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CIVIL 369: STRUCTURAL STABILITY

Problem Set 1

Question 1 – Plastic Analysis and Geometric Nonlinearities (25 points)

The single storey frame structure shown in Figure 4 is subjected to gravity load N=200kN at each column as well as lateral load F. The final geometry and cross-sectional profiles are shown in the figure. The steel material is S355J2 ($G = 200GPa$, $f_y = 355MPa$)

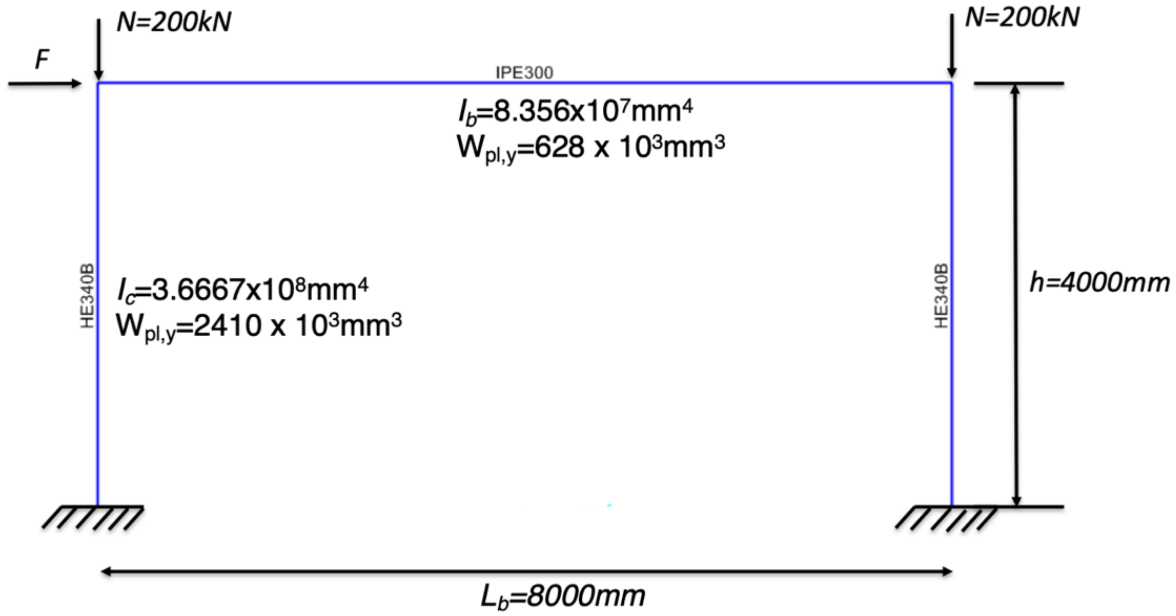


Figure 1 – Stability bracing

1.1. Calculate the collapse load of the structure till it becomes a complete collapse mechanism by neglecting P-Delta effects (i.e., ignore N in this case).
1.2. Calculate the collapse load of the structure by considering P-Delta effects in your step-by-step calculations. Calculate the deflection at which the structure loses its lateral load resistance.
1.3. Calculate the collapse load when $N = 2000kN$ per column.
1.4. Compare the three solutions in an equilibrium path. Discuss your findings.

Note: there are several ways to solve this problem; please discuss your assumptions in detail.

IPE, PEA

IPE- und IPEA-Träger — *Profilés IPE et IPEA*

$A_v = A - 2bt_f + (t_w + 2r)\,t_f$

$A_w = (h - t_f) \cdot t_w$

$S_y = ½\, W_{ply}$

$S_z = ½\, W_{plz}$

$W_{ely} = \dfrac{I_y}{h/2}$

$\overline{W}_y = \dfrac{I_y}{(h - t_f)/2}$

$W_{elz} = \dfrac{I_z}{b/2}$

Maximale Lagerlängen /
Longueurs maximales en stock:
$h \le 180$ 18 m
$h \ge 200$ 24 m
EURONORM 19 – 57,
DIN 1025/5, ASTM A 6,
Werksnorm/*Norme d'usine*

○ Das Verfahren PP nach SIA 263 ist für dieses Profil aus S355 bei reiner Biegung (n = 0) nicht anwendbar!

* Auch in S355J0 oder S355J2 ab Schweizer Lager erhältlich.

○ *La méthode PP selon SIA 263 n'est pas applicable pour ce profilé en acier S355 en flexion simple (n = 0)!*

* *Livrable en S355J0 ou S355J2 du stock suisse.*

IPE	m	Statische Werte / *Valeurs statiques*												
	kg/m	A mm²	A_v mm²	A_w mm²	I_y mm⁴	W_{ely} mm³	$\overline{W}_y$ mm³	W_{ply} mm³	i_y mm	I_z mm⁴	W_{elz} mm³	W_{plz} mm³	i_z mm	$K = I_x$ mm⁴
					x 10⁶	x 10³	x 10³	x 10³		x 10⁶	x 10³	x 10³		x 10⁶
80*	6,0	764	358	284	0,801	20,0	21,4	23,2	32,4	0,085	3,69	5,82	10,5	0,0067
100*	8,1	1030	508	387	1,71	34,2	36,3	39,4	40,7	0,159	5,79	9,15	12,4	0,0115
120*	10,4	1320	631	500	3,18	53,0	55,9	60,7	49,0	0,277	8,65	13,6	14,5	0,0169
140*	12,9	1640	764	626	5,41	77,3	81,3	88,3	57,4	0,449	12,3	19,2	16,5	0,0240
160*	15,8	2010	966	763	8,69	109	114	124	65,8	0,683	16,7	26,1	18,4	0,0353
180*	18,8	2390	1125	912	13,2	146	154	166	74,2	1,01	22,2	34,6	20,5	0,0472
200*	22,4	2850	1400	1070	19,4	194	203	221	82,6	1,42	28,5	44,6	22,4	0,0685
220*	26,2	3340	1588	1240	27,7	252	263	285	91,1	2,05	37,3	58,1	24,8	0,0898
240*	30,7	3910	1914	1430	38,9	324	338	367	99,7	2,84	47,3	73,9	26,9	0,127
270*	36,1	4590	2214	1710	57,9	429	446	484	112	4,20	62,2	97,0	30,2	0,157
300*	42,2	5380	2568	2050	83,6	557	578	628	125	6,04	80,5	125	33,5	0,198
330*	49,1	6260	3081	2390	117,7	713	739	804	137	7,88	98,5	154	35,5	0,276
360*	57,1	7270	3514	2780	162,7	904	937	1020	150	10,4	123	191	37,9	0,371
400*	66,3	8450	4269	3320	231,3	1160	1200	1310	165	13,2	146	229	39,5	0,504
450*	77,6	9880	5085	4090	337,4	1500	1550	1700	185	16,8	176	276	41,2	0,661
500*	90,7	11600	5987	4940	482,0	1930	1990	2190	204	21,4	214	336	43,1	0,886
550	106	13400	7234	5910	671,2	2440	2520	2790	223	26,7	254	401	44,5	1,22
600	122	15600	8378	6970	920,8	3070	3170	3510	243	33,9	308	486	46,6	1,65
750 x	137	17500	9290	8460	1599	4250	4340	○4860	303	51,7	393	614	54,4	1,36
750 x	147	18700	10540	9720	1661	4410	4510	5110	298	52,9	399	631	53,1	1,57
750 x	173	22100	11640	10700	2058	5400	5560	6220	305	68,7	515	810	55,7	2,71
750 x	196	25100	12730	11600	2403	6240	6450	7170	310	81,8	610	959	57,1	4,06
PEA														
120	8,7	1100	542	428	2,57	43,8	45,8	49,9	48,3	0,224	7,00	11,0	14,2	0,0101
140	10,5	1340	620	501	4,35	63,3	66,0	71,6	57,0	0,364	9,98	15,5	16,5	0,0133
160	12,7	1620	780	604	6,89	87,8	91,2	99,1	65,3	0,544	13,3	20,7	18,3	0,0191
180	15,4	1960	920	733	10,6	120	124	135	73,7	0,819	18,0	28,0	20,5	0,0265
200	18,4	2350	1147	855	15,9	162	167	182	82,3	1,17	23,4	36,5	22,3	0,0402
220	22,2	2830	1355	1050	23,2	214	222	240	90,5	1,71	31,2	48,5	24,6	0,0559
240	26,2	3330	1631	1190	32,9	278	288	312	99,4	2,40	40,0	62,4	26,8	0,0820
270	30,7	3920	1875	1420	49,2	368	381	412	112	3,58	53,0	82,3	30,2	0,101
300	36,5	4650	2225	1760	71,7	483	498	542	124	5,19	69,2	107	33,4	0,131
330	43,0	5470	2699	2060	102	626	645	702	137	6,85	85,6	133	35,4	0,190
360	50,2	6400	2972	2280	145	812	839	907	151	9,44	111	172	38,4	0,269
400	57,4	7310	3578	2700	203	1020	1050	1140	167	11,7	130	202	40,0	0,350
450	67,2	8560	4226	3300	298	1330	1370	1490	186	15,0	158	246	41,9	0,462
500	79,4	10100	5047	4050	429	1730	1780	1950	206	19,4	194	302	43,8	0,636
550	92,1	11700	6030	4780	600	2190	2260	2480	226	24,3	232	362	45,5	0,879
600	108	13700	7014	5680	829	2780	2860	3140	246	31,2	283	442	47,7	1,21

