Problem Set 10

CIVIL-425: Continuum Mechanics and Applications

15 May 2025

Exercise 1: Constitutive laws

For the following strain-energy functions, write: a) the stress-strain relations in material form (i. e., in terms of the second PK stress tensor $\mathbf{S}$ and the right Cauchy-Green deformation tensor $\mathbf{C}$) and in spatial form (i. e., in terms of the Cauchy stress tensor $\boldsymbol{\sigma}$ and the left Cauchy-Green deformation tensor $\mathbf{C}$); b) the material and spatial elastic moduli D_{IJKL} and D_{ijkl}, respectively; c) the relation between the Cauchy stress σ and the stretch λ in uniaxial tension. If possible, plot the uniaxial stress against the logarithm of the stretch (i. e., the logarithmic strain). Is the predicted stress-strain behavior sensible?

1. Ideal compressible fluid:
$$W = W(J)$$
2. Saint Venant-Kirchhoff solid:
$$W = \frac{1}{2}\left[\lambda_0 \left(E_{KK}\right)^2 + 2\mu E_{IJ}E_{IJ}\right]$$
3. Neo-Hookean solid:
$$W = \frac{\lambda}{2}\log^2(J) - \mu\log(J) + \frac{\mu}{2}I_1$$
4. *Bonus:* Compressible Mooney-Rivlin solid:
$$W = \frac{\lambda}{2}\log^2(J) - \mu\log(J) + \frac{1}{2}\left[\alpha_1\left(I_1 - 3\right) + \alpha_2\left(I_2 - 3\right)\right]$$

In the above expressions, λ, μ, α_1 and α_2 are constants. Consider the case $\alpha_1 = \alpha_2 = \mu$

Solution

Recall that for a given square matrix $\mathbf{A}$:

$$(\mathbf{A}^{-1})' = \mathbf{A}^{-1}\mathbf{A}'\mathbf{A}^{-1}$$
$$\frac{\partial \det \mathbf{A}}{\partial \mathbf{A}} = \det \mathbf{A}\mathbf{A}^{-1}$$
$$\frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{jl}\,.$$

Moreover if $\mathbf{A}$ is symmetric then

$$\frac{\partial A_{ij}}{\partial A_{kl}} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,.$$

1. $W = W(J)$

a)

$$
\begin{aligned}
P_{iJ} &= \frac{\partial W}{\partial F_{iJ}} = W'(J)\frac{\partial J}{\partial F_{iJ}} = W'(J)JF^{-1}_{Ji} \\
\Rightarrow &\sigma_{ij} = J^{-1}P_{iJ}F_{jJ} \Rightarrow \sigma_{ij} = W'(J)\delta_{ij} = p(J)\delta_{ij} \\
\Rightarrow &S_{IJ} = J\sigma_{ij}F^{-1}_{Ii}F^{-1}_{Jj} \Rightarrow S_{IJ} = JW'(J)C^{-1}_{IJ}
\end{aligned}
$$

b)

$$
\begin{aligned}
D_{IJKL} =& 2\frac{\partial S_{IJ}}{\partial C_{KL}} = 2\left[\left(W'(J) + JW''(J)\right)\frac{\partial J}{\partial C_{KL}}C^{-1}_{IJ} + JW'(J)\frac{\partial C^{-1}_{IJ}}{\partial C_{KL}}\right] \\
&\text{But } \frac{\partial J}{\partial C_{KL}} = \frac{J}{2}C^{-1}_{KL};\ \frac{\partial C^{-1}_{IJ}}{\partial C_{KL}} = -\frac{1}{2}\left(C^{-1}_{IK}C^{-1}_{JL} + C^{-1}_{IL}C^{-1}_{JK}\right) \\
D_{IJKL} =& J\left(W'(J) + JW''(J)\right)C^{-1}_{IJ}C^{-1}_{KL} - JW'(J)\left(C^{-1}_{IK}C^{-1}_{JL} + C^{-1}_{IL}C^{-1}_{JK}\right)
\end{aligned}
$$

If we define:

$$
\begin{aligned}
Dijkl &= J^{-1}F_{iI}F_{jJ}F_{kK}F_{lL}D_{IJKL},\ \text{so that } L_v\sigma = \mathbf{D}:\mathbf{d} \\
D_{ijkl} &= \left(W'(J) + JW''(J)\right)\delta_{ij}\delta_{kl} - W'(J)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)
\end{aligned}
$$

c) The solid cannot sustain uniaxial stress states.

