Week #8 - Excavation of a tunnel

This weeks exercise deals with a classical problem of geotechnical engineering. We will investigate the deformations and stresses induced by the (sudden) excavation of a tunnel. We will compare the undrained displacements and the drained stresses to analytical solutions. Mesh and material properties are provided. We use here quadratic interpolation for both displacement and pore-pressure.

1 Boundary and initial conditions

To save computational cost we will not model the entire area but only a quarter (note that this is possible due to the symmetry of the problem when we assume constant stress and pore pressure fields in the beginning). As such the horizontal displacements will be fixed on the left boundary and the vertical displacements on the bottom boundary. Additionally, the resolution will require to drain at the tunnel boundary and to define stresses on the top and right boundary (the tunnel will be stress free).

- We will let you define the blocked DOF and nodes.

In a next step the initial stress field and corresponding surface tractions need to be defined. For this we refer to Chapter 6 and mainly section 6.2.1 where we introduce the poroelastic constitutive equation:

$$\int_{\Omega} \epsilon_{ij}(\mathbf{v}) c_{ijkl} \epsilon_{kl}(\mathbf{u}) \, \mathrm{d}V - \int_{\Omega} \epsilon_{ij}(\mathbf{v}) \alpha_{ij}(p) \, \mathrm{d}V = -\int_{\Omega} \epsilon_{ij}(\mathbf{v}) \sigma^o_{ij} \mathrm{d}V - \int_{\Omega} \epsilon_{ij}(\mathbf{v}) \alpha_{ij} p^o \mathrm{d}V + \int_{\Gamma_{t_i}} (t^g_i) v_i \, \mathrm{d}S$$

where the right hand side contains three terms. The first one is related to the initial stress field within the volume, the second to the initial pressure field and the third to the surface tractions (which are logically related to the stress field as to be in equilibrium). We set the initial stress field with $\sigma^o_{xx} = -20$[MPa] and $\sigma^o_{yy} = -40$ [MPa], which corresponds to a mean compressive stress $P_o = -30$[MPa] and mean deviatoric stress, $S_o = -10$[MPa] (transformation to nodal force performed via the function *MatrixAssembly.set_stress_field*()). Additionally, we will set the pore pressure uniformly to $p_o = 3$[MPa] everywhere except at the tunnel wall (drained surface).

- Set the initial pore pressure field and the corresponding line forces of the surface tractions.

2 Check for equilibrium

The next part of the code is dedicated to a check of the equilibrium prior to the excavation. This is just a very simple check (more for debugging than anything else). We use the fact that the body containing an initial stress field can be balanced by the corresponding surface tractions (so no displacement will take place theoretically). We need this initial equilibrium to be fulfilled if we model it as there would have been no tunnel excavated. Only with the correct initial condition iti s possible to solve for total stresses. To perform the comparison, we fix the stresses within the tunnel as not to have any stress free boundary.

- Investigate the maximum displacement (which should be in theory zero - and numerically very small) and plot of the deformed mesh to be sure that the equilibrium is effectively satisfied.

3 Assembly of matrices

As for last week's exercise the different parts of the total matrix need now to be assembled. Different from all the previous sessions, we will not provide you all the routines for the assembly. Notably, you will need to code up the element coupling matrix.

- Go to the function *Elements.element_ coupling_ matrix()* and code it up.
- Note: The coupling term is given by $\mathbf{A}^{el} = \int_{\Omega^{el}} \mathbf{B}'\alpha\mathbf{N}\, dV$.

4 Undrained solution ($t = 0^+$)

- As for the last exercise we let you assemble the total matrix.

Similarly to previous sessions, the force vector needs to be defined. Note that we've always transformed fixed displacements and pressures into the corresponding forces onto the other nodes. As we deal here with an initial pore pressure field this procedure needs to be followed.

- Assemble the force vector.
- Note that you have to solve the following system:

$$\begin{bmatrix} \mathbf{K} & -\mathbf{A} \\ -\mathbf{A^T} & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{U_o} \\ \mathbf{p_o} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

We solve the system for you and plot the deformed mesh as well as the pore pressure field. You will recognize that there is a deviatoric contraction of the tunnel combined with a non intuitive undrained pore pressure response. We will now compare the results of this undrained analysis with the analytical solutions for displacement at the top node (the node at angle of $\theta = 90°$ on the tunnel), and for pore pressure along the axis $x = -d_{area}$. Which are given by:

$$u_r(r = r_o, \theta = 90°, t = 0^+) = -\frac{P_o r_o}{2G} - \frac{(3 - 4\nu_u) r_o S_o}{2G} \tag{1}$$

$$p(r, \theta = 0°, t = 0^+) = \frac{4B(1 + \nu_u)}{3} \frac{r_o^2}{r^2} S_o + p_o$$

where r_o is the radius of the tunnel, P_o the far-field mean stress, S_o the far-field deviatoric stress, G shear modulus, ν_u undrained poisson's ratio, B Skempton coefficient and p_o the initial pore pressure field.

- We will let you code the analytical solution for the displacement.

5 Drained solution ($t \geq 0^+$)

The drained solution is the large time solution for the problem. In between, one would observe a transient characterizing the transition from the undrained to the drained state. Note that this transient period is better captured on a logarithmic scale of time, therefore we use a variable time-step using the logspace matlab function to generate a list of time following a log variation - spanning the time range 10^{-5} to 10^2.

- We will let you calculate the time step from the provided time vector t_k and code the update of the corresponding block of the global matrix.

The problem is then solved by the usual procedure in an implicit way. As to check the solution we will plot the evolution of the displacement of the top node and compare to the early time solution (given by equation 1) and the large time solution easily calculated by simply replacing ν_u with ν in equation 1. Finally, we will compare the stresses along the axis $x = -d_{area}$ with the analytical solution given by:

$$\sigma_{xx}(r) = -P_o\left(1 - \frac{r_o^2}{r^2}\right) + S_o\left(1 - 4\frac{r_o^2}{r^2} + 3\frac{r_o^4}{r^4}\right) + \eta p_o\left(1 - \frac{r_o^2}{r^2}\right)$$

- We will let you get the corresponding numerical stresses for the comparison.
- Try now to extend the time-span of the simulation say to 10^3 - explain what you are observing.