IPE- und IPEA-Träger — *Profilés IPE et IPEA* — IPE, PEA

Die Profile PER, IPEo und IPEv sind im Walzprogramm einzelner Werke aufgeführt. PEA 80 und PEA 100 sind ebenfalls normiert, aber kaum wirtschaftlich.

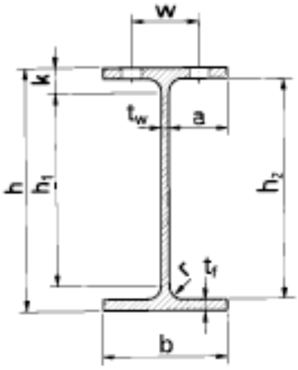
Les profilés PER, IPEo et IPEv figurent dans le programme de laminage de quelques aciéries. Les PEA 80 et PEA 100, également normalisés, sont peu économiques.

▒ Im allgemeinen nur ab Werk lieferbar. Mindestmengen und Termine beachten.

▒ *En général livrable d'usine uniquement. Tenir compte des quantités minimales et des délais.*

Walztoleranzen siehe Seite 116

Tolérances de laminage voir p. 116



IPE	m kg/m	h mm	b mm	t_w mm	t_f mm	r mm	h_1 mm	k mm	a mm	h_2 mm	w mm	$Ø_{max}$	U_m m^2/m	U_t m^2/t	IPE
		Profilmasse *Dimensions de la section*					Konstruktionsmasse *Dimensions de construction*						Oberfläche *Surface*		
80	6,0	80	46	3,8	5,2	5	60	10	21	70			0,328	54,8	80
100	8,1	100	55	4,1	5,7	7	74	13	25	89			0,400	49,5	100
120	10,4	120	64	4,4	6,3	7	92	14	29	107	36	M10	0,475	45,6	120
140	12,9	140	73	4,7	6,9	7	112	14	34	126	38	M10	0,551	42,6	140
160	15,8	160	82	5,0	7,4	9	126	17	38	145	44	M12	0,623	39,4	160
180	18,8	180	91	5,3	8,0	9	146	17	42	164	50	M12	0,698	37,1	180
200	22,4	200	100	5,6	8,5	12	158	21	47	183	56	M12	0,768	34,3	200
220	26,2	220	110	5,9	9,2	12	178	21	52	202	60	M16	0,848	32,4	220
240	30,7	240	120	6,2	9,8	15	190	25	56	220	68	M16	0,922	30,0	240
270	36,1	270	135	6,6	10,2	15	220	25	64	250	72	M20	1,04	28,8	270
300	42,2	300	150	7,1	10,7	15	248	26	71	279	80	M20	1,16	27,5	300
330	49,1	330	160	7,5	11,5	18	270	30	76	307	86	M24	1,25	25,5	330
360	57,1	360	170	8,0	12,7	18	298	31	81	335	90	M24	1,35	23,6	360
400	66,3	400	180	8,6	13,5	21	330	35	85	373	96	M27	1,47	22,2	400
450	77,6	450	190	9,4	14,6	21	378	36	90	421	106	M27	1,61	20,7	450
500	90,7	500	200	10,2	16,0	21	426	37	94	468	110	M27	1,74	19,2	500
550	106	550	210	11,1	17,2	24	468	41	99	516	120	M27	1,88	17,7	550
600	122	600	220	12,0	19,0	24	514	43	104	562	120	M27	2,02	16,6	600
750 x 137		753	263	11,5	17,0	17	685	34	126	719	120	M27	2,51	18,3	750x137
750 x 147		753	265	13,2	17,0	17	685	34	126	719	120	M27	2,51	17,1	750x147
750 x 173		762	267	14,4	21,6	17	685	39	126	719	120	M27	2,53	14,6	750x173
750 x 196		770	268	15,6	25,4	17	685	42	126	719	120	M27	2,55	13,0	750x196
PEA															**PEA**
120	8,7	118	64	3,8	5,1	7	93	12	30	107	36	M10	0,472	54,5	120
140	10,5	137	73	3,8	5,6	7	111	13	34	126	38	M10	0,547	52,1	140
160	12,7	157	82	4,0	5,9	9	127	15	39	145	44	M12	0,619	48,7	160
180	15,4	177	91	4,3	6,5	9	145	16	43	164	50	M12	0,694	45,1	180
200	18,4	197	100	4,5	7,0	12	159	19	47	183	56	M12	0,764	41,5	200
220	22,2	217	110	5,0	7,7	12	177	20	52	202	60	M16	0,843	38,0	220
240	26,2	237	120	5,2	8,3	15	189	24	57	220	68	M16	0,918	35,0	240
270	30,7	267	135	5,5	8,7	15	219	24	64	250	72	M20	1,04	33,9	270
300	36,5	297	150	6,1	9,2	15	247	25	71	279	80	M20	1,16	31,8	300
330	43,0	327	160	6,5	10,0	18	271	28	76	307	86	M24	1,25	29,1	330
360	50,2	357	170	6,6	11,5	18	297	30	81	334	90	M24	1,35	26,9	360
400	57,4	397	180	7,0	12,0	21	331	33	86	373	96	M27	1,46	25,4	400
450	67,2	447	190	7,6	13,1	21	377	35	91	421	106	M27	1,60	23,8	450
500	79,4	497	200	8,4	14,7	21	425	36	95	468	110	M27	1,74	21,9	500
550	92,1	547	210	9,0	15,7	24	467	40	100	516	120	M27	1,88	20,4	550
600	108	597	220	9,8	17,5	24	513	42	105	562	120	M27	2,01	18,6	600

