Problem Set 1

CIVIL-425: Continuum Mechanics and Applications

29 February 2024

Exercise 1, universal static deformations

Look at example 1.1.5 from Ortiz's notes on universal static deformations. Code an interactive notebook with your preferred language (Jupyter notebook, Matlab, Mathematica) to visualize the deformations while changing the parameters.

Exercise 2, cylinder inside out

A cylindrical tube of inner radius a, outer radius b, and length L is turned inside out and subsequently constrained to take the shape of a straight cylindrical tube of the same dimensions. Assuming that the radial fibers do not undergo any stretching, determine the deformation mapping φ. (Hint: use cylindrical coordinates). Show that $\varphi \circ \varphi$ is the identity mapping, i.e. turning the cylinder inside out twice returns it to its initial configuration.

Solution

Let the undeformed tube occupy the region $B_0 = [a, b] \times [0, 2\pi] \times [0, L]$ in polar coordinates, then the deformation mapping is given by :

$$
\begin{aligned}
r &= a + b - R \\
\theta &= \Theta \\
z &= L - Z
\end{aligned}
$$

We confirm that the stretch ratio of radial fibers is:

$$\lambda_r = \frac{dr}{dR} = -1\,,$$

meaning that there is no stretch, but only flipping in the radial direction.

If we apply the map twice:

$$
\begin{aligned}
r' &= a + b - R = a + b - (a + b - R) = R \\
\theta' &= \theta = \Theta \\
z' &= L - z = L - (L - Z) = Z
\end{aligned}
$$

If we choose the coordinate system so that the tube center midpoint along the axis is at coordinate $Z = 0$, then the deformation is simple $z = -Z$.

Exercise 3, plate to cylinder

A plate of thickness h, width L and infinite length is bent into a cylindrical shell section (plane strain conditions, see Figure 1). Assume that the plane $X_1 = 0$ remains unstretched, i.e. is the neutral plane. Determine the deformation mapping φ. Find under what conditions the deformation mapping ceases to be invertible.

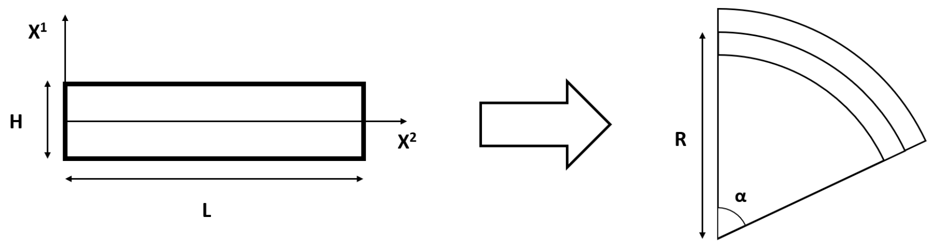

Figure 1: Sketch of deformation of the plate

Solution

The underformed plate occupies the region B_0 in cartesian coordinates.

$$B_0 = [0, L] \times [-h/2, h/2] \times [-\infty, \infty]$$

We can parametrize the deformed plate in cylindrical coordinates as:

$$r = R + X_1 \tag{1}$$

$$\theta = \alpha \frac{X_2}{L} = \frac{X_2}{R}\ , \tag{2}$$

where the neutral axis condition gives $R\alpha = L$. Recalling the transformation from the Cartesian to polar coordinates:

$$r = \sqrt{x_1^2 + x_2^2}$$

$$\tan\theta = \frac{x_2}{x_1}$$

by substituting it in Equations 2 we get the deformation mapping

$$x_1 = (R + X_1)\cos\frac{X_2}{R}$$

$$x_2 = (R + X_1)\sin\frac{X_2}{R}\ ,$$

where we used the identity $\sqrt{1 + \tan^2(X_2/R)} = 1/\cos(x^2/R)$. From the deformation mapping we can compute the deformation gradient:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \begin{pmatrix} \cos\frac{X_2}{R} & -\frac{R+X_1}{R}\sin\frac{X_2}{R} \\ \sin\frac{X_2}{R} & \frac{R+X_1}{R}\cos\frac{X_2}{R} \end{pmatrix} \tag{3}$$

Setting $\det \mathbf{F} = 0$ yields the condition $R + X_1 = 0$. Considering that the smallest value of $X_1 = -h/2$, we find that the critical bending angle to reach interpenetration is

$$\frac{h}{2} = R \iff \alpha_c = \frac{2L}{h}$$

Therefore, the thicker the smaller the compatible bending angle.

