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Exercise #10 –Potential Energy Method of Equilibrium

Problem 1

The structural system shown in Figure 1 consists of a rigid vertical member that is subjected to a compressive force P and two horizontal beams of rigidities EI and $2EI$. The vertical member should not rotate by a certain amount in order to prevent any geometric instability.

Compute the following:

1. The total internal energy U of the system by considering no imperfections.
2. Compute the total external energy V_p of the system by considering no imperfections.
3. Investigate the stability (or instability) of the system based on the total potential energy theorem by using linear and nonlinear theory.

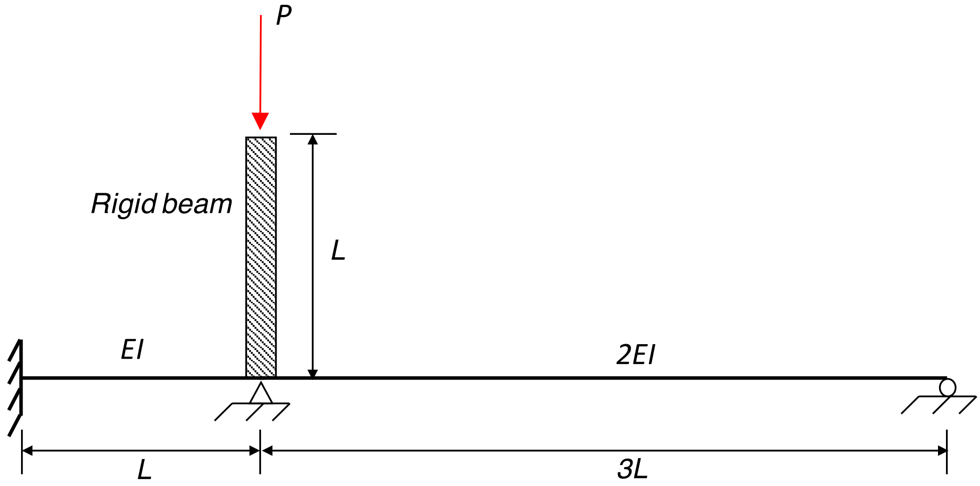


Figure 1. Planar structural system

Note that a beam with length L and end moments can be idealized as follows depending on its boundary conditions.

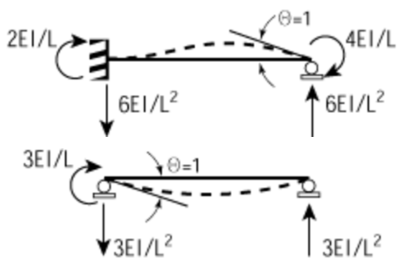


Figure 2. Stiffness coefficients

Problem 2

The steel bridge shown in Figure 3a can be represented by a set of two steel rods of infinite axial stiffness as shown in Figure 3b and a horizontal beam of rigidity EI that should not rotate by a certain amount in order to prevent any stability associated issues. Compute the following:

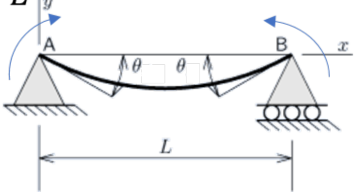
1. The total internal energy U of the system by considering no imperfections.
2. Compute the total external energy V_p of the system by considering no imperfections.
3. Investigate the stability (or instability) of the system based on the total potential energy theorem by using linear and nonlinear theory.

(a) (b)

Figure 3. Steel bridge deck and mathematical model idealization

Note: a beam with end moments can be idealized as follows:

$$M = \frac{2EI}{L}\theta \qquad M = \frac{2EI}{L}\theta$$



Solution

Problem 1

Part 1

First step is the calculation of an equivalent spring stiffness at the bottom node of the infinitely rigid bar:

$$c_{eq}\theta = \left(\frac{4EI}{L} + \frac{3 \cdot 2 \cdot EI}{3L}\right)\theta \Rightarrow c_{eq} = \frac{6EI}{L}$$

The internal energy then is:

$$\mathrm{U} = \frac{1}{2}c_{eq}\theta^2 = \frac{3EI}{L}\theta^2$$

Part 2

The total potential energy is:

$$\mathrm{V_P} = -PL(1 - cos\theta)$$

Part 3

Linear Theory:

