CIVIL 369: "Structural Stability"

School of Architecture, Civil & Environmental Engineering
Civil Engineering Institute
Resilient Steel Structures Laboratory (RESSLab)

-Energy and Dynamic Methods of Equilibrium

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Objectives of This Week's Lecture

To introduce:

- ✧ Analytical methods for assessing stability of characteristic systems
  - ✧ Potential Energy Method
  - ✧ Dynamic Method
- ✧ Illustrative examples to examine structural stability

Total Potential Energy

✧ The total potential Π of an elastic conservative system is defined as follows,

$$\Pi = U + V_p$$

✧ U : elastic strain energy of a conservative system. The work performed by both the internal and the external forces is independent of the path traveled by these forces, and it depends only on the initial and the final positions.

✧ V_p : potential of the external forces, using the original deflected position as a reference V_p is the external work; $V_p = -We$

Computation of Total Potential Energy

- U is the internal work performed by the internal forces (total strain energy); $U = Wi$

- The potential of the external forces is computed as,

$$V_p = -\sum_i \int P_i(q_i) \cdot dq_i \quad \text{or} \quad V_p = -\sum_i P_i \cdot dq_i$$

- q_i are the generalized coordinates (deflections or rotations that define the deformed shape of a system)

- P_i are the corresponding external loads

Equilibrium Criterion

- A conservative system subjected to a static force P is in equilibrium at a deformed configuration q, when the total potential energy Π has a local maximum or local minimum (stationary value) compared to any other deformed configuration near by.

Mathematical Expression of Potential Energy

✧ For a system with one degree of freedom, q, which is subjected to external loads P, the potential energy Π is a single variable function,

$$\Pi(P, q)$$

✧ The equilibrium position, q_o will be as follows,

$$\left.\frac{\partial \Pi(P, q)}{\partial q}\right|_{q=q_0} = 0$$

Mathematical Expression of Potential Energy

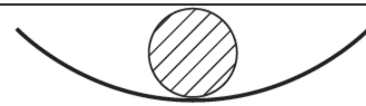
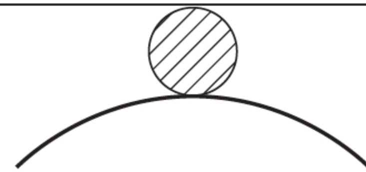
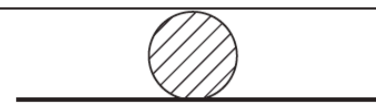
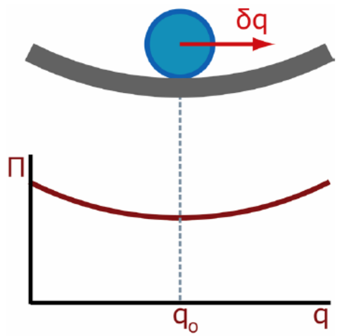
• *Minimum of* $\prod$ • ***Stable equilibrium*** • *Energy must be added to change configuration.*	$\frac{d^2\prod}{d\theta^2} > 0$		*Ball in cup can be disturbed, but it will return to the center.*
• *Maximum of* $\prod$ • ***Unstable equilibrium*** • *Energy is released as configuration is changed.*	$\frac{d^2\prod}{d\theta^2} < 0$		*Ball will roll down if disturbed.*
• *Transition from minimum to maximum* • ***Neutral equilibrium*** • *There is no change in energy.*	$\frac{d^2\prod}{d\theta^2} = 0$		*Ball is free to roll.*

Image Source: Galambos and Surovek 2008

Stability/Instability Criterion

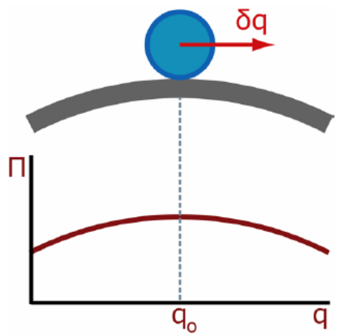
✧ The equilibrium point q_o is stable if the potential energy Π has a local minimum at position q_o, therefore,

$$\left.\frac{\partial^2 \Pi(P,q)}{\partial q^2}\right|_{q=q_0} > 0$$



✧ The equilibrium point q_o is unstable if the potential energy Π has a local maximum at position q_o, therefore,

$$\left.\frac{\partial^2 \Pi(P,q)}{\partial q^2}\right|_{q=q_0} < 0$$

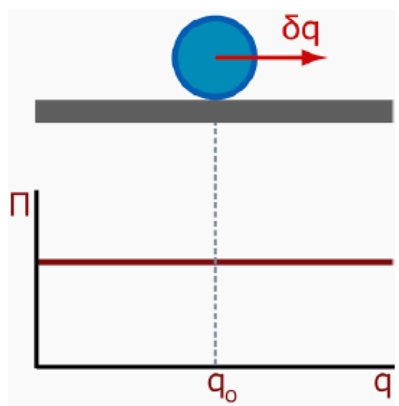


Stability/Instability Criterion

✧ If the second derivative of the potential energy Π at the equilibrium point q_o is zero then we need to find the $k-th$ derivative that,

$$\left.\frac{\partial^{\mathrm{k}}\Pi(P,q)}{\partial q^{k}}\right|_{q=q_0} \neq 0$$

✧ If *k* is an odd number, then the equilibrium is neutral



Stability/Instability Criterion

✧ If *k* is an even number, then Π has a local maximum or minimum.

✧ A local minimum implies that the equilibrium is stable and,

$$\left.\frac{\partial^{\mathrm{k}}\Pi(P,q)}{\partial q^{k}}\right|_{q=q_0} > 0$$

✧ A local maximum implies that the equilibrium is unstable and,

$$\left.\frac{\partial^{\mathrm{k}}\Pi(P,q)}{\partial q^{k}}\right|_{q=q_0} < 0$$

Example 1: Fixed Rod with Vertical Load

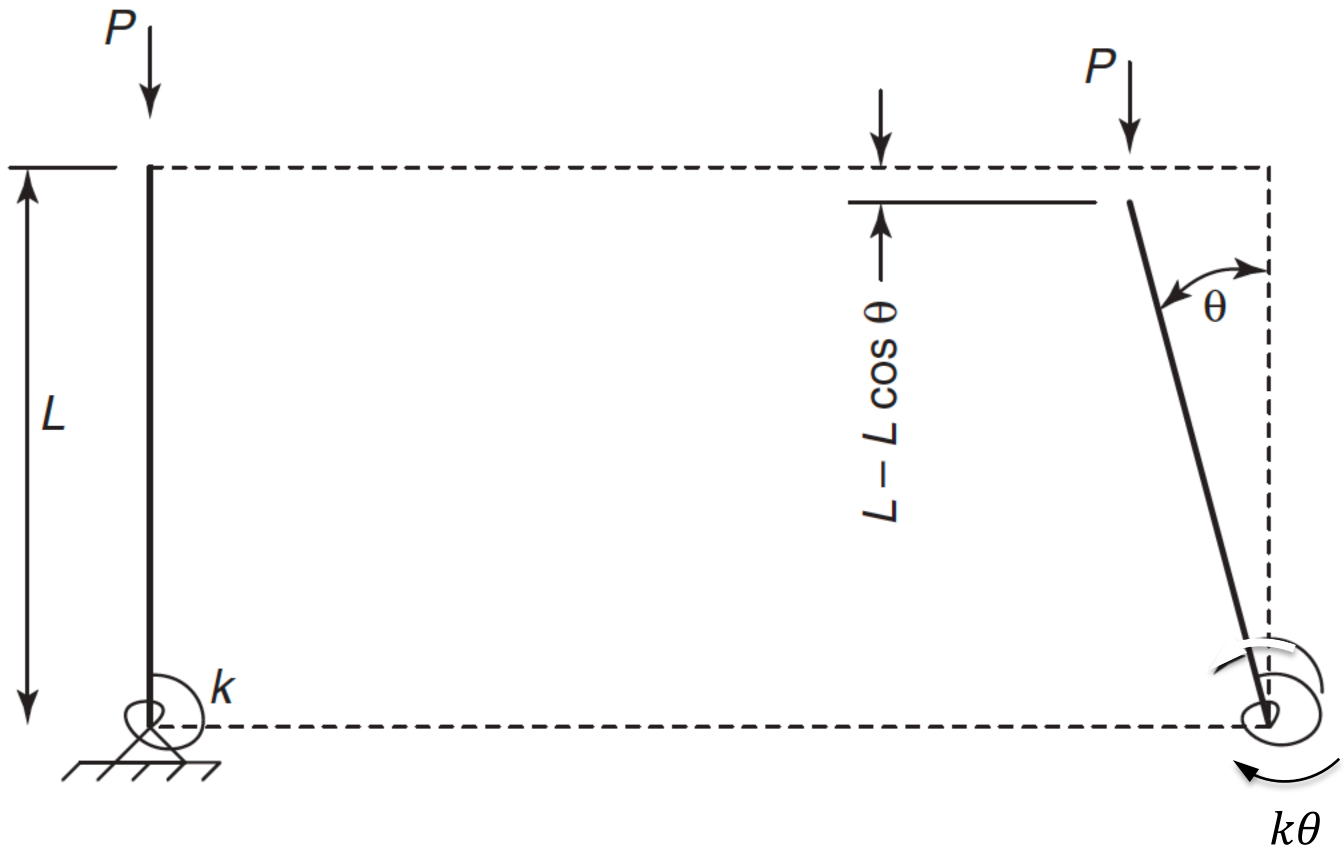


Image Source: Galambos and Surovek 2008

Example 1: Fixed Rod with Vertical Load

Total Internal Energy

$$U = \frac{1}{2} \cdot k \cdot \theta \cdot \theta = \frac{1}{2} \cdot k \cdot \theta^2$$

Total External Energy

$$V_p = -\sum_i P_i \cdot q_i = -P \cdot q$$

$$V_p = -P \cdot (L - L cos(\theta))$$

Total Potential Energy

$$\Pi = \frac{1}{2} \cdot k \cdot \theta^2 - P \cdot L(1 - cos(\theta))$$

Example 1: Linear Buckling Theory

Total Potential Energy

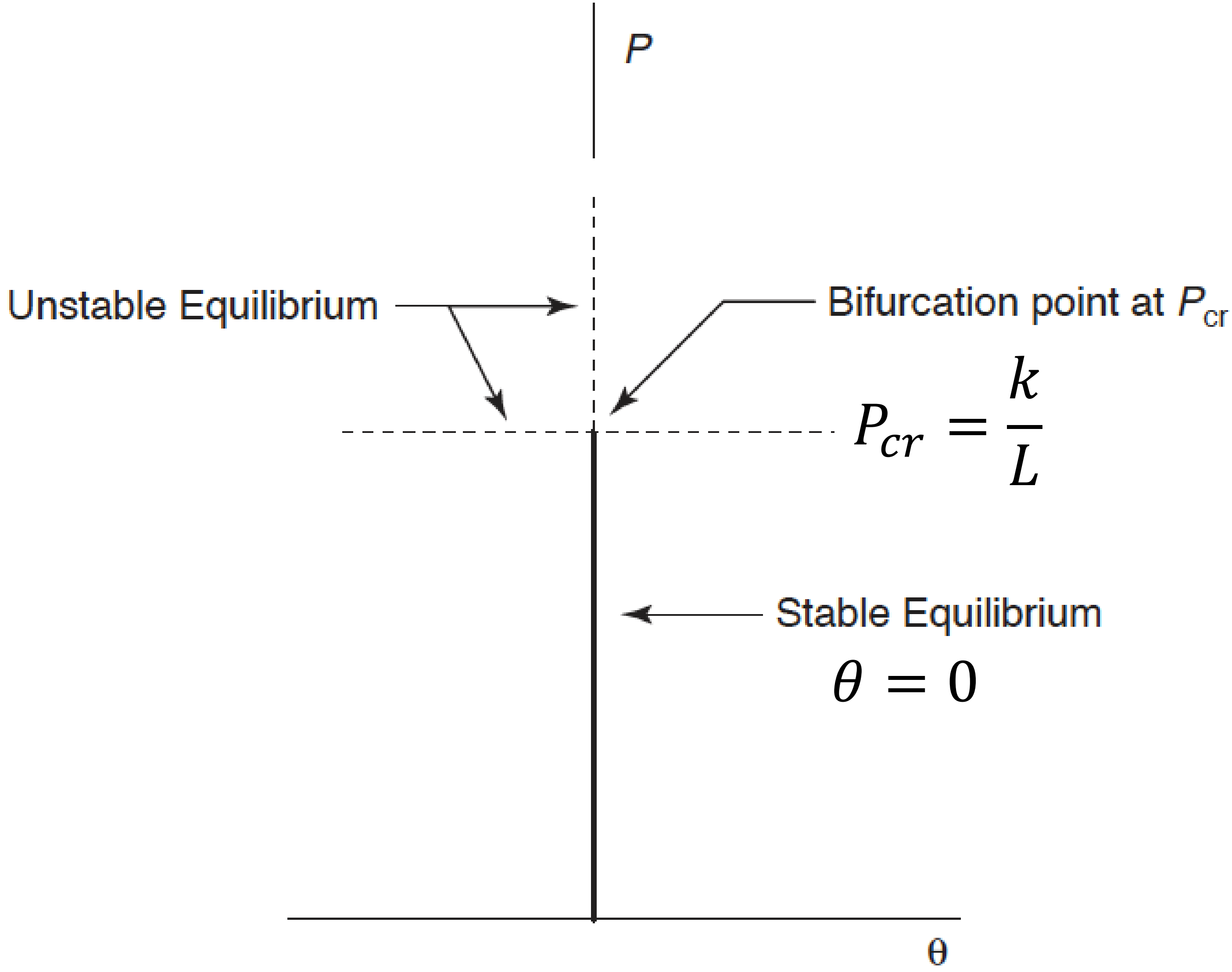
$$\Pi = \frac{1}{2} \cdot k \cdot \theta^2 - P \cdot L[1 - cos(\theta)] \cong \frac{1}{2} \cdot k \cdot \theta^2 - P \cdot L \cdot \frac{\theta^2}{2}$$

$$\frac{\partial \Pi}{\partial q} = k \cdot \theta - P \cdot L \cdot \theta = \theta \cdot (k - PL)$$

Equilibrium point can be found if $\left.\frac{\partial \Pi(P,\theta)}{\partial \theta}\right|_{\theta=\theta_0} = 0$