Alternative solution

The deformation mapping stops being invertible when it is not bijective. Here, it happens when the length of one compressed fiber goes to 0. This starts with the lowest fiber of length l':

$$l' = (R - \frac{h}{2})\alpha = (\frac{L}{\alpha} - \frac{h}{2})\alpha = 0$$

$$\Rightarrow \alpha_c = \frac{2L}{h}$$

The mapping ceases to be invertible for $\alpha > \alpha_c$.

Exercise 4, composition of mappings

Let $B_0 = [0, a] \times [0, a]$, i.e. a square of side a and let $\varphi_1(X) = \{X_1, \lambda X_2, X_3\}$ (uniaxial stretching) and $\varphi_2(X) = \{X_1 + X_2 \tan\alpha, X_2, X_3\}$ (pure shear). Carry out the compositions of mappings $\varphi_2 \circ \varphi_1$ and $\varphi_1 \circ \varphi_2$. Plot and compare the results. Does composition of mappings commute?

Solution

$$\varphi_1(X) = \{X_1, \lambda X_2, X_3\}$$
$$\varphi_2(X) = \{X_1 + X_2 tan\alpha, X_2, X_3\}$$

$$\varphi_2 \circ \varphi_1(X) = \{X_1 + \lambda X_2 tan\alpha, \lambda X_2, X_3\}$$
$$\varphi_1 \circ \varphi_2(X) = \{X_1 + X_2 tan\alpha, \lambda X_2, X_3\}$$
$$\varphi_2 \circ \varphi_1 \neq \varphi_1 \circ \varphi_2$$

Composition of mappings DOES NOT commute. See figure 5.

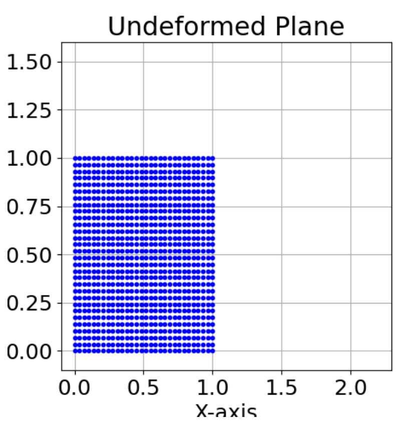

Figure 2: Undeformed (X_1, X_2) plane

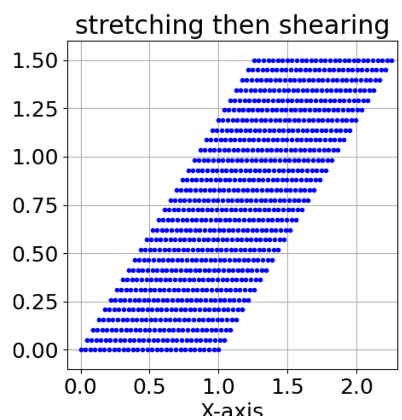

Figure 3: stretching the shearing $(\varphi_2 \circ \varphi_1)$

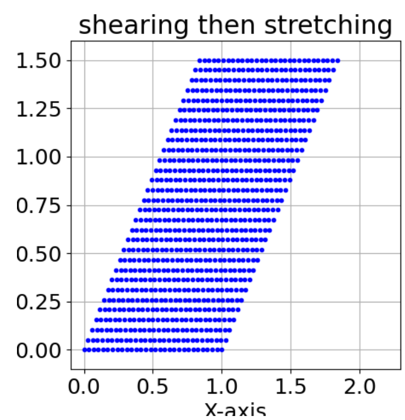

Figure 4: stretching the shearing $(\varphi_1 \circ \varphi_2)$

Figure 5: Illustration of non-commutativity of deformation mappings.

Exercise 5, bug walking on stretching wire

A bug walks on a stretching wire with speed V relative to the material point of the wire on which the bug stands. The bug starts at the pinned end at $t = 0$. The free end of the wire moves with velocity V_0. The initial length of the wire is L. Find the time required for the bug to reach the free end of the wire.