Suggested Solution

1.1-

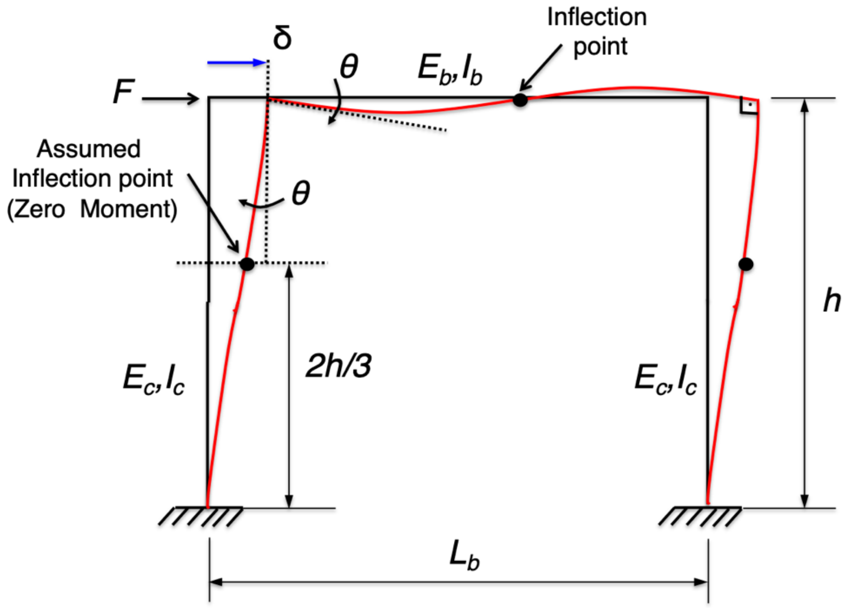


Figure 1.2. Lateral deflection shape and assumed inflection points

NOTE: The inflection point is assumed at $2h/3$ given that this is the first storey of the steel MRF; however, an alternative solution would be acceptable by assuming the inflection point at $h/2$.

The moment diagram prior to the onset of the first plastic hinge is as follows,

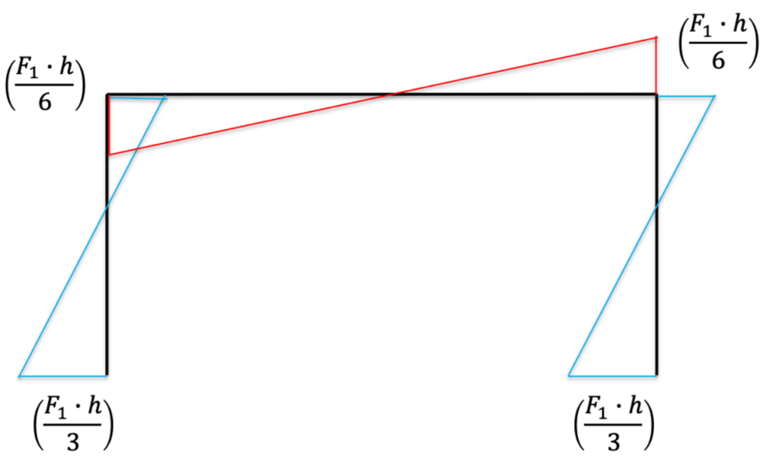


Figure 1.3. Moment diagram at first plastic hinge

$$M_{pl,y}^{(b)} = W_{pl,y}^{(b)} \cdot f_y = 628x10^3 \cdot 0.355 = 222940kNmm \quad (1.1)$$

$$M_{pl,y}^{(c)} = W_{pl,y}^{(c)} \cdot f_y = 2410x10^3 \cdot 0.355 = 855550kNmm \quad (1.2)$$

Therefore, the plastic resistance of the steel column is larger than that of the steel beam; hence, the first plastic hinge will form at the beam based on the beam-to-column joint equilibrium. Therefore,

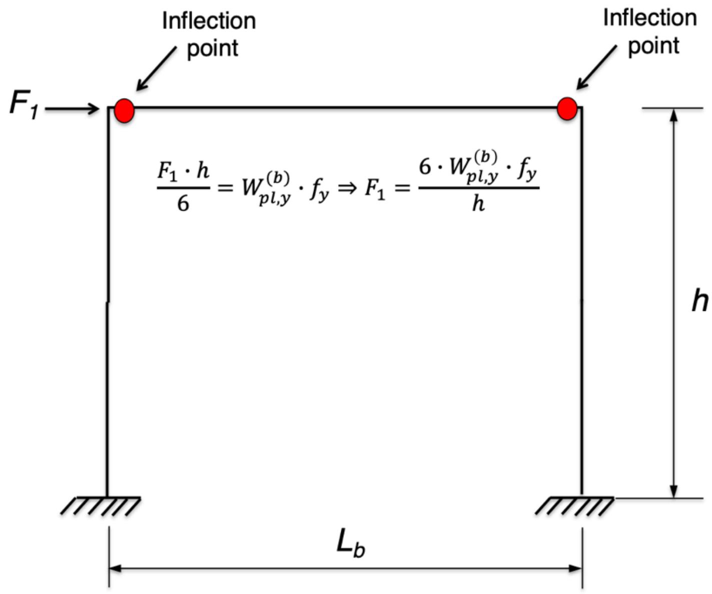


Figure 1.4. Assumed plastic hinge position (beam)

$$F_1 = \frac{6 \cdot M_{pl,y}^{(b)}}{h} = 6 \cdot \frac{222940}{4000} = 334.4kN \tag{1.3}$$

However, we need to check if the first plastic hinge may form at the base of the column before the beam end. This would imply,

$$F_1 = \frac{3 \cdot M_{pl,y}^{(c)}}{h} = 3 \cdot \frac{855550}{4000} = 641.7kN \tag{1.4}$$

Therefore, based on the principle of minimum energy, the minimum load that should be applied to the structure to actually develop a plastic hinge is, $F_1 = 334.4kN$.

The corresponding deflection at this point is as follows,

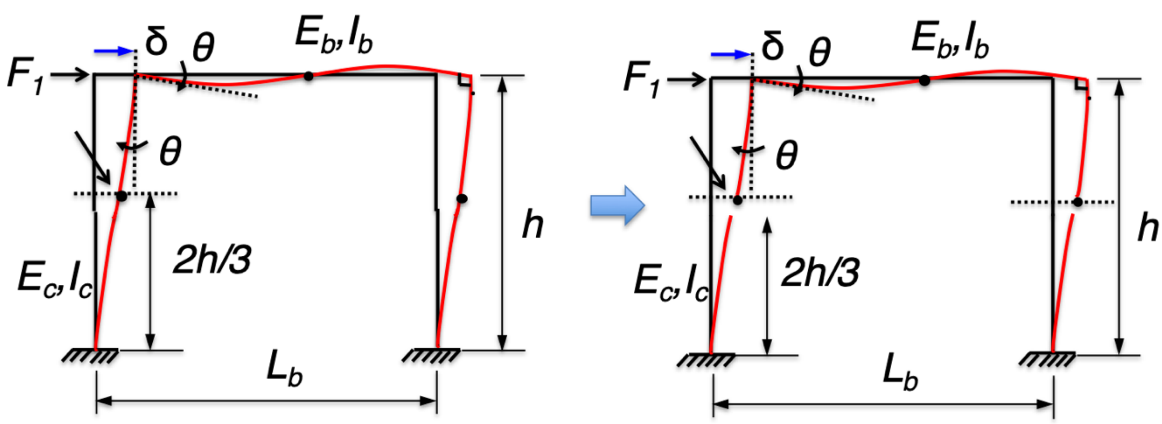


Figure 4.5. decomposition of lateral deflections based on the assumed inflection point

Therefore,

$$\delta_1 = \frac{F_1 \cdot (2 \cdot h/3)^3}{2 \cdot 3 \cdot E \cdot I_c} + \frac{F_1 \cdot (h/3)^2}{6} \cdot \left[\frac{(h/3)}{E \cdot I_c} + \frac{L_b}{2 \cdot E \cdot I_b}\right] =$$

$$334.4 \cdot \frac{(2 \cdot 4000/3)^3}{2 \cdot 3 \cdot 200 \cdot 3.6667x10^8} + 334.4 \cdot \frac{(4000/3)^2}{6} \cdot \left[\frac{4000/3}{200 \cdot 3.6667x10^8} + \frac{8000}{2 \cdot 200 \cdot 8.356x10^7}\right]$$
$$= 39.9mm \quad (1.5)$$