Therefore, $\theta = 0 \; or \; P = \frac{k}{L} = P_{cr}$

Example 1: - Linear Buckling Theory



Example 1: Evaluation of Stability – Linear Theory

$$\frac{\partial^2 \Pi(P,\theta)}{\partial \theta^2} = k - PL$$

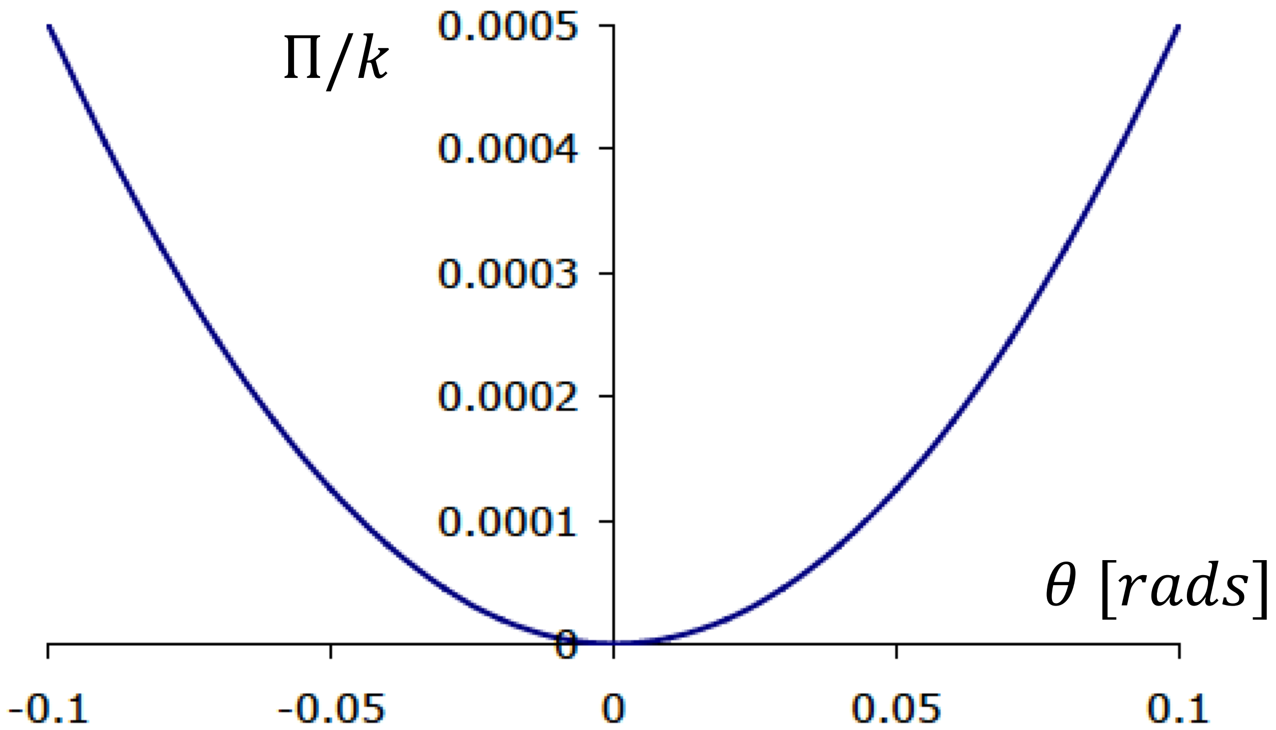
Evaluation of stability of the main equilibrium path,

$$\left.\frac{\partial^2 \Pi(P,\theta)}{\partial \theta^2}\right|_{\theta=0} = k - PL \rightarrow if \begin{cases} P < P_{cr} = \dfrac{k}{L}, then\ stable \\ P > P_{cr} = \dfrac{k}{L}, then\ unstable \\ P = P_{cr} = \dfrac{k}{L}, then\ ?? \end{cases}$$

Example 1: Evaluation of Stability – Linear Theory

Evaluation of stability of the main equilibrium path,

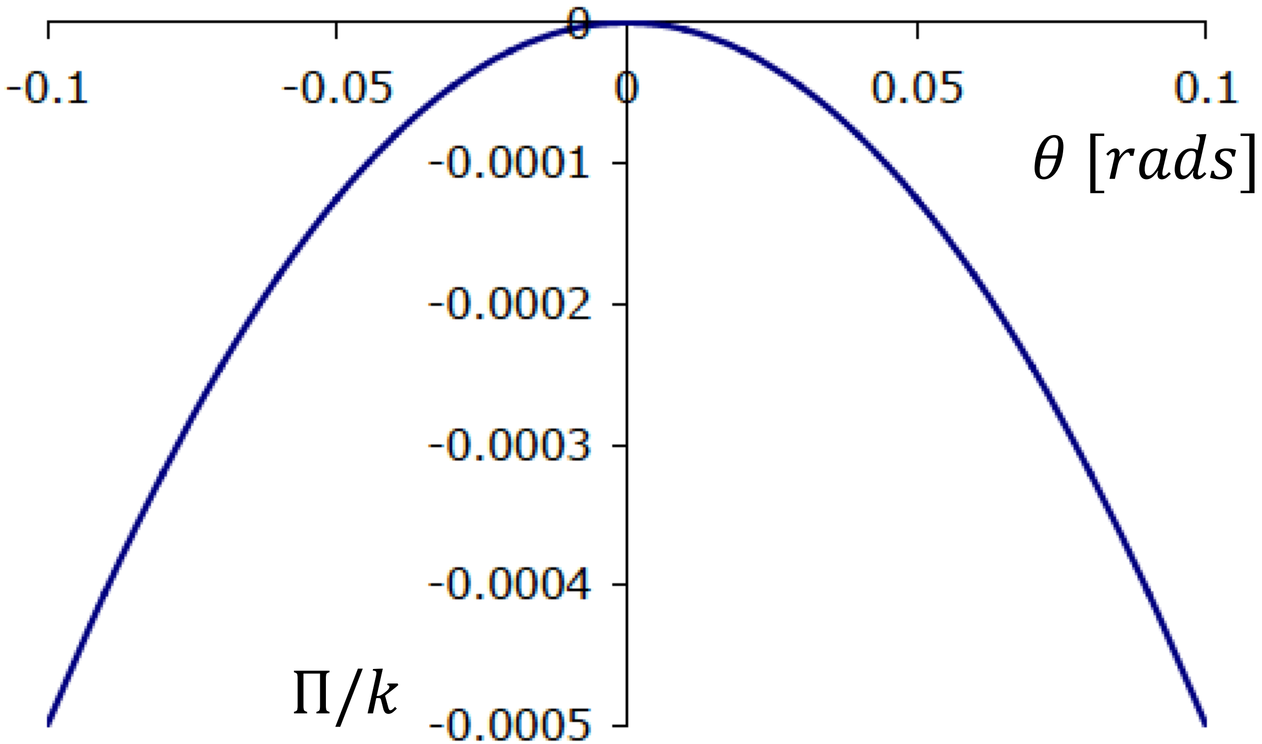
$$\Pi(0.9P_{cr},\theta) = \frac{1}{2}\cdot k\cdot\theta^2 - 0.9P_{cr}\cdot L\cdot\frac{\theta^2}{2} = 0.05\cdot k\cdot\theta^2$$



Example 1: Evaluation of Stability – Linear Theory

Evaluation of stability of the main equilibrium path,

$$\Pi(1.1P_{cr},\theta)=\frac{1}{2}\cdot k\cdot\theta^2-1.1\cdot P_{cr}\cdot L\cdot\frac{\theta^2}{2}=-0.05\cdot k\cdot\theta^2$$



Example 1: - Nonlinear Theory

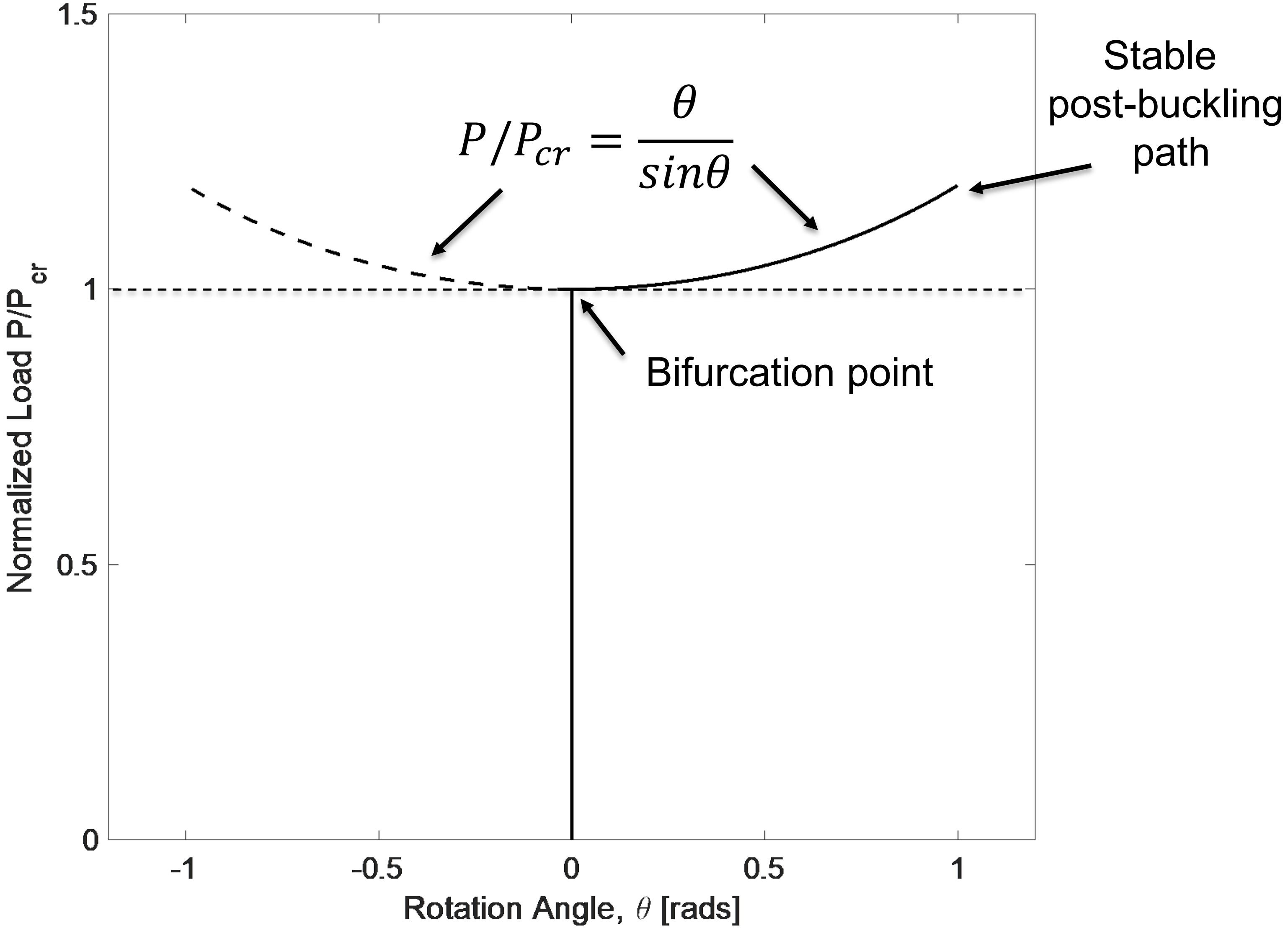
$$\Pi = \frac{1}{2} \cdot k \cdot \theta^2 - P \cdot L(1 - cos(\theta))$$

$$\frac{\partial \Pi}{\partial q} = k \cdot \theta - P \cdot L \cdot \sin(\theta)$$

Equilibrium point can be found if $\left. \frac{\partial \Pi(P,\theta)}{\partial \theta} \right|_{\theta=\theta_0} = 0$

Therefore, $\theta = 0 \; or \; P = \frac{k}{L} \cdot \frac{\theta}{sin\theta}$

Example 1: - Nonlinear Theory



Example 1: Evaluation of Stability – Nonlinear Theory

$$\frac{\partial^2 \Pi(P,\theta)}{\partial \theta^2} = k - PLcos\theta$$

Evaluation of stability of the main equilibrium path,

$$\left.\frac{\partial^2 \Pi(P,\theta)}{\partial \theta^2}\right|_{\theta=0} = k - PL \rightarrow if \begin{cases} P < P_{cr}\ then\ stable \\ P > P_{cr}\ then\ unstable \\ P = P_{cr}\ then\ ?? \end{cases}$$

Example 1: Evaluation of Stability – Nonlinear Theory

Evaluation of stability of bifurcation point,

$$\frac{\partial^3 \Pi(P,\theta)}{\partial \theta^3} = PLsin\theta \Rightarrow \left.\frac{\partial^3 \Pi(P,\theta)}{\partial \theta^3}\right|_{\theta=0,P=P_{cr}} = 0$$

