Problem Set 11

CIVIL-425: Continuum Mechanics and Applications

22 May 2025

Exercise. Bingham pressure-driven flow between two parallel plates.

A Bingham material is a type of non-Newtonian fluid that behaves as rigid at low stresses but flows as a viscous fluid when the second invariant $J_2(\boldsymbol{s})$ of the deviatoric part of stress tensor $\boldsymbol{s}$ exceeds a certain threshold, defined by the yield stress σ_Y. Once the yield stress is exceeded $\sqrt{J_2(\boldsymbol{s})} > \sigma_Y$, the fluid starts to flow and its behaviour can be described by a linear relationship between stress and shear rate, similar to a Newtonian fluid but offset by the yield stress:

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\boldsymbol{\epsilon}} = 0 \text{ when } \sqrt{J_2(\boldsymbol{s})} \leq \sigma_Y$$

$$\dot{\boldsymbol{\epsilon}}^p = \frac{\sqrt{J_2(\boldsymbol{\sigma})} - \sigma_Y}{\eta} \frac{\boldsymbol{s}}{||\boldsymbol{s}||}$$

where ν is fluid viscosity, $\boldsymbol{s}$ is the deviatoric part of the stress tensor $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{s}$ with $p = -\frac{\sigma_{aa}}{3}$ being the fluid pressure.

Question 1: Momentum Balance

- Setup: Two parallel plates separated by $2h$, with a constant pressure gradient $\frac{dp}{dx}$. The flow is steady, incompressible, the fluid velocity is solely in the x-direction, with the velocity field given by $\dot{\boldsymbol{u}} = (\dot{u_x}(y), 0)$ where u_x depends only on the y-coordinate.
- Task: Derive the momentum balance equation for pure shear flow.

Question 2: Shear Stress Profile

- Task: Calculate the shear stress profile $\tau(y)$ across the gap from $y = -h$ to $y = h$.

Question 3: Velocity Profile

- Boundary Conditions: Zero velocity at the walls.
- Objective: Determine the velocity profile $\dot{u}_x(y)$ of the Bingham fluid within the channel.
- Task: Using the shear stress profile from Question 2, identify regions of rigid-like behaviour and fluid flow, then compute the velocity profile $\dot{u}_x(y)$, considering zero slip conditions at the walls.