To calculate the incremental load at which a collapse mechanism forms,

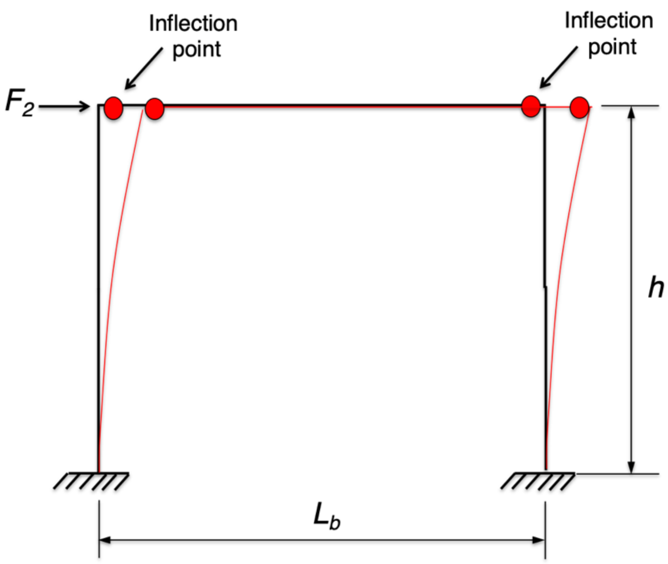


Figure 1.6. Lateral deflection after the formation of the first plastic hinge

The moment diagram in this case is as follows,

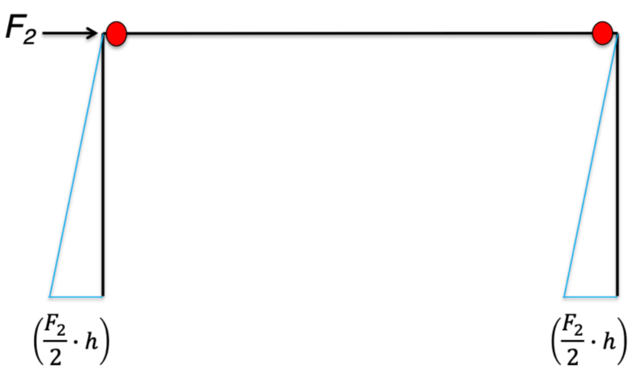


Figure 1.7. Moment diagram after the formation of the first plastic hinge

Therefore,

$$\frac{F_2}{2} \cdot h = M_{pl,y}^{(c)} - F1 \cdot \frac{h}{3} \Rightarrow F_2 = \frac{2 \cdot \left(M_{pl,y}^{(c)} - F1 \cdot \frac{h}{3}\right)}{\text{h}} = \frac{2 \cdot \left(855550 - 334.4 \cdot \frac{4000}{3}\right)}{4000} = 204.8kN \quad (1.6)$$

The corresponding deflection at this point is as follows,

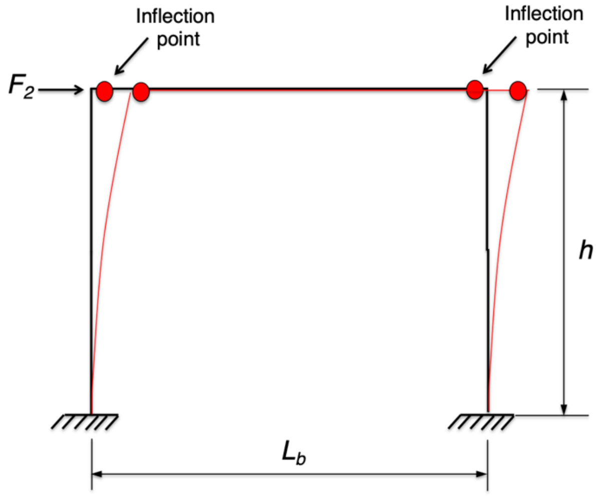


Figure 1.8. Lateral deflection after the formation of the first plastic hinge

$$\delta_2 = \frac{F_2 \cdot h^3}{6 \cdot E \cdot I_c} = \frac{204.8 \cdot 4000^3}{6 \cdot 200 \cdot 3.6667x10^8} = 29.8mm \tag{1.7}$$

1.2-

When P-Delta effects are considered in the analysis, the load deformation equilibrium path rotates by $P \cdot \frac{\delta}{h}$; in this case, the load, P is the total vertical load that is supported by the floor. Therefore, $P = 400kN$.

At the corresponding deflections, δ_1, and $\delta_{tot} = \delta_1 + \delta_2$, the P-Delta forces should be:

$$V_{P-\Delta}^{(1)} = 400 \cdot \frac{39.9}{4000} = 3.99kN \tag{1.8}$$

$$V_{P-\Delta}^{(tot)} = 400 \cdot \frac{39.9 + 29.8}{4000} = 6.97kN \tag{1.9}$$

Therefore, the corresponding collapse load in this case should be:

$$F_c = F_1 + F_2 - V_{P-\Delta}^{tot} = 344.4 + 204.8 - 6.97 = 532.3kN \tag{1.10}$$

Note that the collapse load by including P-Delta effects is close to that without considering P-Delta effects because the stability coefficient is well below 0.10. This may not be the case if the axial load increases.

1.3-

Similarly, if we consider $P = 4000kN$.

At the corresponding deflections, δ_1, and $\delta_{tot} = \delta_1 + \delta_2$, the P-Delta forces should be:

$$V_{P-\Delta}^{(1)} = 4000 \cdot \frac{39.9}{4000} = 39.9kN \tag{1.11}$$

$$V_{P-\Delta}^{(tot)} = 4000 \cdot \frac{39.9 + 29.8}{4000} = 69.7kN \tag{1.12}$$

Therefore, the corresponding collapse load in this case should be:

$$F_c = F_1 + F_2 - V_{P-\Delta}^{tot} = 344.4 + 204.8 - 69.7 = 479.5kN \tag{1.13}$$

1.4-

The solutions from the previous questions are compared in an equilibrium path. Notice that the higher the axial load the lower the collapse load. Moreover, notice that once the collapse load is attained, the equilibrium path has a negative stiffness; therefore, it is unstable.

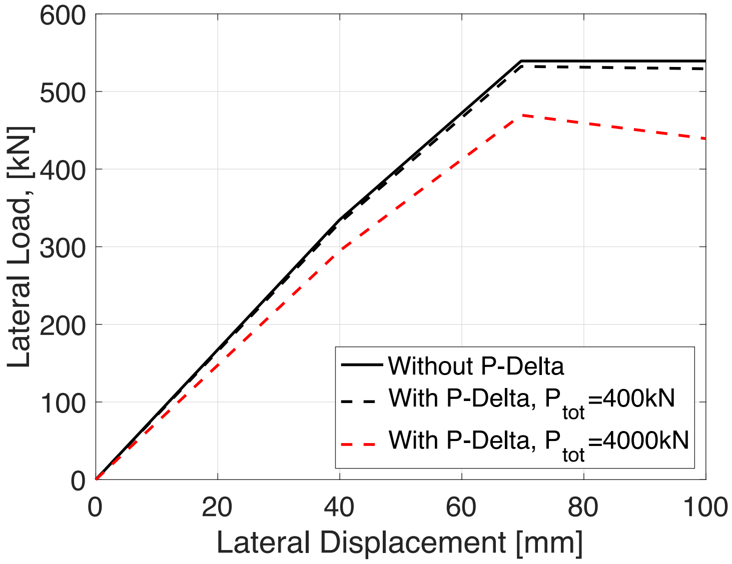


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Figure 1.9. Equilibrium path of the structure by excluding and including P-Delta effects

Question 2 – Buckling analysis (40 points)

Calculate the critical load of the steel column ($E = 210kN/mm^2$) shown below (see Figure 2) by solving the differential equation that describes the deflection of the column along its length. For your calculations consider the following: $c = 8{,}00m$ and $I_0 = 18260cm^4$.