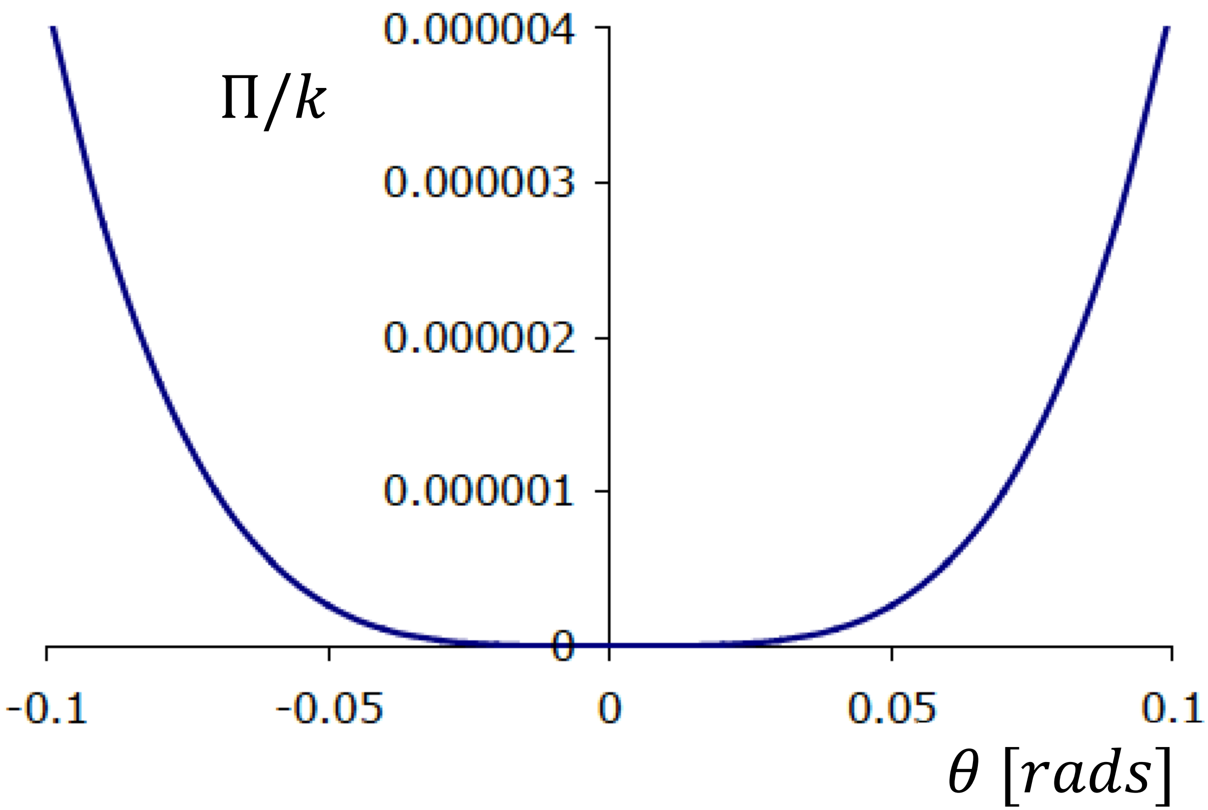
$$\frac{\partial^4 \Pi(P,\theta)}{\partial \theta^4} = PLcos\theta \Rightarrow \left.\frac{\partial^4 \Pi(P,\theta)}{\partial \theta^4}\right|_{\theta=0,P=P_{cr}} = k > 0$$

Therefore the bifurcation point is stable

Example 1: Evaluation of Stability – Nonlinear Theory

Evaluation of stability of the bifurcation point,

$$\Pi(P_{cr}, \theta) = \frac{1}{2} \cdot k \cdot \theta^2 - k \cdot (1 - cos\theta)$$



Example 1: Evaluation of Stability – Nonlinear Theory

Evaluation of stability for secondary equilibrium path,

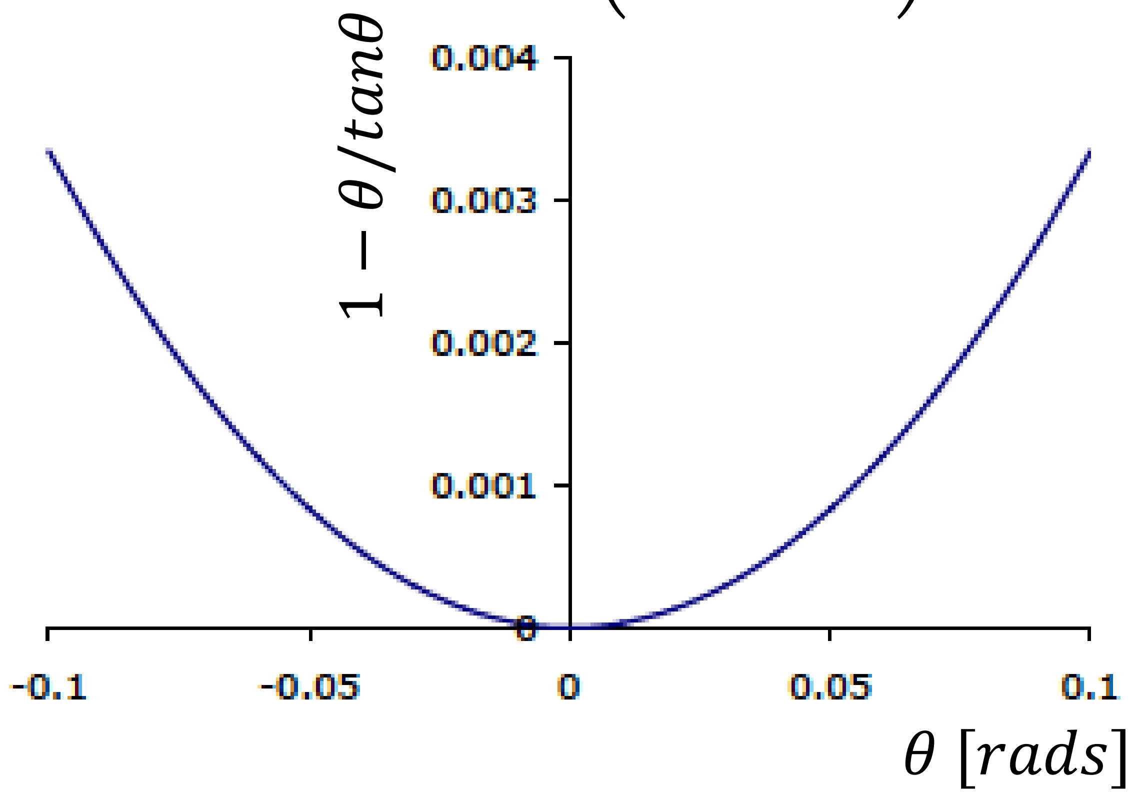
$$\left.\begin{aligned}\frac{\partial^2\Pi(P,\theta)}{\partial\theta^2} &= k - PL\cos\theta \\ P &= \frac{k}{L}\cdot\frac{\theta}{\sin\theta}\end{aligned}\right\} \quad \frac{\partial^2\Pi(P,\theta)}{\partial\theta^2} = k - k\cdot\frac{\theta}{\sin\theta}\cdot\cos\theta$$

$$\frac{\partial^2\Pi(P,\theta)}{\partial\theta^2} = k - k\cdot\frac{\theta}{\sin\theta}\cdot\cos\theta = k\cdot\left(1-\frac{\theta}{\tan\theta}\right)$$

Example 1: Evaluation of Stability – Nonlinear Theory

Evaluation of stability for secondary equilibrium path,

$$\frac{\partial^2 \Pi(P,\theta)}{\partial \theta^2} = k \cdot \left(1 - \frac{\theta}{tan\theta}\right) \geq 0$$

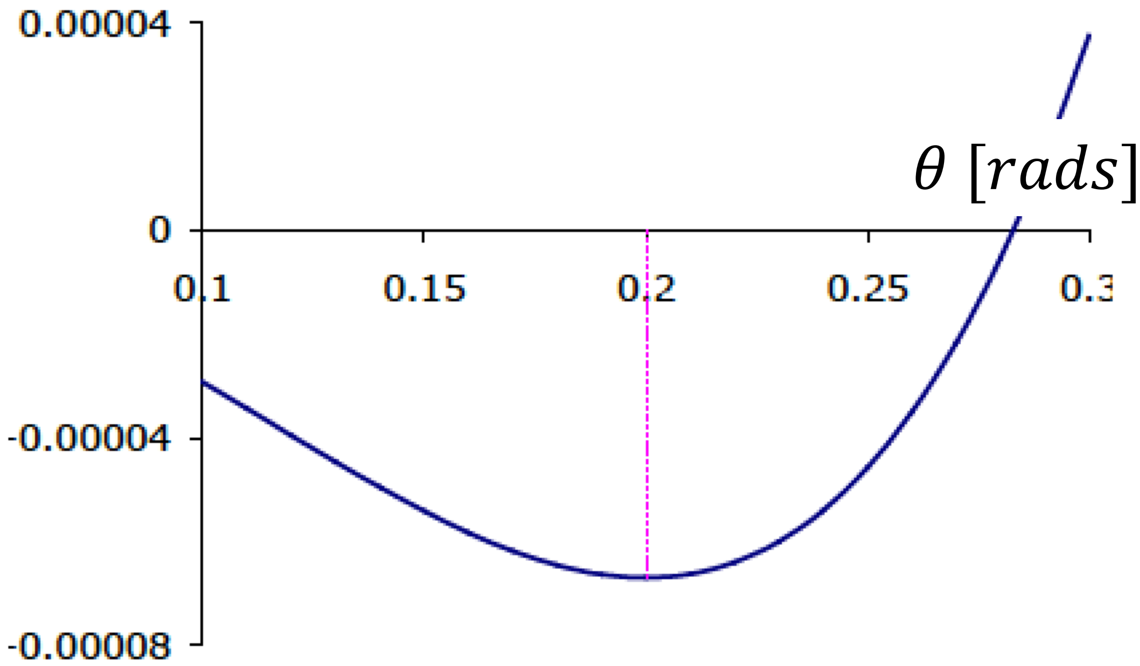


Example 1: Evaluation of Stability – Nonlinear Theory

Random position on the secondary equilibrium path,

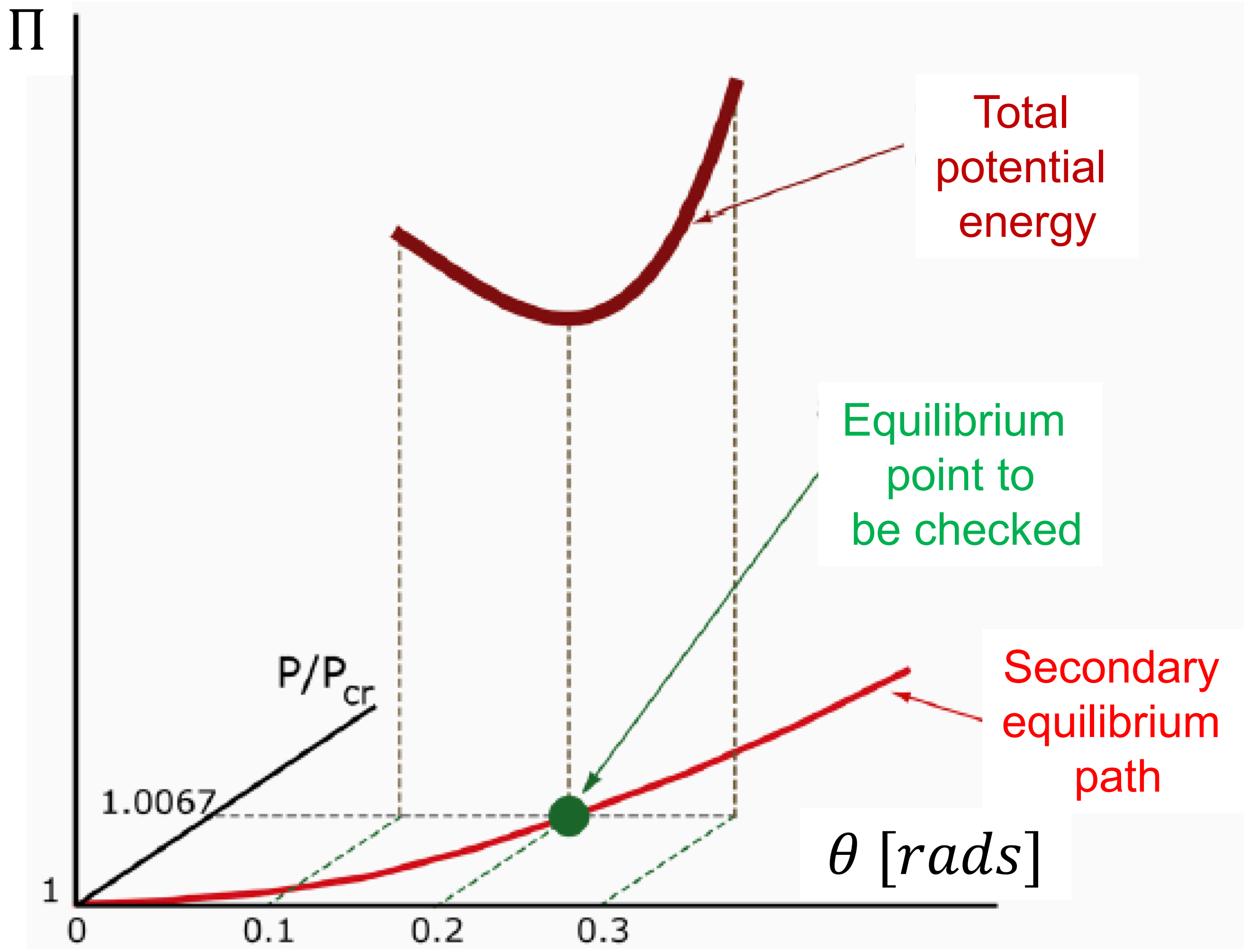
$$(P = 1.00670k/L, \theta\text{=0.2rad})$$

$$\Pi(1.00670k/L, \theta) = \frac{1}{2} \cdot k \cdot \theta^2 - 1.00670k \cdot (1 - cos\theta)$$



