=0} = \int_{\Omega} \sqrt{\left(\frac{\partial X_1}{\partial S}\right)^2 + \left(\frac{\partial X_2}{\partial S}\right)^2} dS = \int_{\Omega} |\boldsymbol{T}| dS = \int_{\Omega} 1 \cdot dS = L \,. \tag{29}$$