Hint: The column is comprised of two members with different geometric characteristics. You should derive the deflection equations for each member. The critical load should be calculated based on the system of differential equations and the buckling determinant.

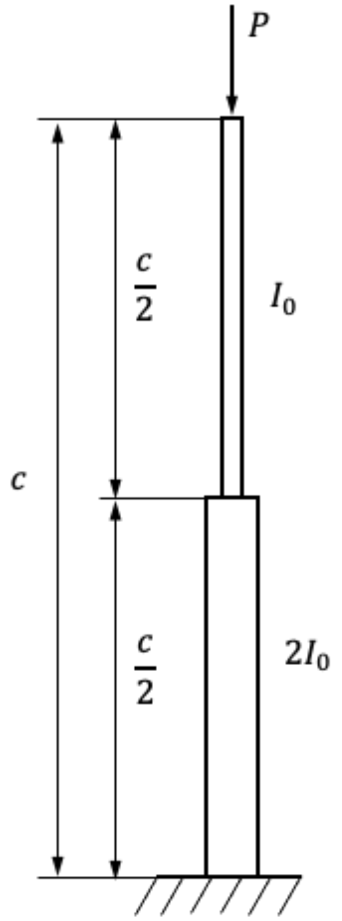


Figure 2 – Steel column stability

Problem definition:
Calculate the critical buckling load of the structure in Fig. 2.1

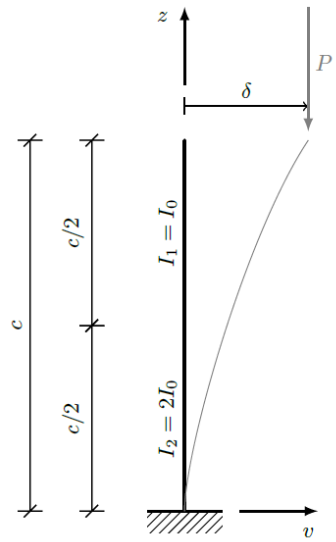


Figure 2.1- Cantilever with change in cross-section inertia at $c/2$ mark. Region above $c/2$ mark is designated by region 1 and below by region 2. In grey is the deformed shape according and corresponding load point for demand P. The lateral displacement at the top of the cantilever is defined as δ, that is $v(c) = \delta$

Solution

$$\begin{cases} EI_1 \cdot \dfrac{\partial^2 v_1}{\partial z^2} = P \cdot (\delta - v_1) \\ EI_2 \cdot \dfrac{\partial^2 v_2}{\partial z^2} = P \cdot (\delta - v_2) \end{cases} \rightarrow \begin{cases} \dfrac{\partial^2 v_1}{\partial z^2} + \dfrac{P}{EI_1} v_1 = \dfrac{P}{EI_1}\delta \\ \dfrac{\partial^2 2}{\partial z^2} + \dfrac{P}{EI_2} v_2 = \dfrac{P}{EI_2}\delta \end{cases} \tag{2.1}$$

For convenience, let us define:

$$k_1^2 = \frac{P}{EI_1} \text{ and } k_2^2 = \frac{P}{EI_2} \tag{2.2}$$

Which yields:

$$\begin{cases} \dfrac{\partial^2 v_1}{\partial z^2} + k_1^2 \cdot v_1 = k_1^2 \delta \\ \dfrac{\partial^2 v_2}{\partial z^2} + k_2^2 \cdot v_1 = k_2^2 \delta \end{cases} \tag{2.3}$$

The system of differential equations above has homogeneous and particular solutions of the form:

$$\begin{cases} v_1 = A \cos(k_1 z) + B \sin(k_1 z) + \delta \\ v_2 = C \cos(k_2 z) + D \sin(k_2 z) + \delta \end{cases} \tag{2.4}$$

Let us now impose the boundary conditions of our problem. With respect to bar #2:

$$\begin{cases} v_2(0) = 0 \to D = -\delta \\ \dfrac{\partial v_2}{\partial z}(0) \to C \cdot k_2 = 0 \end{cases} \to v_2 = -\delta \cdot \cos(k_2 z) + \delta \to v_2 = -\delta(1 - \cos(k_2 z)) \qquad (2.5)$$

As for bar #1:

$$\begin{cases} v_1(c) = \delta \to A\ \cos(k_1 c) + B \sin(k_1 c) + \delta = \delta \\ v_1\left(\dfrac{c}{2}\right) = v_2\left(\dfrac{c}{2}\right) \to A\ \cos\left(\dfrac{k_1 c}{2}\right) + B \sin\left(\dfrac{k_1 c}{2}\right) + \delta = \delta\left(1 - \cos\left(\dfrac{k_2 c}{2}\right)\right) \end{cases}$$

$$\to \begin{cases} A\ \cos(k_1 c) + B \sin(k_1 c) = 0 \\ A\ \cos\left(\dfrac{k_1 c}{2}\right) + B \sin\left(\dfrac{k_1 c}{2}\right) = -\delta \cdot \cos\left(\dfrac{k_2 c}{2}\right) \end{cases} \qquad (2.6)$$

Let us now solve the system of equation using Cramer’s Rule:

$$D = \begin{vmatrix} \cos(k_1 c) & \sin(k_1 c) \\ \cos\left(\dfrac{k_1 c}{2}\right) & \sin\left(\dfrac{k_1 c}{2}\right) \end{vmatrix} = \cos(k_1 c) \sin\left(\dfrac{k_1 c}{2}\right) - \sin(k_1 c) \cos\left(\dfrac{k_1 c}{2}\right)$$

$$= \sin\left(k_1\left(\frac{c}{2} - c\right)\right) = -\sin\left(k_1 \cdot \frac{c}{2}\right) \qquad (2.7)$$

$$D_A = \begin{vmatrix} 0 & \sin(k_1 c) \\ -\delta \cdot \cos\left(\dfrac{k_2 c}{2}\right) & \sin\left(\dfrac{k_1 c}{2}\right) \end{vmatrix} = \delta \cdot \sin(k_1 c) \cos\left(\frac{k_2 c}{2}\right) \qquad (2.8)$$

$$D_B = \begin{vmatrix} \cos(k_1 c) & 0 \\ \cos\left(\dfrac{k_1 c}{2}\right) & -\delta \cdot \cos\left(\dfrac{k_2 c}{2}\right) \end{vmatrix} = -\delta \cdot \cos(k_1 c) \cos\left(\frac{k_2 c}{2}\right) \qquad (2.9)$$

$$A = \frac{D_A}{D} = -\delta \frac{\sin(k_1 c) \cos\left(\frac{k_2 c}{2}\right)}{\sin\left(k_1 \cdot \frac{c}{2}\right)} = B \cdot \tan(k_1 c) \qquad (2.10)$$

$$B = \frac{D_B}{D} = \delta \frac{\cos(k_1 c) \cos\left(\frac{k_2 c}{2}\right)}{\sin\left(k_1 \cdot \frac{c}{2}\right)} \qquad (2.11)$$

Imposing now a fifth boundary condition, that the tangent at the beginning of bar 1 and end of bar 2 have to be the same, i.e.:

$$\frac{\partial v_1}{\partial z}\left(\frac{c}{2}\right) = \frac{\partial v_2}{\partial z}\left(\frac{c}{2}\right) \qquad (2.12)$$

We have:

$$-Ak_1 \sin\left(k_1 \cdot \frac{c}{2}\right) + Bk_1 \cos\left(\frac{k_1 c}{2}\right) = \delta k_2 \sin\left(k_2 \cdot \frac{c}{2}\right) \tag{2.13}$$

