Problem Set 11

CIVIL-425: Continuum Mechanics and Applications

22 May 2025

Exercise. Bingham fluid flow between two parallel plates.

Question 1: Momentum Balance Under steady-state assumption momentum balance is

$$\nabla \cdot \boldsymbol{\sigma} = 0$$

or

$$-\nabla p + \nabla \cdot \boldsymbol{s} = 0$$

The expanded form in two dimensions is

$$-\frac{\partial p}{\partial x} + \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y} = 0$$

$$-\frac{\partial p}{\partial y} + \frac{\partial s_{xy}}{\partial x} + \frac{\partial s_{yy}}{\partial y} = 0$$

where $s_{xx} = 0$ and $s_{yy} = 0$ because $\dot{\boldsymbol{u}} = (\dot{u}_x(y), 0)$. Assuming that $\frac{\partial p}{\partial y}$ is very small in a narrow channel $w << L$, we obtain

$$-\frac{\partial p}{\partial x} + \frac{\partial s_{xy}}{\partial y} = 0$$

$$\frac{\partial s_{xy}}{\partial x} = 0$$

Question 2: Shear Stress Profile Integrating momentum balance we obtain

$$s_{xy} = \frac{\mathrm{d}p}{\mathrm{d}x} y$$

This linear profile indicates a maximum and minimum at the plates, depending on the sign of $\frac{\mathrm{d}p}{\mathrm{d}x}$.

Question 3: Velocity Profile The fluid starts to flow if $\sqrt{J_2(\boldsymbol{s})} > \sigma_Y$. Deviatoric part of stress tensor is such that $tr(\boldsymbol{s}) = 0$, therefore $\sqrt{J_2(\boldsymbol{s})} = |s_{xy}|$. The flow condition becomes

$$|s_{xy}| > \sigma_Y$$

In pure shear, plastic strain rate can be written as

$$\dot{\boldsymbol{\epsilon}}^p = \begin{bmatrix} 0 & \dot{\gamma} \\ \dot{\gamma} & 0 \end{bmatrix}$$

where $\dot{\gamma} = \frac{\mathrm{d}u_x}{\mathrm{d}y}$.

For $|y| \leq y_c$ where y_c is where $|s_{xy}(y_c)| = \sigma_Y$ ($=> y_c = \pm\frac{\sigma_Y}{|\mathrm{d}p/\mathrm{d}x|}$), the fluid behaves as rigid and $\dot{\gamma} = \frac{\mathrm{d}\dot{u}_x}{\mathrm{d}y} = 0$, which leads to $\dot{u}_x = A = const$.

On the other hand, when $|y| > y_c$, we have

$$\dot{\gamma} = \frac{\mathrm{d}\dot{u}_x}{\mathrm{d}y} = \frac{|s_{xy}| - \sigma_Y}{\eta} \frac{s_{xy}}{|s_{xy}|}$$

recalling that $s_{xy} = \frac{\mathrm{d}p}{\mathrm{d}x} y$, we obtain

$$\dot{\gamma} = \frac{\mathrm{d}\dot{u}_x}{\mathrm{d}y} = \frac{\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right| |y| - \sigma_Y}{\eta} sign(y)$$

First, assume, that $y > y_c$, then

$$\dot{\gamma} = \frac{\mathrm{d}\dot{u}_x}{\mathrm{d}y} = \frac{\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right| y - \sigma_Y}{\eta}$$

integrating and taking into account $\dot{u}_x(y = +h) = 0$ we obtain

$$\dot{u}_x(y) = \frac{1}{\eta}\left(\frac{1}{2}\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right|(y^2 - h^2) + \sigma_Y(h - y)\right)$$

Analogously for $y < -y_c$

$$\dot{\gamma} = \frac{\mathrm{d}\dot{u}_x}{\mathrm{d}y} = \frac{\frac{\mathrm{d}p}{\mathrm{d}x} y + \sigma_Y}{\eta}$$

integrating and taking into account $\dot{u}_x(y = -h) = 0$ we obtain

$$\dot{u}_x(y) = \frac{1}{\eta}\left(\frac{1}{2}\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right|(y^2 - h^2) + \sigma_Y(y + h)\right)$$

Now we can compute constant velocity at $y = \pm y_c$

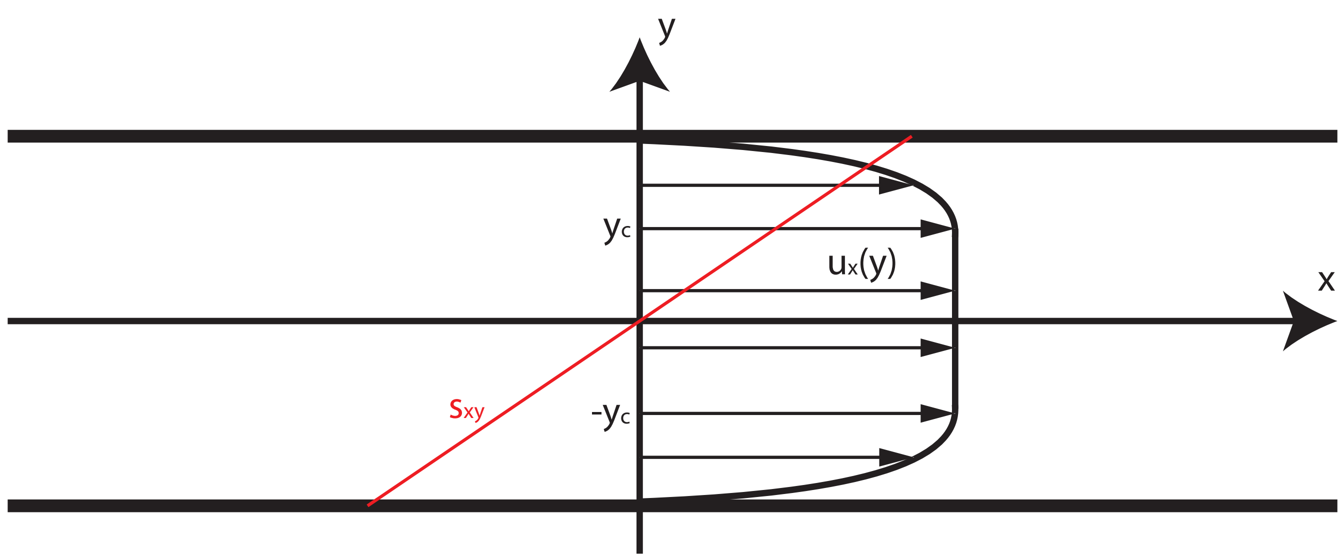
$$\dot{u}_x(y) = \frac{1}{\eta}\left(\frac{1}{2}\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right|(y_c^2 - h^2) + \sigma_Y(h - y_c)\right)$$

We can summarise our solution

$$\dot{u}_x(y) = \frac{1}{\eta}\left(\frac{1}{2}\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right|(y^2 - h^2) + \sigma_Y(h - y)\right) \text{ for } y > y_c$$

$$\dot{u}_x(y) = \frac{1}{\eta}\left(\frac{1}{2}\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right|(y_c^2 - h^2) + \sigma_Y(h - y_c)\right) \text{ for } -y_c \leq y \leq y_c$$

$$\dot{u}_x(y) = \frac{1}{\eta}\left(\frac{1}{2}\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right|(y^2 - h^2) + \sigma_Y(y + h)\right) \text{ for } y < -y_c$$


y
y_c
$u_x(y)$
x
s_{xy}
$-y_c$