Example 1: Evaluation of Stability – Nonlinear Theory

Random position on the secondary equilibrium path,



Example 1: System with Imperfections

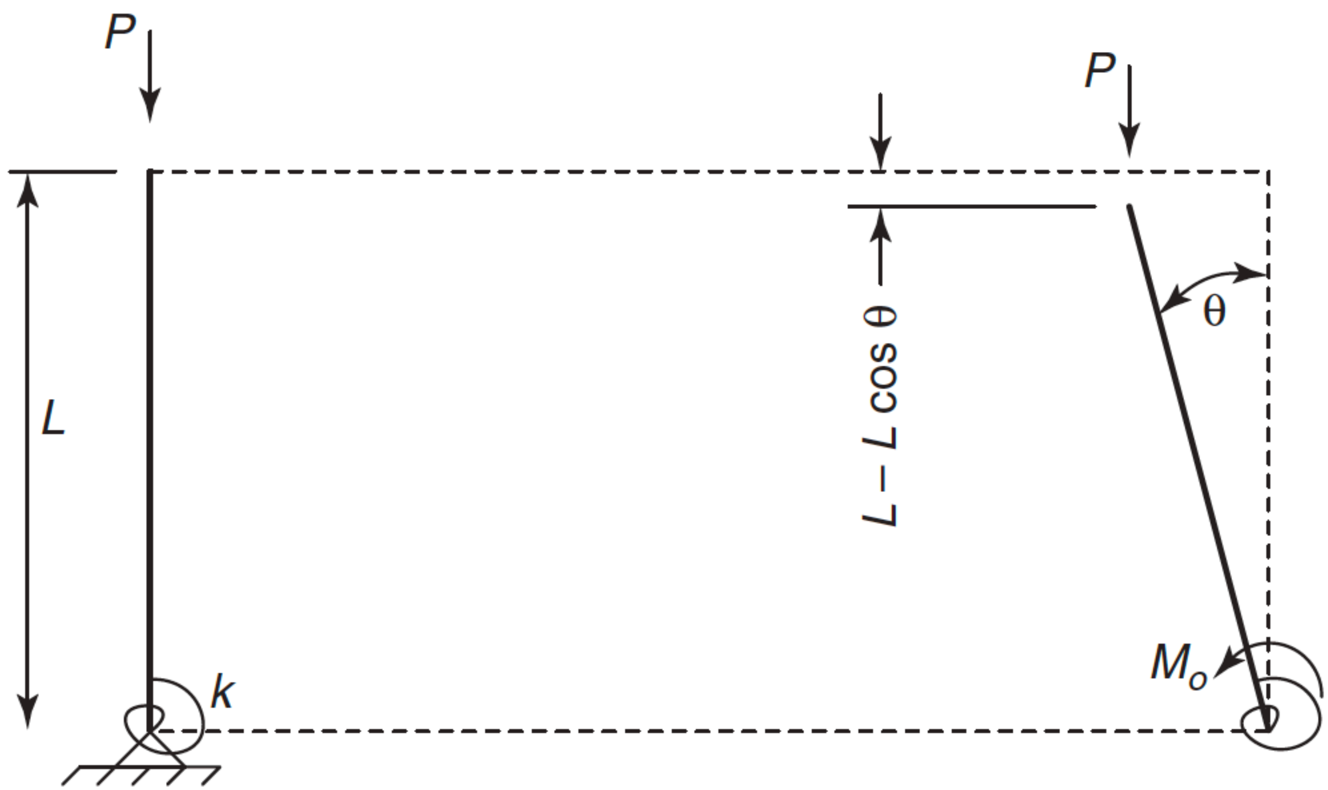


Image Source: Galambos and Surovek 2008

Example 1: System with Imperfections

Total Internal Energy

$$U = \frac{1}{2} \cdot k \cdot (\theta - \theta_o)^2$$

Total External Energy

$$V_p = \sum_i P_i \cdot q_i^{init.} - \sum_i P_i \cdot q_i^{fin.} = P \cdot q^{\theta_o} - P \cdot q^{\theta}$$

$$V_p = P \cdot L(cos\theta_o - cos\theta)$$

Total Potential Energy

$$\Pi = \frac{1}{2} \cdot k \cdot (\theta - \theta_o)^2 - P \cdot L(cos\theta_o - cos\theta)$$

Example 1: System with Imperfections
-Nonlinear Theory

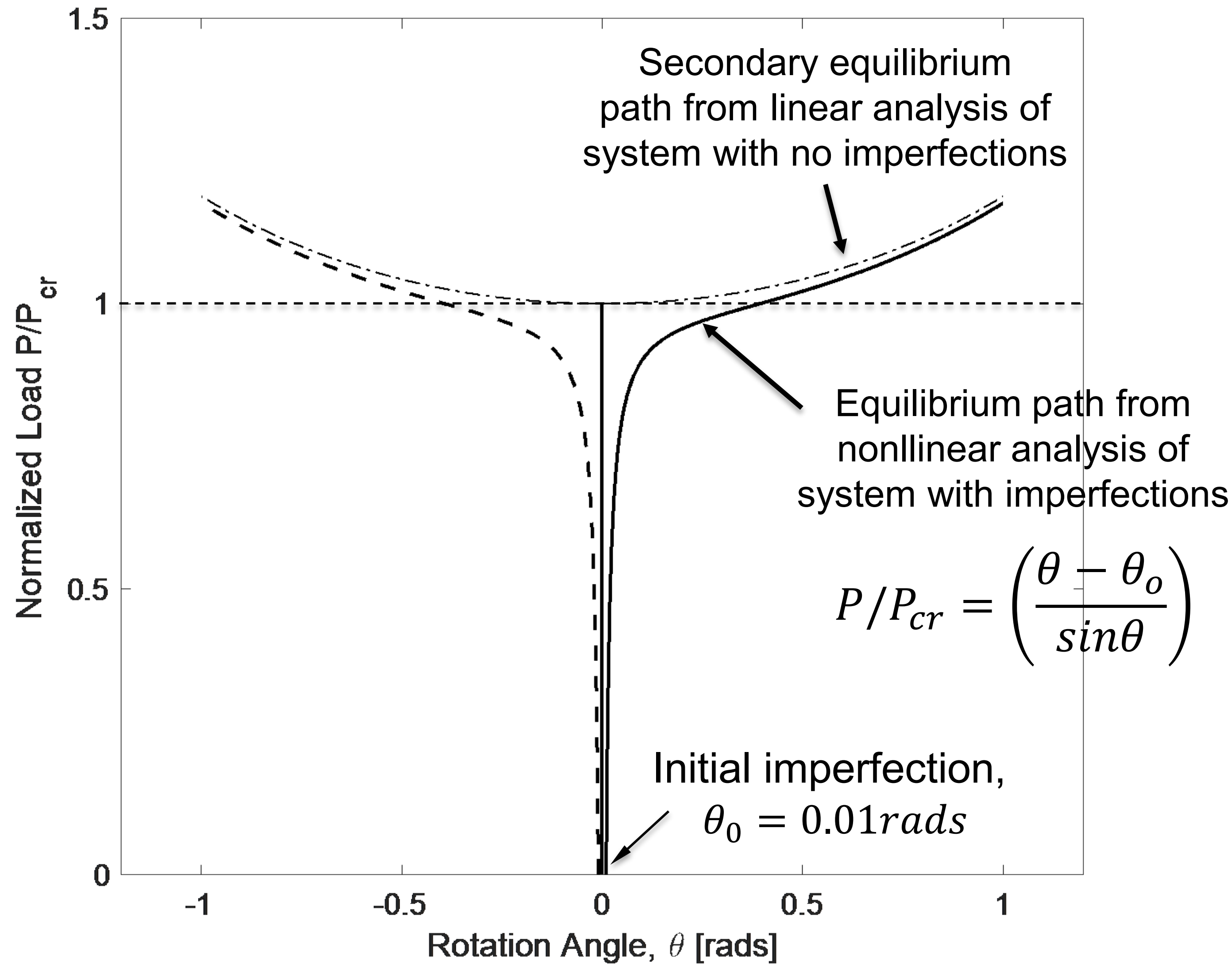
Total Potential Energy

$$\Pi = \frac{1}{2} \cdot k \cdot (\theta - \theta_o)^2 - P \cdot L(cos\theta_o - cos\theta)$$

$$\frac{\partial \Pi}{\partial q} = k \cdot (\theta - \theta_o) - P \cdot L \cdot \sin(\theta)$$

$$\frac{\partial \Pi}{\partial q} = 0 \Longrightarrow P = \frac{k}{L} \cdot \frac{\theta - \theta_o}{\sin\theta}$$

Example 1: System with Imperfections – Nonlinear Theory



Example 1: System with Imperfections
-Nonlinear Theory

$$\frac{\partial^2 \Pi(P,\theta)}{\partial \theta^2} = k - PLcos\theta$$

Evaluation of stability of the continuous equilibrium path,

$$\left.\begin{aligned} &\frac{\partial^2 \Pi(P,\theta)}{\partial \theta^2} = k - PLcos\theta \\ &P = \frac{k}{L}\cdot\frac{\theta-\theta_o}{\sin\theta} \end{aligned}\right\}$$

$$\frac{\partial^2 \Pi(P,\theta)}{\partial \theta^2} = k\left(1 - \frac{\theta-\theta_o}{tan\theta}\right)$$

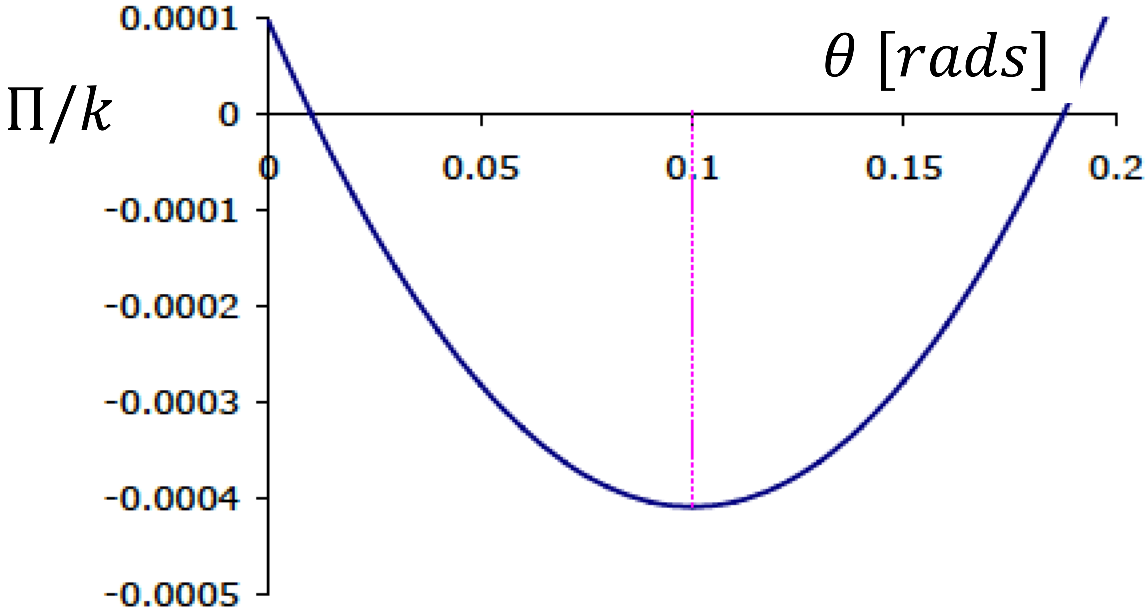
$$\frac{\partial^2 \Pi(P,\theta)}{\partial \theta^2} = k\left(\frac{tan\theta - \theta + \theta_o}{tan\theta}\right)$$

Example 1: System with Imperfections
-Nonlinear Theory

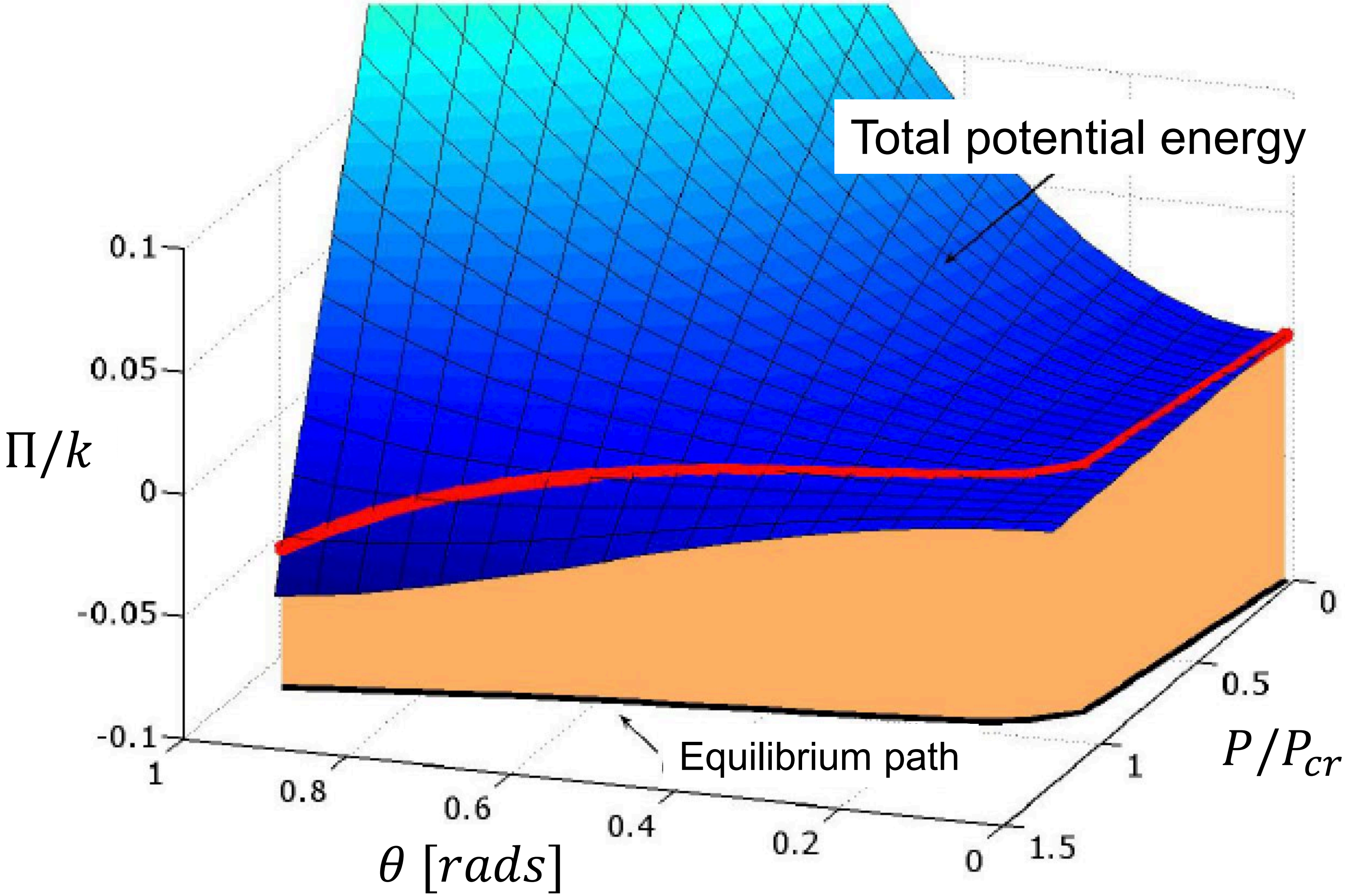
Random position on the secondary equilibrium path,

$(P = 0.901502k/L, \theta$=0.1rad), assume $\theta_o = 0.01rads$

$$\Pi(0.901502k/L, \theta)$$
$$= \frac{1}{2} \cdot k \cdot (\theta - 0.01)^2 - 0.901502 \cdot k \cdot (\cos(0.01) - cos\theta)$$