Replacing A and B by the solutions of the system we solved with Cramer's rule, yields:

$$\delta k_1 \frac{\sin(k_1 c)\cos\left(\frac{k_2 c}{2}\right)}{\sin\left(k_1 \cdot \frac{c}{2}\right)} \sin\left(k_1 \cdot \frac{c}{2}\right) + k_1 \delta \frac{\cos(k_1 c)\cos\left(\frac{k_2 c}{2}\right)}{\sin\left(k_1 \cdot \frac{c}{2}\right)} \cos\left(\frac{k_1 c}{2}\right) = \delta k_2 \sin\left(k_2 \cdot \frac{c}{2}\right) \tag{2.14}$$

From the above, one can see that δ can be completely removed from the equation, meaning that no matter the displacement, the stability conditions must hold. Simplifying gives thus:

$$\frac{\sin(k_1 c)\cos\left(\frac{k_2 c}{2}\right)\sin\left(k_1 \cdot \frac{c}{2}\right)}{\sin\left(k_1 \cdot \frac{c}{2}\right)\sin\left(k_2 \cdot \frac{c}{2}\right)} + \frac{\cos(k_1 c)\cos\left(\frac{k_2 c}{2}\right)\cos\left(\frac{k_1 c}{2}\right)}{\sin\left(k_1 \cdot \frac{c}{2}\right)\cos\left(\frac{k_1 c}{2}\right)} = \frac{k_2}{k_1} \tag{2.15}$$

$$\cot\left(\frac{k_2 c}{2}\right) \frac{\sin(k_1 c)\sin\left(k_1 \cdot \frac{c}{2}\right) + \cos(k_1 c)\cos\left(\frac{k_2 c}{2}\right)}{\sin\left(k_1 \cdot \frac{c}{2}\right)} = \frac{k_2}{k_1} \tag{2.16}$$

$$\frac{\cot\left(\frac{k_2 c}{2}\right)}{\sin\left(\frac{k_1 c}{2}\right)} \cos\left(c - \frac{c}{2}\right) = \frac{k_2}{k_1} \tag{2.17}$$

$$\cot\left(\frac{k_2 c}{2}\right)\cot\left(\frac{k_1 c}{2}\right) = \frac{k_2}{k_1} \rightarrow \tan\left(\frac{k_2 c}{2}\right)\tan\left(\frac{k_1 c}{2}\right) = \frac{k_1}{k_2} \tag{2.18}$$

In our case, we have a fixed ratio for the inertia, which implies that:

$$\frac{k_1}{k_2} = \frac{\sqrt{\frac{P}{EI_0}}}{\sqrt{\frac{P}{2EI_0}}} = \sqrt{2} \tag{2.19}$$

Giving:

$$\tan\left(\frac{k_2 c}{2}\right)\tan\left(\frac{k_1 c}{2}\right) - \sqrt{2} = 0 \tag{2.20}$$

Since a change in variable will make things easier to compute, let's say that $\frac{k_2 c}{2} = \alpha$, and then, using $k_1 = \sqrt{2} k_2$, we can express the previous equation as:

$$\tan(\alpha)\tan\left(\sqrt{2}\,\alpha\right) - \sqrt{2} = 0 \tag{2.21}$$

With this equation one needs only to find the angle for the root. You can use any number of methods here, like the Newton-Raphson or the Bi-Section methods. The solution to this equation is around 41.18° or 0.229 radians.

The critical buckling load us then obtained by using this root back into our change of variable:

$$k_2 = \frac{2\alpha}{c} \rightarrow \sqrt{\frac{P}{2EI_0}} = \frac{2\alpha}{c} \rightarrow P_{cr} \approx 4.14\frac{EI_0}{c^2} \tag{2.22}$$

Question 3 – Interaction of axial load and bending (25 points)

Calculate the critical load of the $HEA\ 300$ steel beam ($E = 210 kN/mm^2$) shown below (see Figure 3).

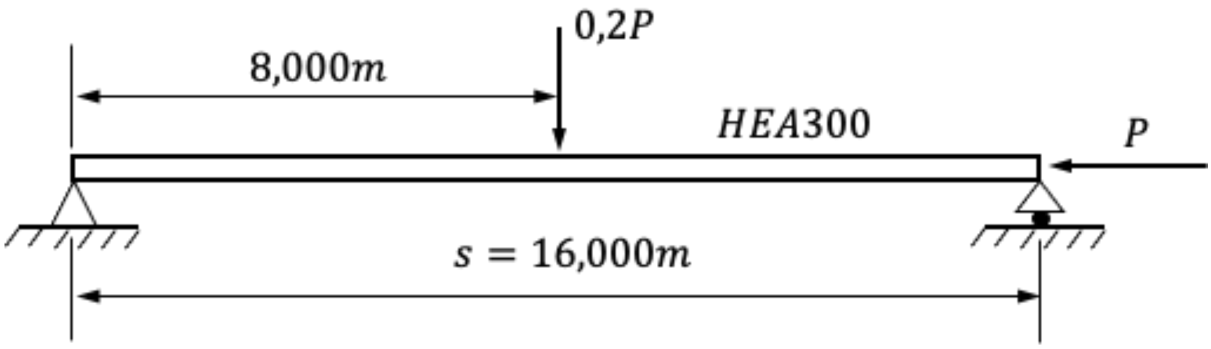


Figure 3 – steel beam under axial load and bending

HEA

Breitflanschträger HEA — *Profilés à larges ailes* HEA

$A_v = A - 2bt_f + (t_w + 2r)\,t_f$

$A_w = (h - t_f) \cdot t_w$ $\quad W_{ely} = \dfrac{I_y}{h/2}$

$S_y = ½\, W_{ply}$

$S_z = ½\, W_{plz}$ $\quad \overline{W}_y = \dfrac{I_y}{(h-t_f)/2}$

$W_{elz} = \dfrac{I_z}{b/2}$

Maximale Lagerlängen / *Longueurs maximales en stock:*
h ≤ 180 18 m
h ≥ 200 24 m

EURONORM 53 – 62, DIN 1025/3

Andere Bezeichnungen / *Autres désignations* } DIE, IPBl

Schlankheitskriterien nach SIA 263 für dieses Profil aus S355 bei reiner Biegung (n = 0) nicht erfüllt für
○ Verfahren PP
● Verfahren EP

* Auch in S355J0 oder S355J2 ab Schweizer Lager erhältlich.

Critères d'élancement selon SIA 263 pour ce profilé en acier S355 en flexion simple (n = 0) non remplies pour
○ *méthode PP*
● *méthode EP*