2. $W = \frac{1}{2}\left[\lambda_0\left(E_{KK}\right)^2 + 2\mu E_{IJ}E_{IJ}\right], E_{IJ} = \frac{1}{2}\left(F_{iI}F_{iJ} - \delta_{IJ}\right)$

a)

$$
\begin{aligned}
S_{IJ} &= \frac{\partial W}{\partial E_{IJ}} = \lambda_0 E_{KK}\delta_{IJ} + 2\mu E_{IJ} \Rightarrow \\
\tau_{ij} &= \lambda_0\frac{1}{2}(\mathrm{tr}(B) - 3)B_{ij} + 2\mu\left(B^2_{ij} - B_{ij}\right)
\end{aligned}
$$

b)

$$
D_{IJKL} = \frac{\partial S_{IJ}}{\partial E_{KL}} = \lambda_0\delta_{IJ}\delta_{KL} + \mu\left(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}\right)
$$

If we define:

$$
\begin{aligned}
Dijkl &= F_{iI}F_{jJ}F_{kK}F_{lL}D_{IJKL},\ \text{so that } L_v\tau = \mathbf{D}:\mathrm{d} \\
Dijkl &= \lambda_0 B_{ij}B_{kl} + \mu\left(B_{ik}B_{jl} + B_{il}B_{jk}\right)
\end{aligned}
$$

c) Uniaxial stress in X_1 direction:

$$
\begin{aligned}
S_{11} &= \left(\lambda_0 + 2\mu\right)E_{11} + \lambda_0 E_{22} + \lambda_0 E_{33} \\
S_{22} &= \lambda_0 E_{11} + \left(\lambda_0 + 2\mu\right)E_{22} + \lambda_0 E_{33} = 0 \\
S_{33} &= \lambda_0 E_{11} + \lambda_0 E_{22} + \left(\lambda_0 + 2\mu\right)E_{33} = 0 \\
&\text{and } E_{22} = E_{33} = -\frac{\lambda_0}{2\left(\lambda_0 + \mu\right)}E_{11} \Rightarrow \\
S_{11} &= \frac{3\mu}{\left(\lambda_0 + \mu\right)}\left(\lambda_0 + \frac{2}{3}\mu\right)E_{11} \\
&\text{But } E_{11} = \frac{1}{2}\left(\lambda^2 - 1\right), P_{11} = \lambda S_{11} \Rightarrow \\
P_{11} &= \frac{3\mu}{\left(\lambda_0 + \mu\right)}\left(\lambda_0 + \frac{2}{3}\mu\right)\frac{\lambda}{2}\left(\lambda^2 - 1\right)
\end{aligned}
$$

The model predicts that the applied load goes to zero as a uniaxial specimen is compressed to zero length, which is unreasonable. See Figure 1.

3. $W = \frac{\lambda}{2}\log^2(J) - \mu\log(J) + \frac{\mu}{2}I_1$

a)

$$S_{IJ} = \lambda_0 \log J C^{-1}_{IJ} + \mu \left(\delta_{IJ} - C^{-1}_{IJ}\right)$$
$$\Rightarrow \tau_{ij} = \lambda_0 \log J \delta_{ij} + \mu \left(B_{ij} - \delta_{ij}\right)$$

b)