Express the total potential energy:

$$\Pi = \mathrm{U} + \mathrm{V_P} = \frac{3EI}{L}\theta^2 - PL(1 - cos\theta) \approx \frac{3EI}{L}\theta^2 - PL\frac{\theta^2}{2} = \left(\frac{6EI}{L} - PL\right)\frac{\theta^2}{2}$$

Variations of potential energy with respect to the rotation are:

$$\frac{\partial \Pi}{\partial \theta} = \left(\frac{6EI}{L} - PL\right)\theta$$

In the limit that small variations of θ will yield no change in the potential energy – i.e. $\frac{\partial \Pi}{\partial \theta} = 0$, then either $\theta = 0$ or

$$P = \frac{6EI}{L^2}$$

After this point, if the load is maintained, for any change in θ the total potential energy will stay constant. No increase in load is possible because it would be an unstable behavior. This can be seen with the second derivative of the potential energy:

$$\frac{\partial^2 \Pi}{\partial \theta^2} = \frac{6EI}{L} - PL \Rightarrow \begin{cases} P < \dfrac{6EI}{L^2} \Rightarrow \dfrac{\partial^2 \Pi}{\partial \theta^2} > 0 - stable \\ P = \dfrac{6EI}{L^2} \Rightarrow \dfrac{\partial^2 \Pi}{\partial \theta^2} = 0 - limit \\ P > \dfrac{6EI}{L^2} \Rightarrow \dfrac{\partial^2 \Pi}{\partial \theta^2} < 0 - unstable \end{cases}$$

Nonlinear Theory:

$$\Pi = \mathrm{U} + \mathrm{V_P} = \frac{3EI}{L}\theta^2 - PL(1 - cos\theta)$$

Again, taking the first derivative of the total potential energy,

$$\frac{\partial \Pi}{\partial \theta} = \frac{6EI}{L}\theta - PLsin\theta$$

So, again, to have no change in the total potential energy as a function of the rotation θ, i.e. $\frac{\partial \Pi}{\partial \theta} = 0$, yields

$$P = \frac{6EI}{L^2}\frac{\theta}{sin\theta}$$

For small rotations $sin\theta \approx \theta$, but as rotations increase the sine function will not increase at the same rate as θ and for sufficiently large rotations it will even decrease. As a consequence, the ratio $\frac{\theta}{sin\theta}$ is always greater than 1 and so the load will always increase after the bifurcation point at $\frac{6EI}{L^2}$. In the limit the load can be ∞ when $\theta \rightarrow \pi$. This is a stable nonlinear behavior.

Problem 2
Part 1:

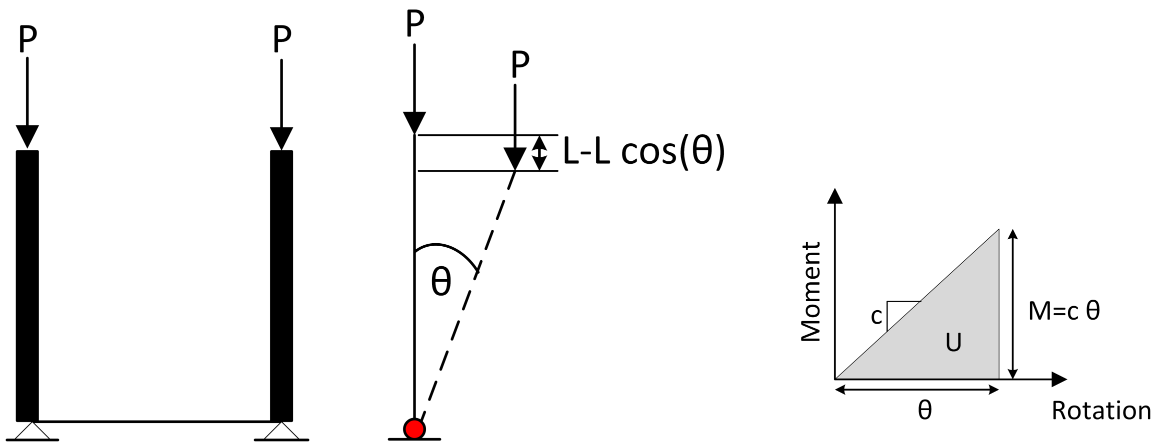


Figure 5. Supporting figures for Exercise #10 – Problem 2.