* *Livrable en S355J0 ou S355J2 du stock suisse.*

<table>
<tr><th rowspan="2">HEA</th><th rowspan="2">m
kg/m</th><th colspan="14">Statische Werte / Valeurs statiques</th></tr>
<tr><th>A
mm²</th><th>A_v
mm²</th><th>A_w
mm²</th><th>I_y
mm⁴</th><th>W_{ely}
mm³</th><th>W̄_y
mm³</th><th>W_{ply}
mm³</th><th>i_y
mm</th><th>I_z
mm⁴</th><th>W_{elz}
mm³</th><th>W_{plz}
mm³</th><th>i_z
mm</th><th>K = I_x
mm⁴</th></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td>x 10⁶</td><td>x 10³</td><td>x 10³</td><td>x 10³</td><td></td><td>x 10⁶</td><td>x 10³</td><td>x 10³</td><td></td><td>x 10⁶</td></tr>
<tr><td>100*</td><td>16,7</td><td>2120</td><td>756</td><td>440</td><td>3,49</td><td>72,8</td><td>79</td><td>83,0</td><td>40,6</td><td>1,34</td><td>26,8</td><td>41,2</td><td>25,1</td><td>0,0520</td></tr>
<tr><td>120*</td><td>19,9</td><td>2530</td><td>846</td><td>530</td><td>6,06</td><td>106</td><td>114</td><td>119</td><td>48,9</td><td>2,31</td><td>38,5</td><td>58,9</td><td>30,2</td><td>0,0696</td></tr>
<tr><td>140*</td><td>24,7</td><td>3140</td><td>1012</td><td>685</td><td>10,3</td><td>155</td><td>166</td><td>173</td><td>57,3</td><td>3,89</td><td>55,6</td><td>84,8</td><td>35,2</td><td>0,0803</td></tr>
<tr><td>160*</td><td>30,4</td><td>3880</td><td>1321</td><td>858</td><td>16,7</td><td>220</td><td>234</td><td>245</td><td>65,7</td><td>6,16</td><td>76,9</td><td>118</td><td>39,8</td><td>0,118</td></tr>
<tr><td>180*</td><td>35,5</td><td>4530</td><td>1447</td><td>969</td><td>25,1</td><td>294</td><td>311</td><td>○325</td><td>74,5</td><td>9,25</td><td>103</td><td>○157</td><td>45,2</td><td>0,147</td></tr>
<tr><td>200*</td><td>42,3</td><td>5380</td><td>1808</td><td>1170</td><td>36,9</td><td>389</td><td>410</td><td>○429</td><td>82,8</td><td>13,4</td><td>134</td><td>○204</td><td>49,8</td><td>0,204</td></tr>
<tr><td>220*</td><td>50,5</td><td>6430</td><td>2067</td><td>1390</td><td>54,1</td><td>515</td><td>544</td><td>○568</td><td>91,7</td><td>19,5</td><td>178</td><td>○271</td><td>55,1</td><td>0,281</td></tr>
<tr><td>240*</td><td>60,3</td><td>7680</td><td>2518</td><td>1640</td><td>77,6</td><td>675</td><td>712</td><td>○745</td><td>101</td><td>27,7</td><td>231</td><td>○352</td><td>60,0</td><td>0,410</td></tr>
<tr><td>260*</td><td>68,2</td><td>8680</td><td>2876</td><td>1780</td><td>104,5</td><td>836</td><td>881</td><td>●920</td><td>110</td><td>36,7</td><td>282</td><td>●430</td><td>65,0</td><td>0,520</td></tr>
<tr><td>280*</td><td>76,4</td><td>9730</td><td>3174</td><td>2060</td><td>136,7</td><td>1010</td><td>1060</td><td>●1110</td><td>119</td><td>47,6</td><td>340</td><td>●518</td><td>70,0</td><td>0,614</td></tr>
<tr><td>300*</td><td>88,3</td><td>11300</td><td>3728</td><td>2350</td><td>182,6</td><td>1260</td><td>1320</td><td>●1380</td><td>127</td><td>63,1</td><td>421</td><td>●641</td><td>74,9</td><td>0,842</td></tr>
<tr><td>320*</td><td>97,6</td><td>12400</td><td>4113</td><td>2650</td><td>229,3</td><td>1480</td><td>1560</td><td>○1630</td><td>136</td><td>69,9</td><td>466</td><td>○710</td><td>74,9</td><td>1,09</td></tr>
<tr><td>340*</td><td>105</td><td>13300</td><td>4495</td><td>2980</td><td>276,9</td><td>1680</td><td>1770</td><td>1850</td><td>144</td><td>74,4</td><td>496</td><td>756</td><td>74,6</td><td>1,29</td></tr>
<tr><td>360*</td><td>112</td><td>14300</td><td>4896</td><td>3320</td><td>330,9</td><td>1890</td><td>1990</td><td>2090</td><td>152</td><td>78,9</td><td>526</td><td>802</td><td>74,3</td><td>1,51</td></tr>
<tr><td>400*</td><td>125</td><td>15900</td><td>5733</td><td>4080</td><td>450,7</td><td>2310</td><td>2430</td><td>2560</td><td>168</td><td>85,6</td><td>571</td><td>873</td><td>73,4</td><td>1,91</td></tr>
<tr><td>450</td><td>140</td><td>17800</td><td>6578</td><td>4820</td><td>637,2</td><td>2900</td><td>3040</td><td>3220</td><td>189</td><td>94,7</td><td>631</td><td>966</td><td>72,9</td><td>2,49</td></tr>
<tr><td>500</td><td>155</td><td>19800</td><td>7472</td><td>5600</td><td>869,7</td><td>3550</td><td>3730</td><td>3950</td><td>210</td><td>103,7</td><td>691</td><td>1060</td><td>72,4</td><td>3,18</td></tr>
<tr><td>550</td><td>166</td><td>21200</td><td>8372</td><td>6450</td><td>1119</td><td>4150</td><td>4340</td><td>4620</td><td>230</td><td>108,2</td><td>721</td><td>1110</td><td>71,5</td><td>3,61</td></tr>
<tr><td>600</td><td>178</td><td>22600</td><td>9321</td><td>7340</td><td>1412</td><td>4790</td><td>5000</td><td>5350</td><td>250</td><td>112,7</td><td>751</td><td>1160</td><td>70,5</td><td>4,08</td></tr>
<tr><td>650</td><td>190</td><td>24200</td><td>10320</td><td>8290</td><td>1752</td><td>5470</td><td>5710</td><td>6140</td><td>269</td><td>117,2</td><td>782</td><td>1200</td><td>69,7</td><td>4,59</td></tr>
<tr><td>700</td><td>204</td><td>26000</td><td>11700</td><td>9610</td><td>2153</td><td>6240</td><td>6490</td><td>7030</td><td>288</td><td>121,8</td><td>812</td><td>1260</td><td>68,4</td><td>5,23</td></tr>
<tr><td>800</td><td>224</td><td>28600</td><td>13880</td><td>11400</td><td>3034</td><td>7680</td><td>7960</td><td>8700</td><td>326</td><td>126,4</td><td>843</td><td>1310</td><td>66,5</td><td>6,10</td></tr>
<tr><td>900</td><td>252</td><td>32100</td><td>16330</td><td>13800</td><td>4221</td><td>9480</td><td>9820</td><td>10800</td><td>363</td><td>135,5</td><td>903</td><td>1410</td><td>65,0</td><td>7,51</td></tr>
<tr><td>1000</td><td>272</td><td>34700</td><td>18460</td><td>15800</td><td>5538</td><td>11190</td><td>11550</td><td>12800</td><td>400</td><td>140,0</td><td>934</td><td>1470</td><td>63,5</td><td>8,37</td></tr>
</table>

Breitflanschträger HEA — *Profilés à larges ailes HEA* — HEA

Die Profile HEAA sind im Walzprogramm einzelner Werke aufgeführt.

Anstelle des nicht mehr gewalzten Profils HEA 1100 können HL-Profile verwendet werden, siehe Seiten 40/41.

w_1 mit $Ø_{max}$ nur für versetzte Schrauben.

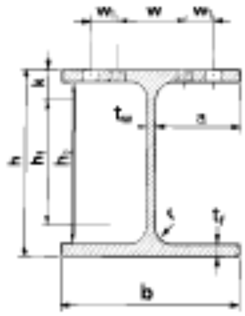
Walztoleranzen siehe Seite 116

Les profilés HEAA figurent dans le programme de laminage de quelques aciéries.

Au lieu du profilé HEA 1100 qui n'est plus laminé, on utilisera des profilés HL (voir pages 40/41).

w_1 avec $Ø_{max}$ seulement pour boulons décalés.