$$D_{IJKL} = \lambda_0 C^{-1}_{IJ} C^{-1}_{KL} + \left(\mu - \lambda_0 \log J\right) \left(C^{-1}_{IK} C^{-1}_{JL} + C^{-1}_{IL} C^{-1}_{JK}\right)$$
$$D_{ijkl} = \lambda_0 \delta_{ij} \delta_{kl} + \left(\mu - \lambda_0 \log J\right) \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}\right)$$

c) Let $\mathbf{F} = \mathrm{diag}\left(\lambda_1, \lambda_2, \lambda_2\right) \Rightarrow J = \lambda_1 \lambda_2^2 \Rightarrow$

$$\tau_{11} = \lambda_0 \log \lambda_1 \lambda_2^2 + \mu \left(\lambda_1^2 - 1\right)$$
$$\tau_{22} = \tau_{33} = \lambda_0 \log \lambda_1 \lambda_2^2 + \mu \left(\lambda_2^2 - 1\right) = 0$$

On the incompressibility limit $\lambda_0 \to \infty, J \to 1$ can be treated explicitly:

$$\tau_{11} = p + \mu \left(\lambda_1^2 - 1\right)$$
$$p + \mu \left(\lambda_2^2 - 1\right) = \mu \left(\lambda_1^2 - \lambda_2^2\right)$$
$$\tau_{11} = \mu \left(\lambda_1^2 - \lambda_1^{-1}\right)$$
$$P_{11} = \lambda_1^{-1} \tau_{11} = \mu \left(\lambda_1 - \lambda_1^{-2}\right)$$

The model predicts the right limits as $\lambda_1 \to 0, \lambda_1 \to \infty$. See Fig. 1.

4. $W = \frac{\lambda_0}{2} \log^2(J) - \mu \log(J) + \frac{1}{2}\left[(I_1 - 3) + (I_2 - 3)\right]$

a)

$$S_{IJ} = \lambda_0 \log J C^{-1}_{IJ} + \mu \left(\delta_{IJ} - C^{-1}_{IJ}\right) + \mu \left(I_1 \delta_{IJ} - C_{IJ}\right)$$
$$\Rightarrow \tau_{ij} = \lambda_0 \log J \delta_{ij} + \mu \left(B_{ij} - \delta_{ij}\right) + \mu \left[\left(\mathrm{tr}(\mathbf{B}) B_{ij} - B^2_{ij}\right]\right.$$

b)

$$\begin{aligned} D_{IJKL} &= \lambda_0 C^{-1}_{IJ} C^{-1}_{KL} + \left(\mu - \lambda_0 \log J\right) \left(C^{-1}_{IK} C^{-1}_{JL} + C^{-1}_{IL} C^{-1}_{JK}\right) + \\ &+ \mu \left[2 \delta_{IJ} \delta_{KL} - \left(\delta_{IK} \delta_{JL} + \delta_{IL} \delta_{JK}\right)\right] \\ D_{ijkl} &= \lambda_0 \delta_{ij} \delta_{kl} + \left(\mu - \lambda_0 \log J\right) \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}\right) + \\ &+ \mu \left[2 B_{ij} B_{kl} - \left(B_{ik} B_{jl} + B_{il} B_{jk}\right)\right] \end{aligned}$$

c) Let $\mathbf{F} = \mathrm{diag}\left(\lambda_1, \lambda_2, \lambda_2\right) \Rightarrow J = \lambda_1 \lambda_2^2 \Rightarrow$

$$\tau_{11} = \lambda_0 \log \lambda_1 \lambda_2^2 + \mu \left(\lambda_1^2 - 1\right) + \mu \left[\left(\lambda_1^2 + 2\lambda_2^2\right) \lambda_1^2 - \lambda_1^4\right]$$
$$\tau_{22} = \tau_{33} = \lambda_0 \log \lambda_1 \lambda_2^2 + \mu \left(\lambda_2^2 - 1\right) + \mu \left[\left(\lambda_1^2 + 2\lambda_2^2\right) \lambda_2^2 - \lambda_2^4\right] = 0$$