Example 1: System with Imperfections
-Nonlinear Theory



Multi-Degree of Freedom Systems

-Potential Energy Method for Estimating P_{cr}

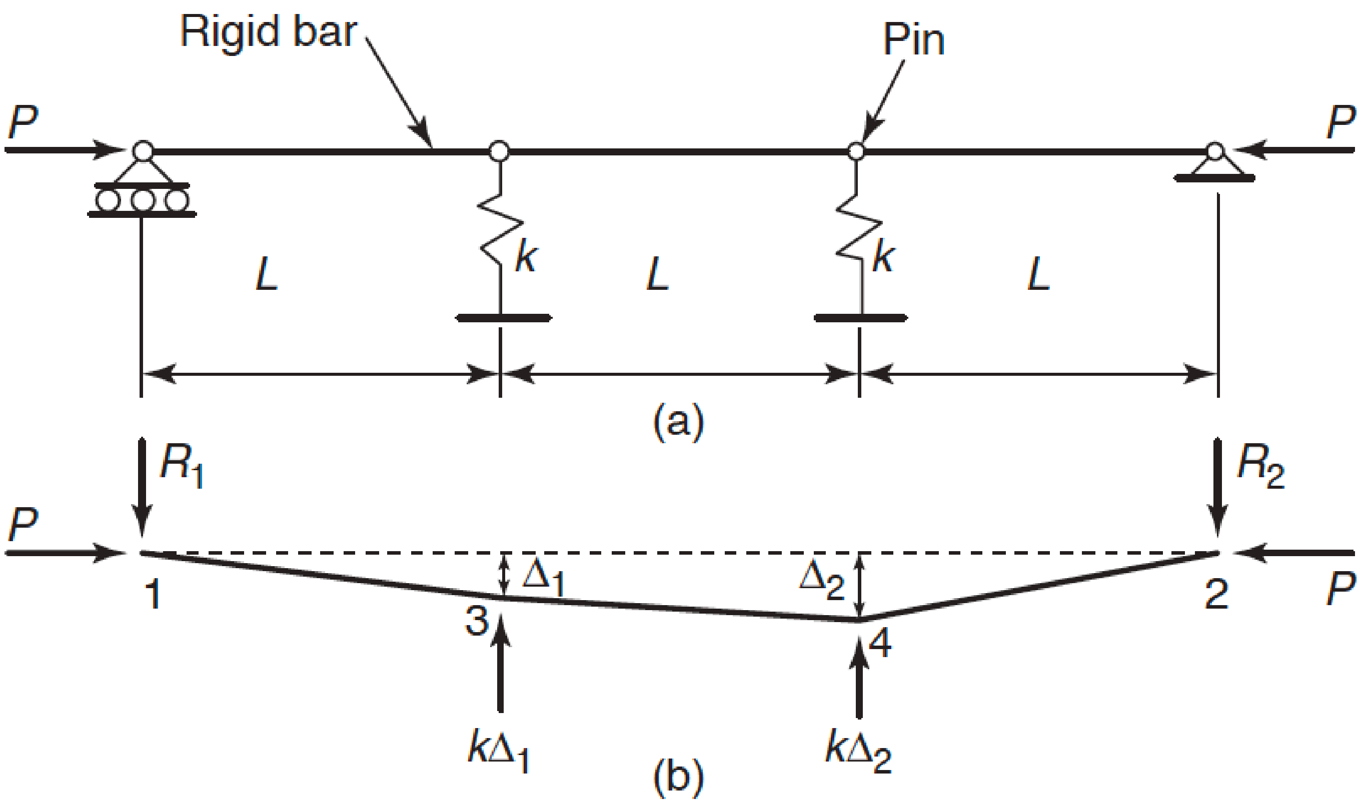
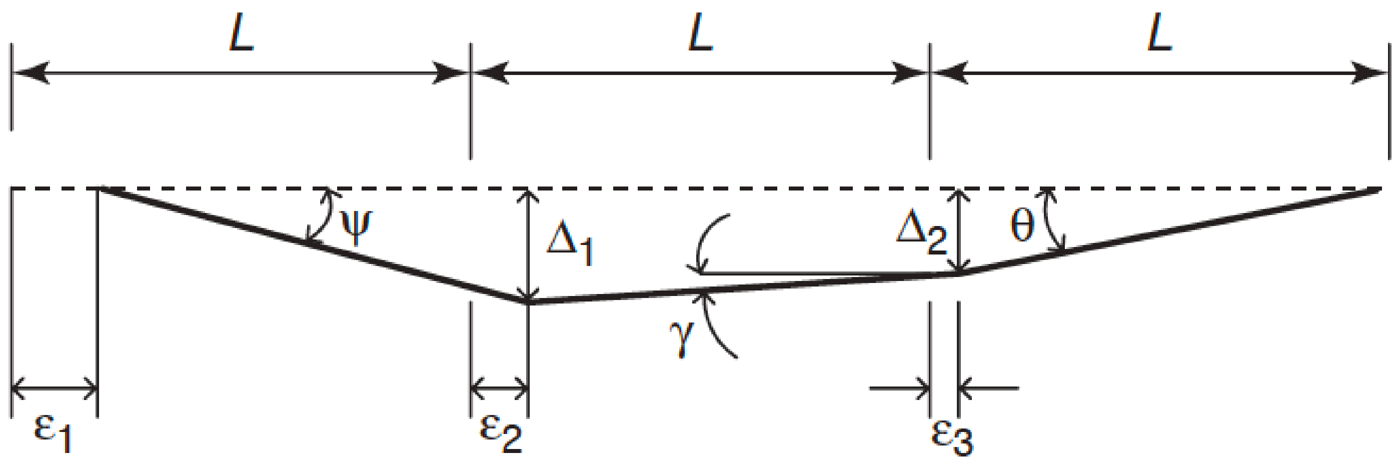


Image Source: Galambos and Surovek 2008

Multi-Degree of Freedom Systems

The energy method can be used for arriving at a solution. The necessary geometric relationships are illustrated herein



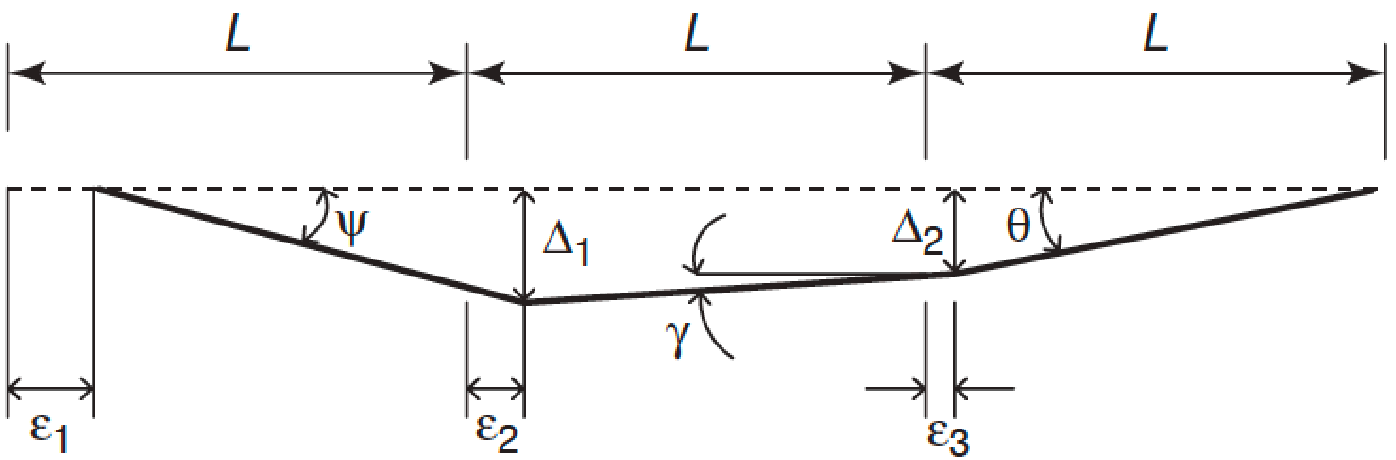
$$\Delta_1 = \psi L \qquad \Delta_2 = \theta L$$

$$\frac{\Delta_1 - \Delta_2}{L} = \gamma = \psi - \theta$$

Image Source: Galambos and Surovek 2008

Multi-Degree of Freedom Systems

The energy method can be used for arriving at a solution. The necessary geometric relationships are illustrated herein



$$\varepsilon_3 = L - L\cos\theta \approx \frac{L\theta^2}{2}$$

$$\varepsilon_2 = \varepsilon_3 + L[1 - \cos(\psi - \theta)] = \frac{L}{2}(2\theta^2 + \psi^2 - 2\psi\theta)$$

$$\varepsilon_1 = \varepsilon_2 + \frac{L\psi^2}{2} = L(\theta^2 + \psi^2 - \psi\theta)$$

Multi-Degree of Freedom Systems

The strain energy equals

$$U_p = \frac{k}{2}(\Delta_1^2 + \Delta_2^2) = \frac{kL^2}{2}(\psi^2 + \theta^2)$$

The potential of the external forces equals

$$V_p = -P\varepsilon_1 = -PL(\theta^2 + \psi^2 - \psi\theta)$$

Multi-Degree of Freedom Systems

For equilibrium, we take the derivatives with respect to the two angular rotations:

$$\frac{\vartheta \Pi}{\vartheta \psi} = 0 = \frac{kL^2}{2}(2\psi) - 2PL\psi + PL\theta$$

$$\frac{\vartheta \Pi}{\vartheta \theta} = 0 = \frac{kL^2}{2}(2\theta) - 2PL\theta + PL\psi$$

Multi-Degree of Freedom Systems

Rearranging we get

$$\begin{bmatrix} (kL^2 - 2PL) & PL \\ PL & (kL^2 - 2PL) \end{bmatrix} \begin{Bmatrix} \theta \\ \psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Setting the determinant of the coefficients equal to zero results in the critical loads of the problem.

$$3\left(\frac{P}{kL}\right)^2 - \frac{4P}{kL} + 1 = 0$$

$$P_{cr_1} = kL \qquad P_{cr_2} = \frac{kL}{3}$$

Multi-Degree of Freedom Systems

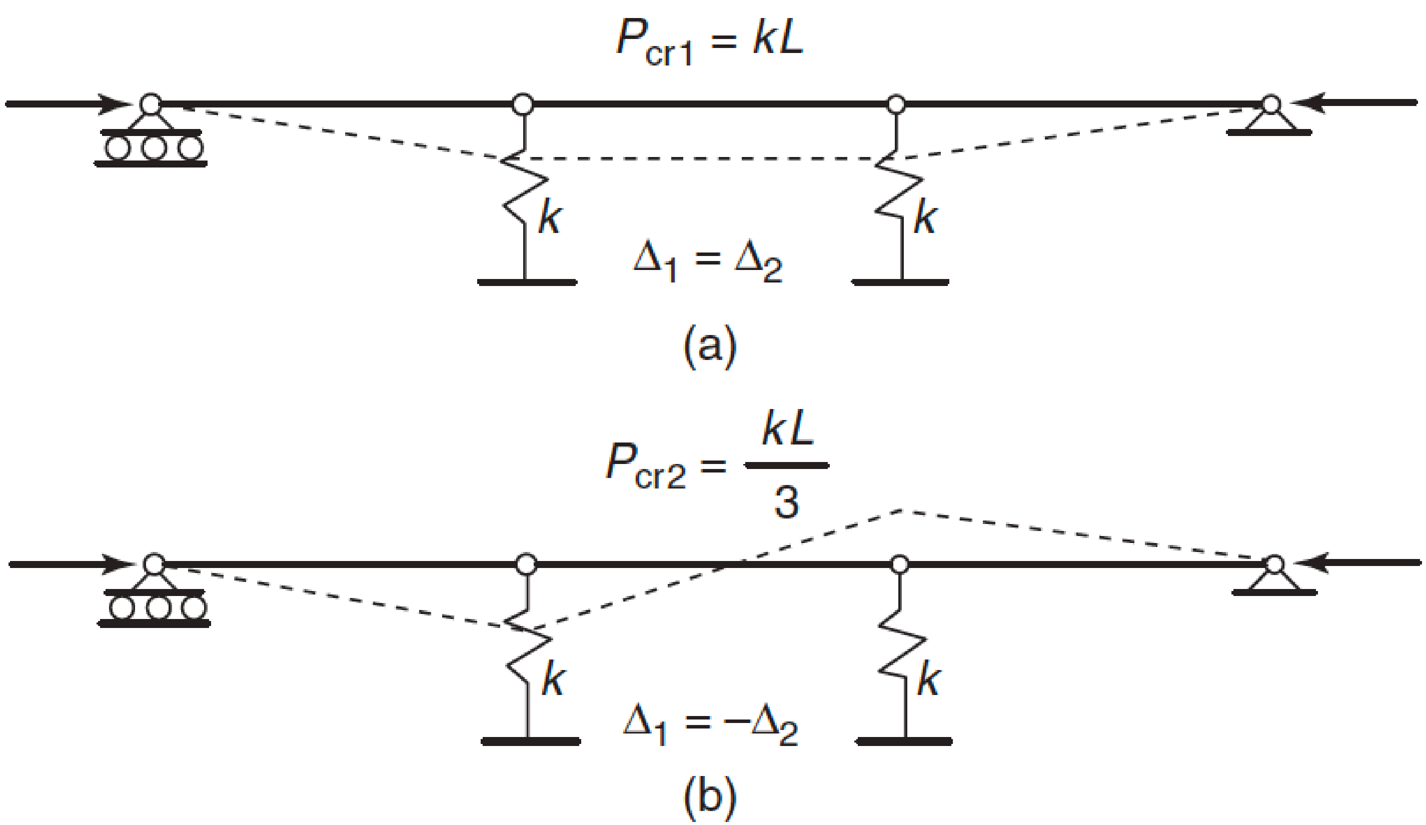
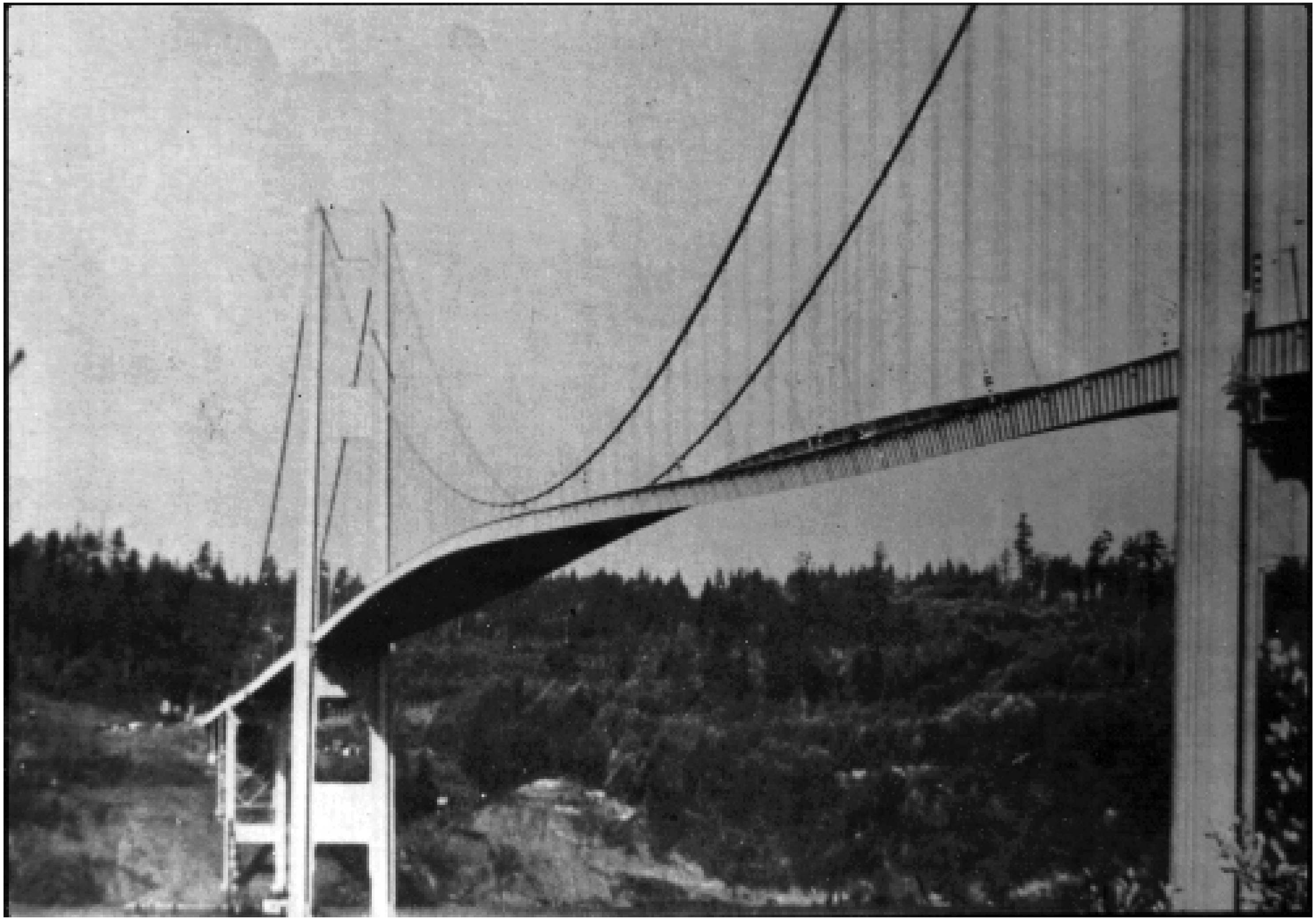