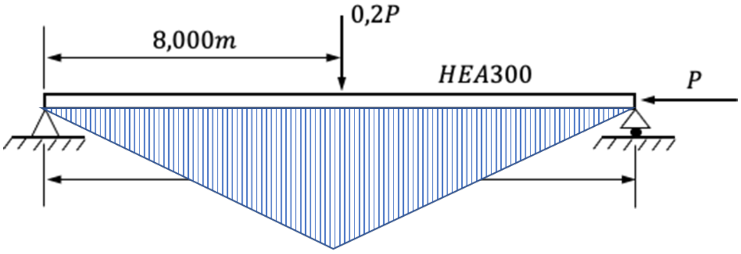
Tolérances de laminage voir page 116



HEA	m	Profilmasse *Dimensions de la section*					Konstruktionsmasse *Dimensions de construction*							Oberfläche *Surface*		HEA
	kg/m	h mm	b mm	t_w mm	t_f mm	r mm	h_1 mm	k mm	a mm	h_2 mm	w mm	w_1 mm	$Ø_{max}$	U_m m²/m	U_t m²/t	
100	16,7	96	100	5	8	12	56	20	47	80	56		M12	0,561	33,6	100
120	19,9	114	120	5	8	12	74	20	57	98	66		M16	0,677	34,0	120
140	24,7	133	140	5,5	8,5	12	91	21	67	116	76		M20	0,794	32,1	140
160	30,4	152	160	6	9	15	104	24	77	134	86		M20	0,906	29,8	160
180	35,5	171	180	6	9,5	15	121	25	87	152	100		M24	1,02	28,7	180
200	42,3	190	200	6,5	10	18	134	28	96	170	110		M24	1,14	26,9	200
220	50,5	210	220	7	11	18	152	29	106	188	120		M24	1,26	24,9	220
240	60,3	230	240	7,5	12	21	164	33	116	206	94	35	M24	1,37	22,7	240
260	68,2	250	260	7,5	12,5	24	176	37	126	225	100	40	M24	1,48	21,7	260
280	76,4	270	280	8	13	24	196	37	136	244	110	45	M24	1,60	21,0	280
300	88,3	290	300	8,5	14	27	208	41	145	262	120	45	M27	1,72	19,5	300
320	97,6	310	300	9	15,5	27	224	43	145	279	120	45	M27	1,76	18,0	320
340	105	330	300	9,5	16,5	27	242	44	145	297	120	45	M27	1,79	17,1	340
360	112	350	300	10	17,5	27	260	45	145	315	120	45	M27	1,83	16,4	360
400	125	390	300	11	19	27	298	46	144	352	120	45	M27	1,91	15,3	400
450	140	440	300	11,5	21	27	344	48	144	398	120	45	M27	2,01	14,4	450
500	155	490	300	12	23	27	390	50	144	444	120	45	M27	2,11	13,6	500
550	166	540	300	12,5	24	27	438	51	143	492	120	45	M27	2,21	13,3	550
600	178	590	300	13	25	27	486	52	143	540	120	45	M27	2,31	13,0	600
650	190	640	300	13,5	26	27	534	53	143	588	120	45	M27	2,41	12,7	650
700	204	690	300	14,5	27	27	582	54	142	636	120	45	M27	2,50	12,3	700
800	224	790	300	15	28	30	674	58	142	734	130	40	M27	2,70	12,0	800
900	252	890	300	16	30	30	770	60	142	830	130	40	M27	2,90	11,5	900
1000	272	990	300	16,5	31	30	868	61	141	928	130	40	M27	3,10	11,4	1000

Solution:

Figure 3.1 shows the bending moment diagram of the beam:



$$M_{Ed} = \frac{0{,}2PL}{4}$$

We assume that the beam bending occurs in the strong axis direction. For notation, we denote Y the strong axis and Z the weak axis.

Section classification:
In order to classify our HEA 300 cross section, we assume that it made of S355 ($f_y = 355$ MPa) steel.
Moreover, since in this case we have combined compressive axial load and a linear bending moment diagram, we will use Tables 5a and 5b of SIA 263 in the case of only compressive load.
-Web slenderness: $\frac{h_1}{t_w} = \frac{208}{8.5} = 24.47 \leq 33 \cdot \varepsilon = 33 \cdot 0.814 = 26.86 \rightarrow$ Class 1

-Flange slenderness: $\frac{a-r}{t_f} = \frac{145-27}{14} = 8.43 \leq 14 \cdot \varepsilon = 11.4 \rightarrow$ Class 3

Therefore, the HEA 300 is class 3 and therefore, we should use $M_{Rd} = M_{el}$ for our computations.

Since we assumed that the bending was with respect to the strong axis, we have:

$$M_{Rd,y} = \frac{W_{el,y} \cdot f_y}{\gamma_{M_1}} = \frac{1260 \cdot 10^3 \cdot 355}{1.05} = 426\ kNm \tag{3.1}$$

$$N_{cr,y} = \frac{\pi^2 E I_y}{L_{cr,y}^2} = \frac{\pi^2 \cdot 210000 \cdot 182.6 \cdot 10^6}{16000^2} = 1478\ kN \tag{3.2}$$

Since the critical lengths for both the weak and strong axis are equal, the weak axis will be the determinant case for the computation of the buckling resistance $N_{k,Rd,z}$
According to the SIA 263, § 4.5.2.2, we have:

$$N_{k,Rd,z} = \frac{\chi_k f_y A}{\gamma_{M1}} \tag{3.3}$$

With:

$$\chi_k = \frac{1}{\phi_k + \sqrt{\phi_k^2 - \bar{\lambda}_k^2}} \leq 1 \tag{3.4}$$

$$\phi_k = 0.5\left(1 + \alpha_k\left(\bar{\lambda}_k - 0.2\right) + \bar{\lambda}_k^2\right) \tag{3.5}$$

$$\bar{\lambda}_k = \sqrt{\frac{f_y}{\sigma_{cr,k,z}}} \tag{3.6}$$

With:

$$\sigma_{cr,k,z} = \frac{\pi^2 E I_z}{L_{cr,z}^2 \cdot A} = \frac{\pi^2 \cdot 210000 \cdot 63.1 \cdot 10^6}{16000^2 \cdot 11300} = 45.2MPa \tag{3.7}$$

Therefore:

$$\bar{\lambda}_k = \sqrt{\frac{355}{45.2}} = 2.80 \tag{3.8}$$

For the imperfection factor α_k, based on Figure 7 of the SIA 263, we have:

$$\frac{h - t_f}{b} = \frac{290 - 14}{300} = 0.92 > 1.2 \tag{3.9}$$

And

$$t_f \leq 100mm \tag{3.10}$$

Therefore, we use the buckling curve c, and we have $\alpha_k = 0.49$, and therefore:

$$\phi_k = 0.5(1 + 0.49(2.8 - 0.2) + 2.8^2) = 5.06 \tag{3.11}$$

$$\chi_k = \frac{1}{5.06 + \sqrt{5.06^2 - 2.8^2}} = 0.108 \leq 1 \tag{3.12}$$

Which finally gives:

$$N_{k,Rd,z} = \frac{0.108 \cdot 355 \cdot 11300}{1.05} = 412\ kN \tag{3.13}$$

The verification of the stability of the beam is performed using the SIA 263 §5.1.10.1:

$$\frac{N_{Ed}}{N_{k,Rd,z}} + \frac{w_y}{1 - \frac{N_{Ed}}{N_{cr,y}}} \cdot \frac{M_{Ed}}{M_{D,Rd}} \leq 1 \tag{3.14}$$

With:

$$w_y = 0.6 + 0.4\left(\frac{M_{Ed,min}}{M_{Ed,max}}\right) = 0.6 + 0.4\left(\frac{0}{M_{Ed,max}}\right) = 0.6 > 0.4 \tag{3.15}$$

Since we do not consider lateral torsional buckling, we have:

$$M_{D,Rd} = M_{Rd,y} \tag{3.16}$$

And finally, the load P should satisfy:

$$\frac{P}{412} + \frac{0.6}{1 - \frac{P}{1478}} \cdot \frac{0.8P}{426} \leq 1 \tag{3.17}$$

We can solve this equation, and we obtain $P_{max} = 263\ kN$