On the incompressibility limit $\lambda_0 \to \infty, J \to 1:$

$$\tau_{11} = \mu \left(\lambda_1^2 - \lambda_1^{-1}\right) + \mu \left[\left(\lambda_1^2 + 2\lambda_1^{-1}\right) \left(\lambda_1^2 - \lambda_1^{-1}\right) - \left(\lambda_1^4 - \lambda_1^{-2}\right)\right]$$
$$P_{11} = \lambda_1^{-1} \tau_{11} = \mu \left(\lambda_1 - \lambda_1^{-2}\right) + \mu \left[\left(\lambda_1 + 2\lambda_1^{-2}\right) \left(\lambda_1 - \lambda_1^{-2}\right) - \left(\lambda_1^3 - \lambda_1^{-3}\right)\right]$$

Model predicts the right limits as $\lambda_1 \to 0$. The maximum in the load elongation curve is a geometrical effect related to the reduction in cross-sectional area. See Fig. 1.

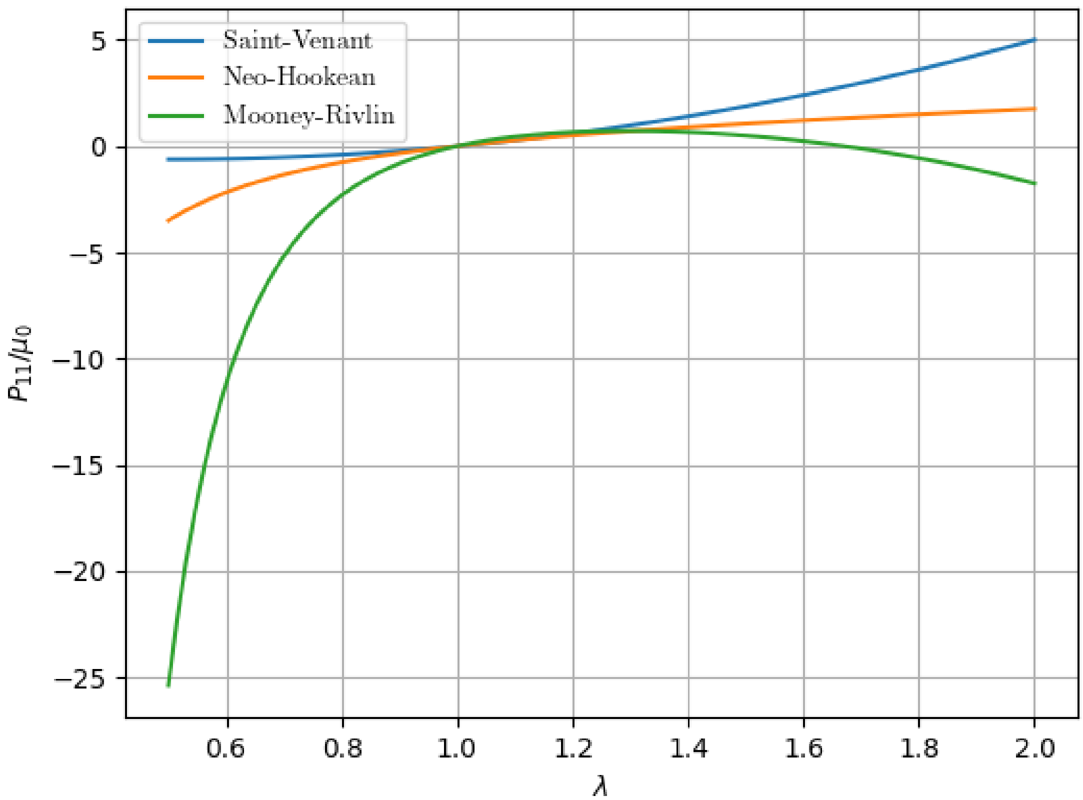

Figure 1: Comparision of common hyperelastic constitutive laws