Instability in the system will lead to the vertical rigid bars rotating by a certain amount, θ. Assuming symmetry of the system shown in Figure 5a, this can be idealized by a rigid column with a vertical load P applied at the top, and a rotational spring at the bottom, as shown in Figure 5b. According to the information provided, the stiffness of this spring will be

$$c = \frac{M}{\theta} = \frac{2EI}{L} \tag{1}$$

Since the column is rigid, it does not contribute to the total strain energy. Assuming that the spring will behave in a linear elastic fashion as shown in Figure 5c, the total internal energy, *U*, will be as follows

$$U = \frac{1}{2} c \cdot \theta^2 = \frac{2EI}{L*2} \cdot \theta^2 = \frac{EI}{L} \cdot \theta^2 \tag{2}$$

Part 2:

The total external potential energy will be the vertical displacement δ, multiplied by P. By trigonometry, $\delta = L - L * \cos\theta$, therefore

$$V_p = P \cdot L(1 - \cos\theta) \tag{3}$$

Part 3:

In order the investigate the stability, the total potential energy, Π, is required.

$$\Pi = U - V_p = \frac{EI}{L} \cdot \theta^2 - P \cdot L(1 - \cos\theta) \tag{4}$$

First the critical load at which the secondary equilibrium path will be obtained must be calculated. This is done by taking the first derivative of the total potential energy of the system with respect to the displacement variable, θ:

$$\frac{\partial \Pi}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{EI}{L}\cdot\theta^2 - P\cdot L(1-\cos\theta)\right)$$
$$= \frac{EI}{L}\cdot 2\theta - P\cdot L(0+\sin\theta) \quad (5)$$
$$= \frac{2EI}{L}\theta - PL\sin\theta$$

Now this is set equal to zero, and the limit as $\theta \to 0$ is taken

$$\lim_{\theta\to 0}\left(\frac{2EI}{L}\theta - PL\sin\theta = 0\right) \quad (6)$$

By L'Hopital's Rule: $\lim_{\theta\to 0}(\theta/\sin\theta) = 1$

$$P_{cr} = \frac{2EI}{L^2} \quad (7)$$

In order to determine if the secondary equilibrium is stable or unstable, the higher order derivatives of Π with respect to *θ* must be taken. If the term is less than zero, the system is unstable. The system is stable for terms greater than zero.

$$\frac{\partial^2 \Pi}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{2EI}{L}\theta - PL\sin\theta\right)$$
$$= \frac{2EI}{L} - PL\cos\theta \quad (8)$$

Replacing P with P_{cr} and evaluating at $\theta\ =\ 0$,

$$\left.\frac{\partial^2 \Pi}{\partial \theta^2}\right|_{\theta=0} = \frac{2EI}{L} - \frac{2EI}{L^2}L\cos\theta = \frac{2EI}{L}(1-\cos 0) = 0 \quad (9)$$

Since the second order term is equal to zero, the higher order terms must be explored. First, examining the third derivative,

$$\frac{\partial^3 \Pi}{\partial \theta^3} = \frac{\partial}{\partial \theta}\left(\frac{2EI}{L}\theta - PL\cos\theta\right)$$
$$= 0 + PL\sin\theta \quad (11)$$

Replacing P with P_{cr} and evaluating at $\theta = 0$,

$$\left.\frac{\partial^3 \Pi}{\partial \theta^3}\right|_{\theta=0} = \frac{2EI}{L}\sin 0 = 0 \quad (12)$$

Again, another term must be explored, examining the fourth derivative:

$$\frac{\partial^4 \Pi}{\partial \theta^4} = \frac{\partial}{\partial \theta}(PL\sin\theta)$$
$$= PL\cos\theta \quad (13)$$

Replacing P with P_{cr} and evaluating at $\theta\ =\ 0$,

$$\left.\frac{\partial^4 \Pi}{\partial \theta^4}\right|_{\theta=0} = \frac{2EI}{L}\cos 0 = \frac{2EI}{L} > 0. \quad (14)$$

Since all the lower order terms are equal to zero, and the fourth order term is greater than 0, the system is stable.