1. write the relationship between the spatial and reference configuration for the stretching wire
2. write the material velocity of a point of the wire
3. write the equation of motion for the bug, with spatial coordinate $x_b(t)$
4. given the solution $x_b(t) = \frac{V}{V_0}(L + V_0 t) \ln\left(\frac{L+V_0 t}{L}\right)$, find the time required for the bug to reach the free end of the wire

Alternative: Solution in the spatial frame

The relationship between the spatial and reference configuration for the stretching wire reads:

$$x(X,t) = \varphi(X,t) = X\left(1 + \frac{v_0}{L}t\right)\,, \tag{4}$$

so that the material velocity of a point on the wire is:

$$v(X,t) = \frac{d\varphi(X,t)}{dt} = \frac{X}{L_0}V_0\,. \tag{5}$$

By substituting $X = X(x,t)$ from eq. 4 into eq. 5 we obtain the spatial velocity of the wire:

$$v(x,t) = \frac{x}{L + V_0 t}V_0\,. \tag{6}$$

As the bug is moving with constant velocity V with respect to the point on the wire it is walking on, calling $x_b(t)$ and $v_b(t)$ respectively the position and the velocity of the bug in space, the equation of motion is:

$$v_b(t) \equiv \frac{dx_b(t)}{dt} = \frac{x_b}{L + V_0 t}V_0 + V\,.$$

The solution to the first order differential equation is:

$$x_b(t) = \frac{V}{V_0}(L + V_0 t)\ln\left(\frac{L + V_0 t}{L}\right).$$

The bug reaches the end of the wire in the spatial frame once $x_b(t^*) = L'$. This yields, $t^* = (L/V_0)(\exp{(V_0/V)} - 1)$ the same solution as in the previous case.

In addition we can also compute the distance of the bug from the opposite end of the wire:

$$L' - x_b(t) = (L + V_0 t)\left[1 - \frac{V}{V_0}\ln\left(1 + \frac{V_0}{L}t\right)\right]\,. \tag{7}$$

A graphical representation is given in figure 6.

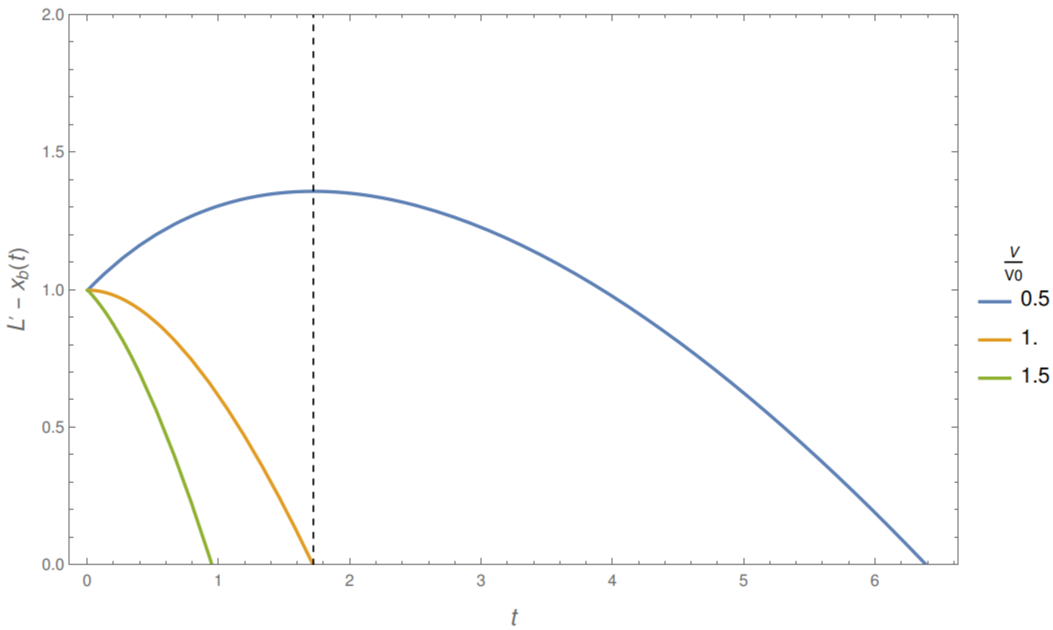

Figure 6: Plot of eq. 7 for $L = 1$, $V_0 = 1$, and different values of V. If the velocity of the bug is larger than that of the free end ($V > V_0$), the bug will see the distance from the end always decreasing. On the contrary if ($V < V_0$), the bug initially loses ground, but it is still making progress. The instant when the bug is further away from the end corresponds to the dashed line (point of depression). *Is this the metaphor of life?*