Image Source: Galambos and Surovek 2008

Dynamic Approach for Structural Stability
-Flutter: Wind Vibrations

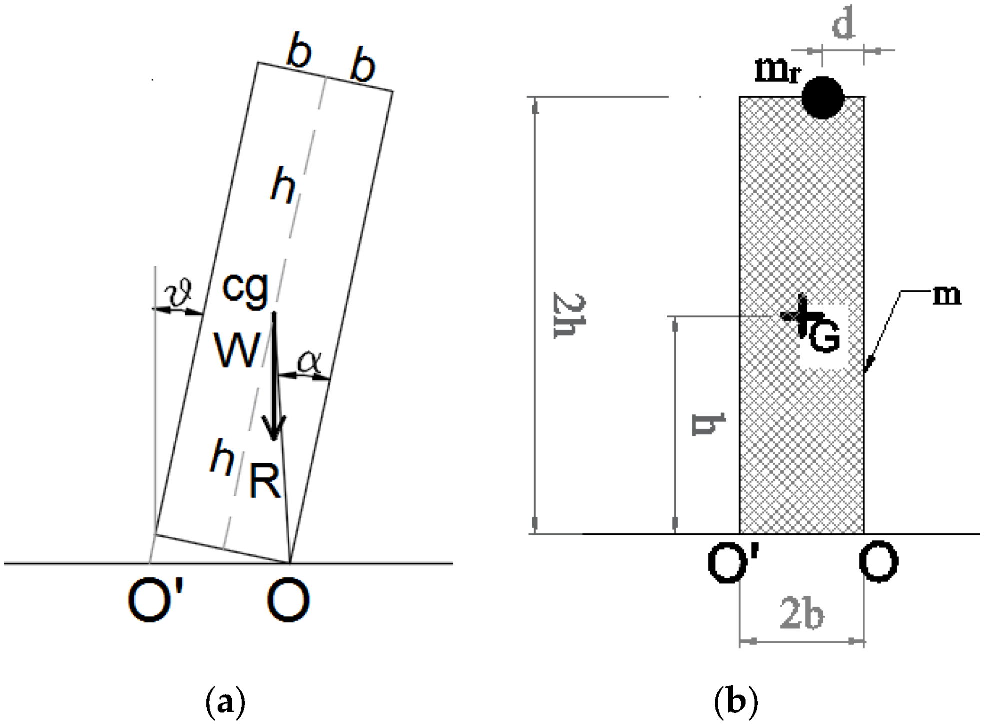


Source: Tacoma Narrows collapse

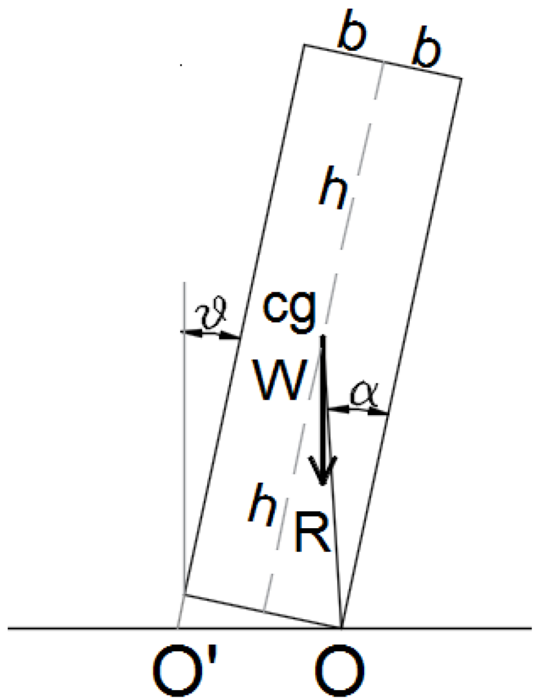
Dynamic Approach for Structural Stability

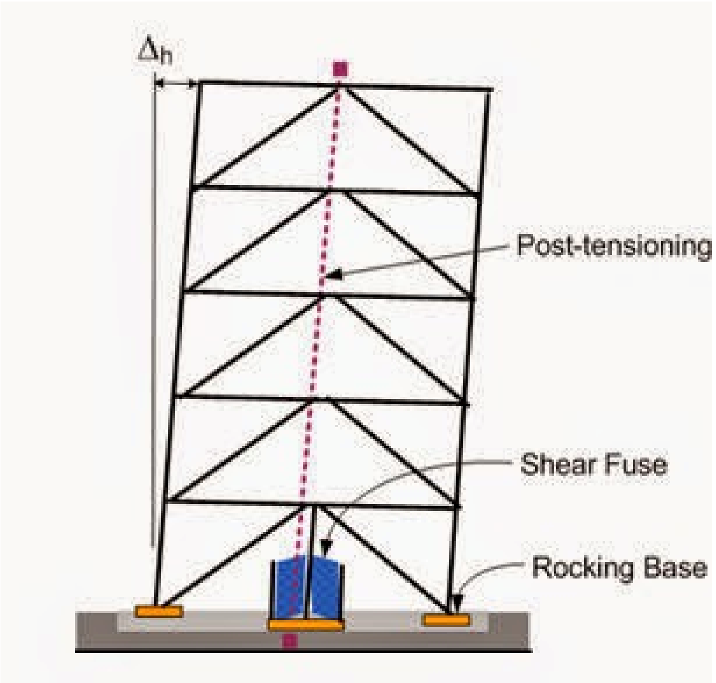
-Other Applications of Dynamic Stability - Rocking

Dynamic Approach for Structural Stability
-Other Applications of Dynamic Stability – Rocking Objects



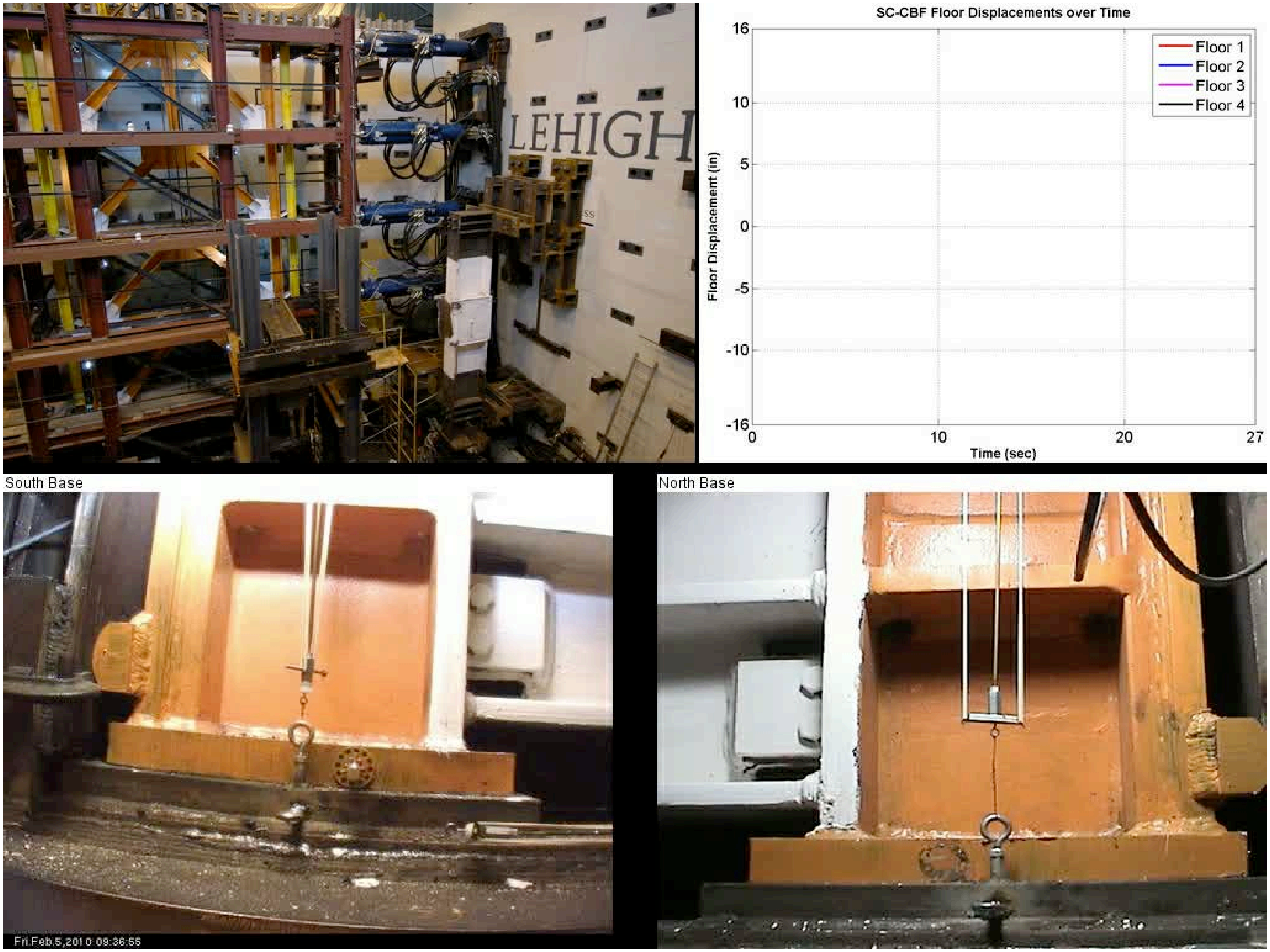
Dynamic Approach for Structural Stability
-Other Applications of Dynamic Stability – Rocking in Buildings





Source: Sause and Ricles 2006

Dynamic Approach for Structural Stability



Source: Sause and Ricles 2006

Dynamic Approach for Checking Stability

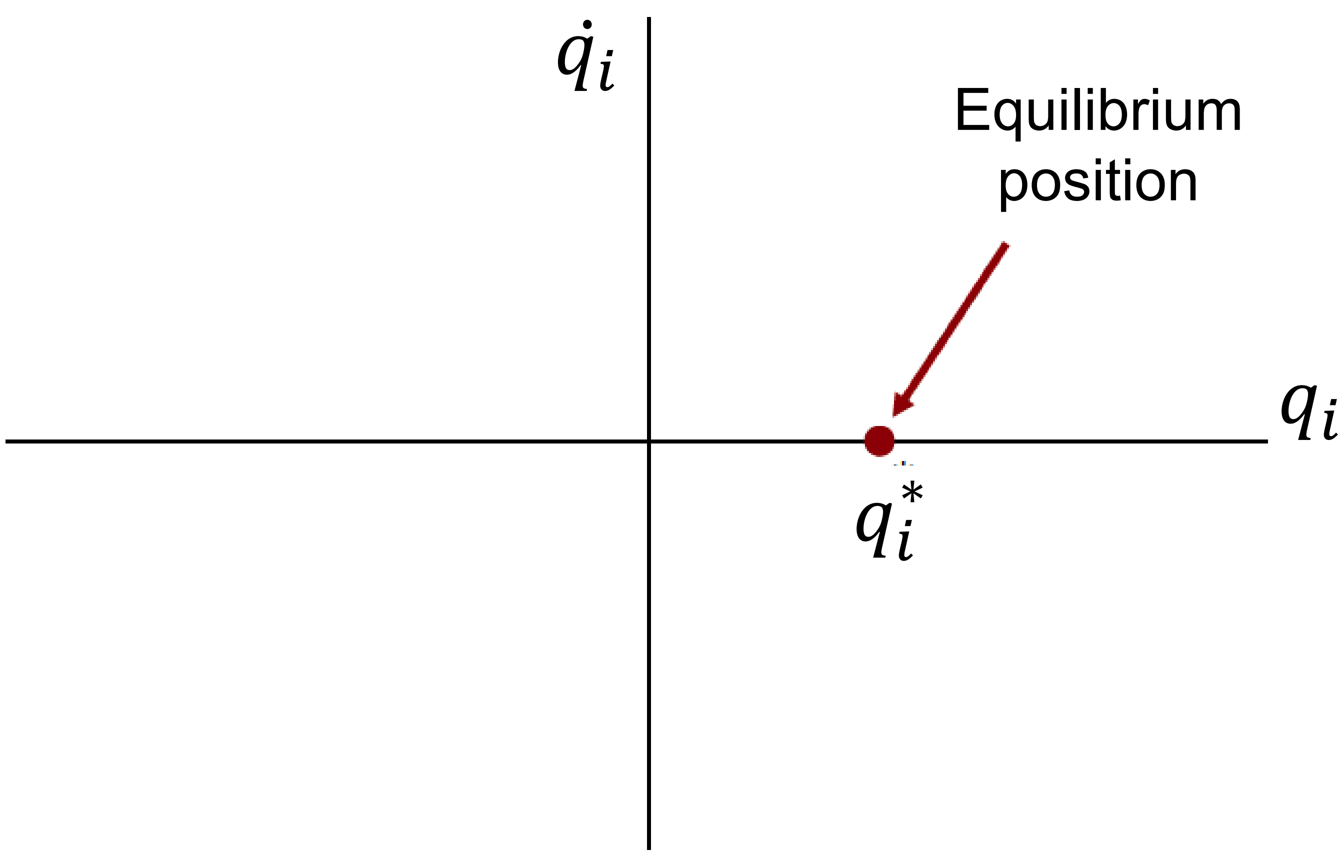
- Assume a structural system with n degrees-of-freedom (DOF) that their generalized coordinates are defined as follows:

$$q_i = (i = 1,2,3,4,5, \dots, n)$$

- Suppose that an equilibrium state is reached such that:

$$q_i = q_i^*, (i = 1,2,3,4,5, \dots, n)$$

Dynamic Approach for Checking Stability
-Phase Diagram



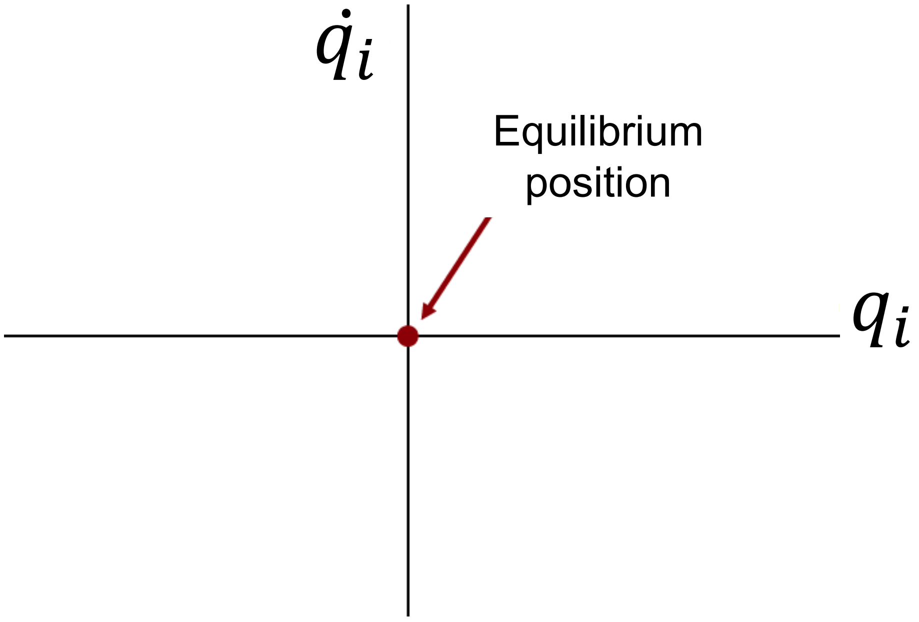
Dynamic Approach for Checking Stability

✧ Select a coordinate system such that:

$$q_i^* = 0, (i = 1,2,3,4,5, \dots, n)$$

✧ Therefore the equilibrium point of the system becomes,

$$q_i = 0, (i = 1,2,3,4,5, \dots, n)$$



Dynamic Approach for Checking Stability

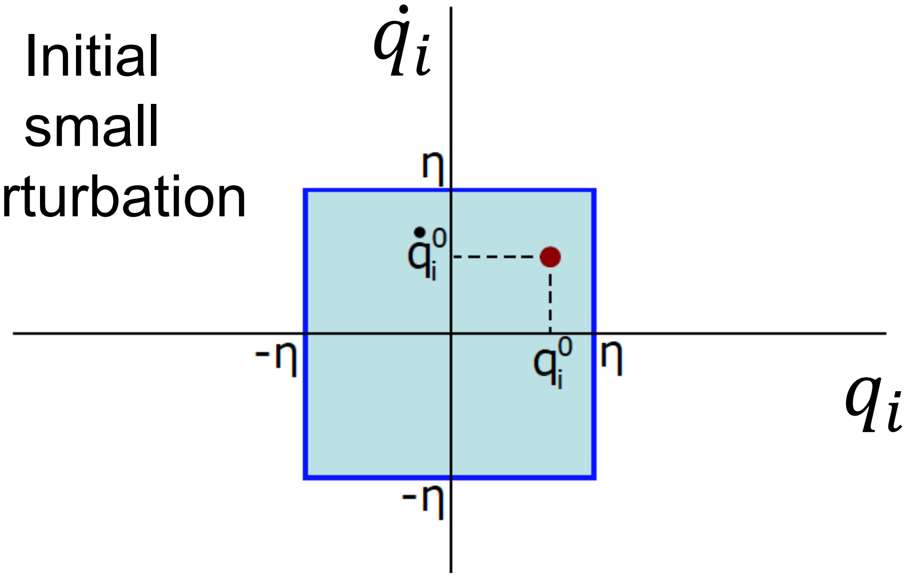
✧ Subject the system to an initial small perturbation

$$(q_i)^0, \left(\frac{\partial q_i}{\partial t}\right)^0 \; (i = 1,2,3,4,5, \dots, m)$$

✧ Therefore,

$$\|(q_i)^0\| \leq \text{n}, \left\| \left(\frac{\partial q_i}{\partial t}\right)^0 \right\| \leq= \text{n} \; (i = 1,2,3,4,5, \dots, m)$$

Initial small perturbation



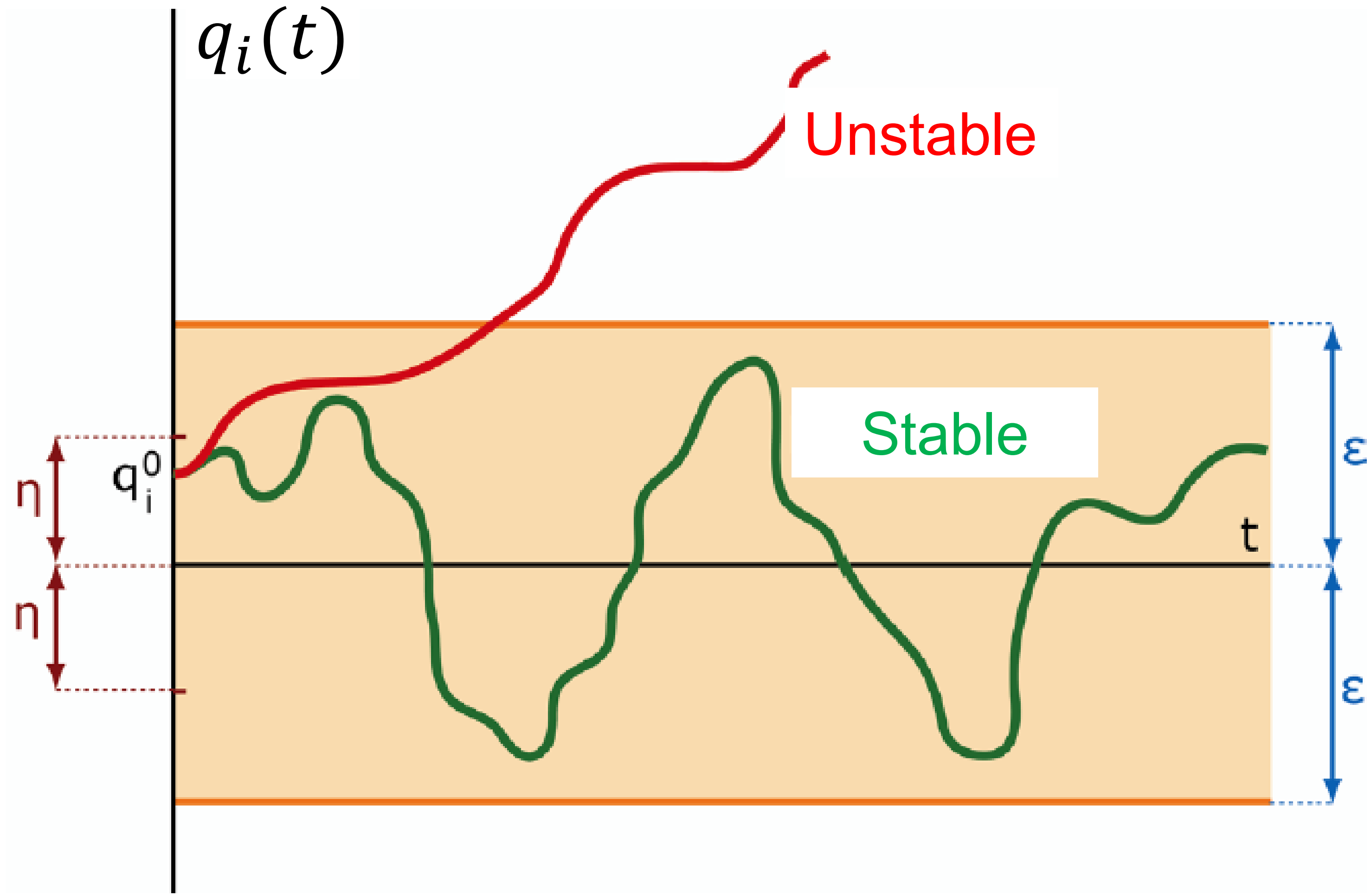
Dynamic Approach for Checking Stability
-Stability Criterion

- ✧ An equilibrium position is stable, if the vibration that the system is subjected to after the perturbation is bounded

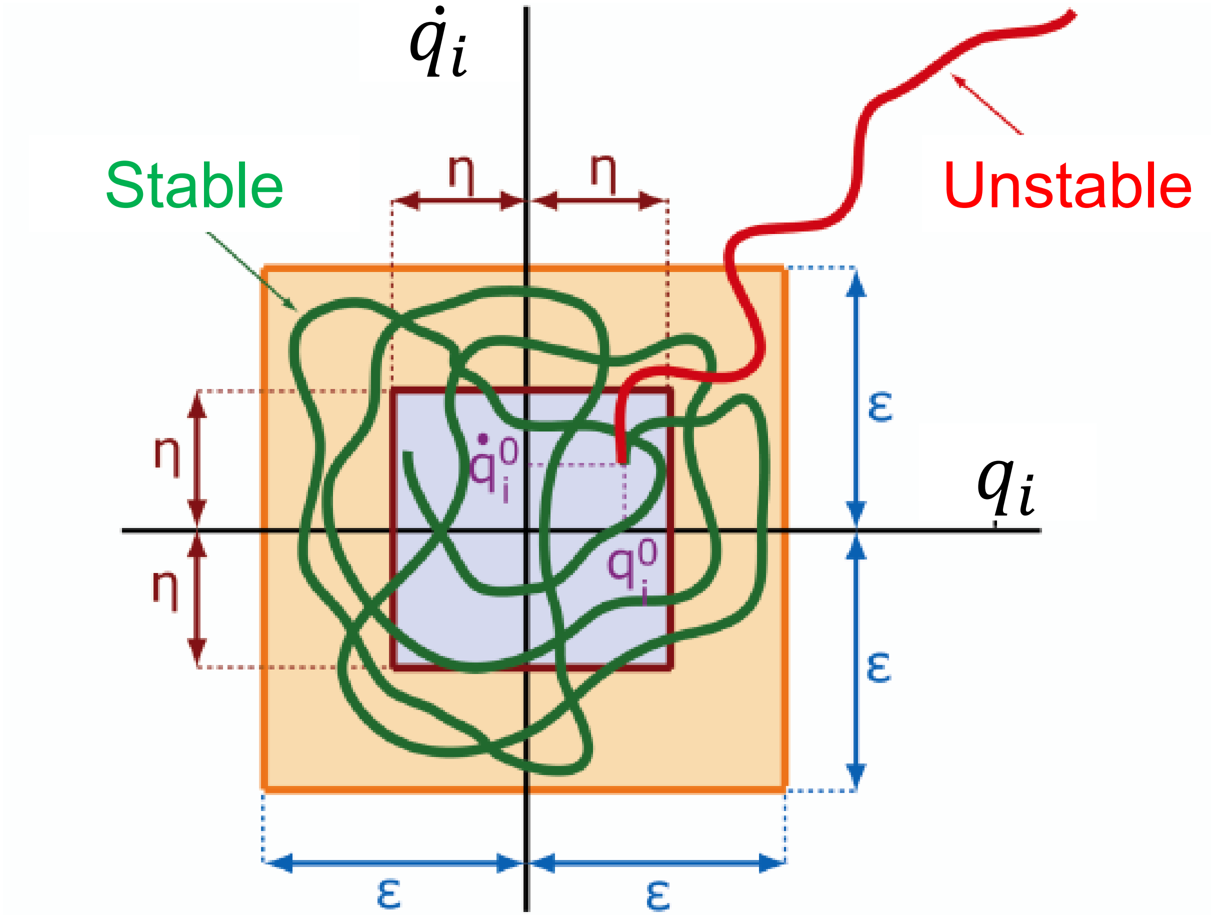
$$\forall n > 0, \exists \varepsilon = \varepsilon(n): |q_i(t)| < \varepsilon \ and \ \left|\frac{\partial q_i(t)}{\partial t}\right| < \varepsilon(i = 1,2,3,4,5, \dots, n)$$

- ✧ Else the equilibrium position is unstable

Dynamic Approach for Checking Stability
-Stability Criterion: Graphical Representation



Dynamic Approach for Checking Stability
-Graphical Representation – Phase Diagram



Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Undamped Situation

✧ Suppose you have a system with mass m and stiffness k, its equation of motion is as follows:

$$m \cdot \frac{\partial^2 q}{\partial t^2} + k \cdot q = 0$$

✧ Assume that the solution will be: $q = C \cdot \mathrm{e}^{\rho t}$

✧ By substitution,

$$mC\rho^2 \mathrm{e}^{\rho t} + kC\mathrm{e}^{\rho t} = 0 \Longrightarrow \left(\rho^2 + \frac{k}{m}\right) C \cdot \mathrm{e}^{\rho t} = 0$$

✧ Characteristic Equation,

$$\rho^2 + \frac{k}{m} = 0$$

Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Undamped Situation

✧ Characteristic Equation,

$$\rho^2 + \frac{k}{m} = 0$$

✧ If $\frac{k}{m} = \omega^2 > 0$ eigen frequencies are positive and real

$$q(t) = C_1 \cdot sin\omega t + C_2 \cdot cos\omega t$$

$$\dot{q}(t) = C_1\omega \cdot cos\omega t - C_2\omega \cdot sin\omega t$$

$$q(0) = q_0 \Rightarrow C_2 = q_0 \qquad \dot{q}(0) = \dot{q}_0 \Rightarrow C_1 = \frac{\dot{q}_0}{\omega}$$

q_0 Initial displacement

$\dot{q}_0$ Initial velocity

Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Undamped Situation

✧ Bounded vibration

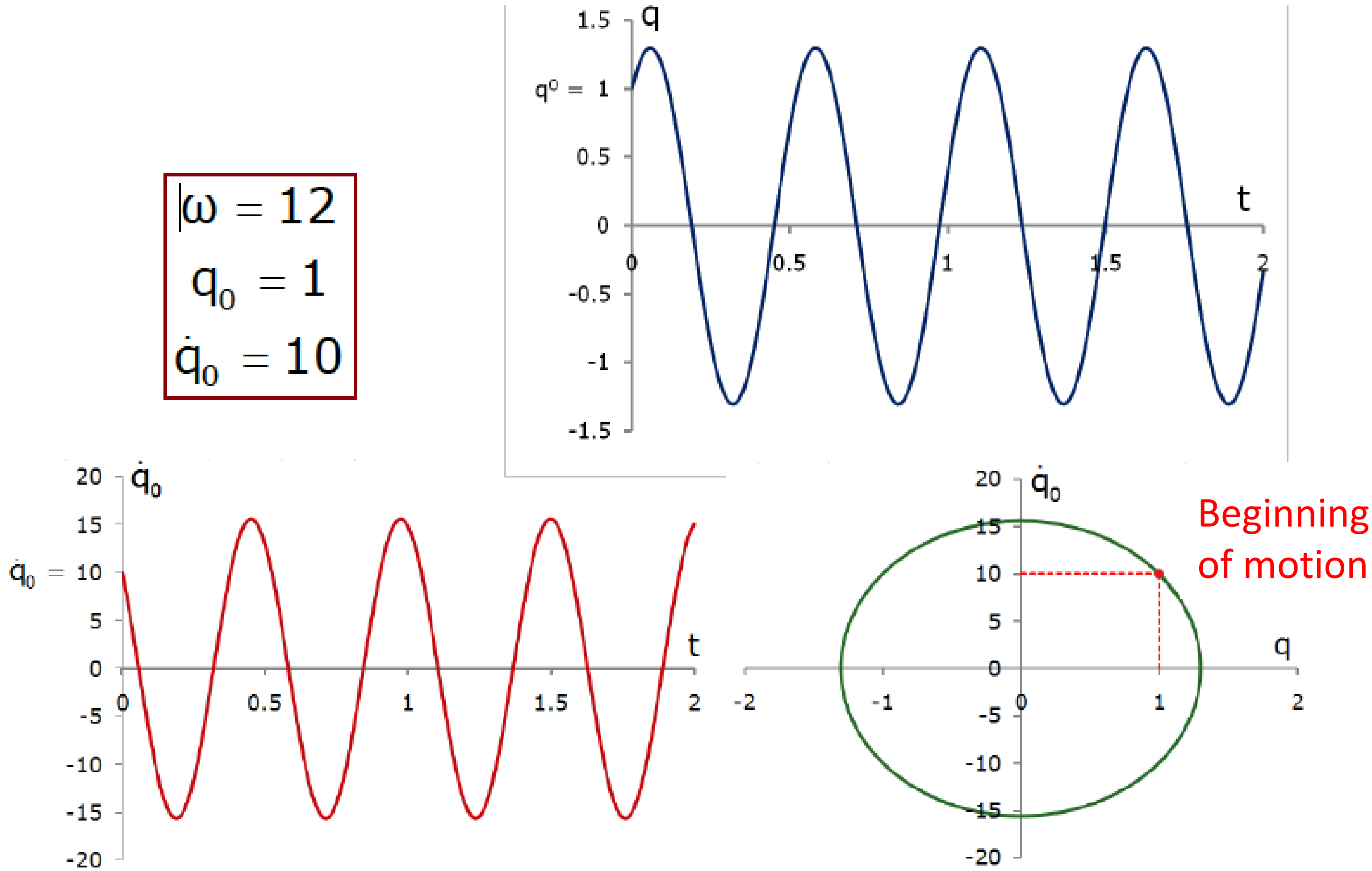
$$\max\{q(t)\} = \sqrt{(q_0)^2 + \left(\frac{\dot{q}_0}{\omega}\right)^2}$$

$$\max\{\dot{q}(t)\} = \sqrt{(q_0\omega)^2 + (\dot{q}_0)^2}$$

Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Undamped Situation

✧ Positive Eigen frequency (bounded vibration)

$$\boxed{\begin{aligned} \omega &= 12 \\ q_0 &= 1 \\ \dot{q}_0 &= 10 \end{aligned}}$$



Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Undamped Situation

✧ Characteristic Equation,

$$\rho^2 + \frac{k}{m} = 0$$

✧ If $\frac{k}{m} = \omega^2 = 0$

$$q(t) = C_1 + C_2 \cdot t$$

$$\dot{q}(t) = C_2$$

$$q(0) = q_0 \Rightarrow C_1 = q_0 \qquad \dot{q}(0) = \dot{q}_0 \Rightarrow C_2 = \dot{q}_0$$

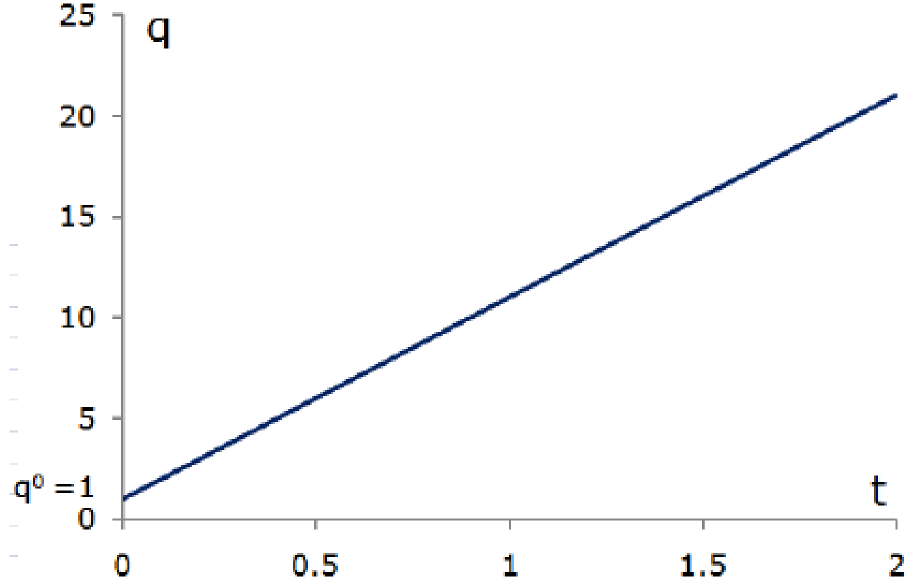
q_0 Initial displacement

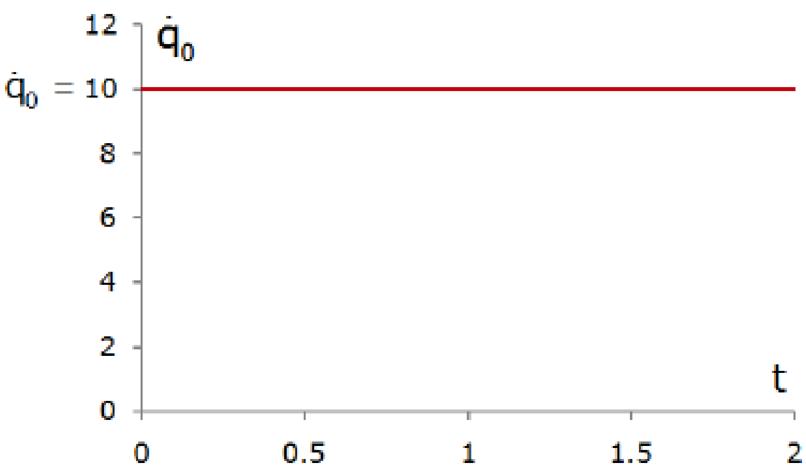
$\dot{q}_0$ Initial velocity

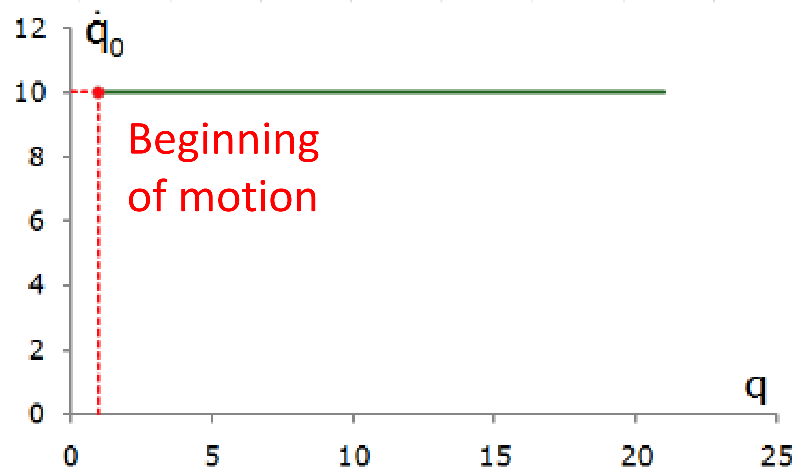
Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Undamped Situation

- Zero Eigen frequency unbounded vibration linearly increasing amplitude

$$\boxed{\begin{aligned} \omega &= 0 \\ q_0 &= 1 \\ \dot{q}_0 &= 10 \end{aligned}}$$







Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Undamped Situation

✧ Characteristic Equation,

$$\rho^2 + \frac{k}{m} = 0$$

✧ If $\frac{k}{m} = \omega^2 < 0$ The eigen frequency of the system is imaginary, $\rho = \pm\omega$

$$q(t) = C_1 \cdot e^{\rho_1 t} + C_2 \cdot e^{\rho_2 t} = C_1 \cdot e^{\omega t} + C_2 \cdot e^{-\omega t}$$

$$\dot{q}(t) = C_1\omega \cdot e^{\omega t} - C_2\omega \cdot e^{-\omega t}$$

$$q(0) = q_0 \Rightarrow C_1 + C_2 = q_0 \quad \dot{q}(0) = \dot{q}_0 \Rightarrow C_1 - C_2 = \frac{\dot{q}_0}{\omega}$$

q_0 Initial displacement

$\dot{q}_0$ Initial velocity

$$C_1 = \frac{1}{2}\left(q_0 + \frac{\dot{q}_0}{\omega}\right) \quad C_2 = \frac{1}{2}\left(q_0 - \frac{\dot{q}_0}{\omega}\right)$$

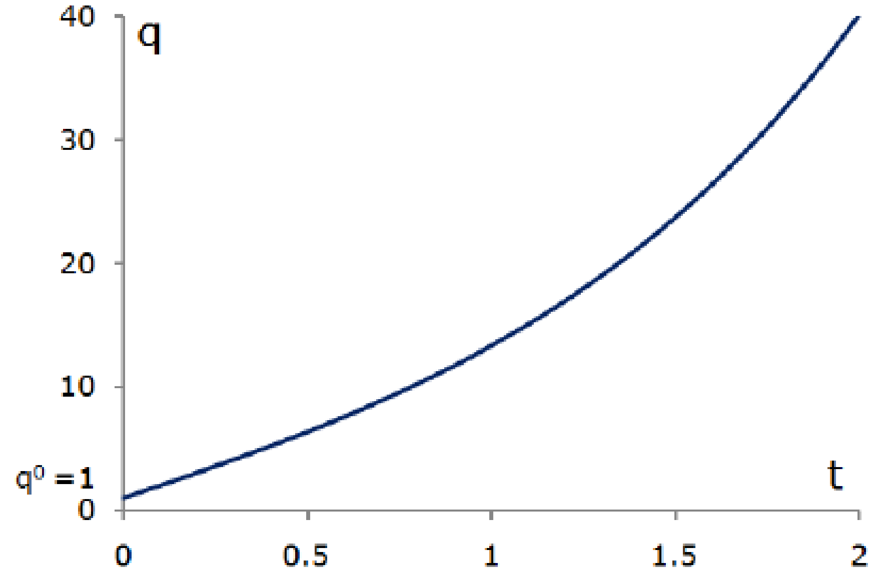
Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Undamped Situation

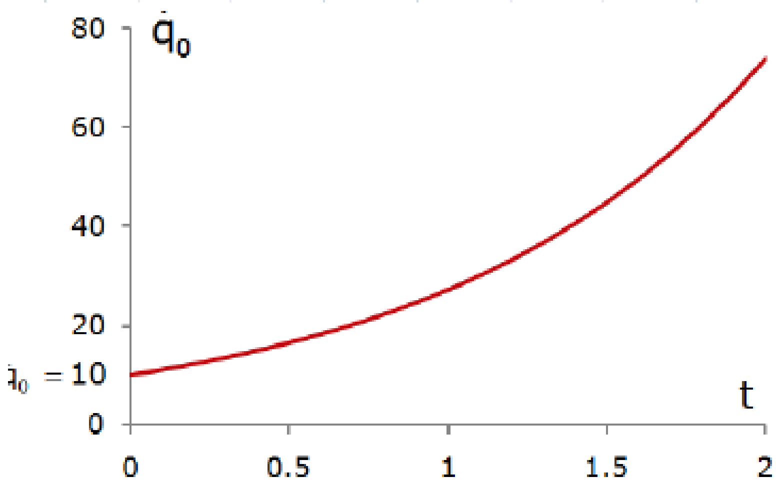
Imaginary eigen frequency

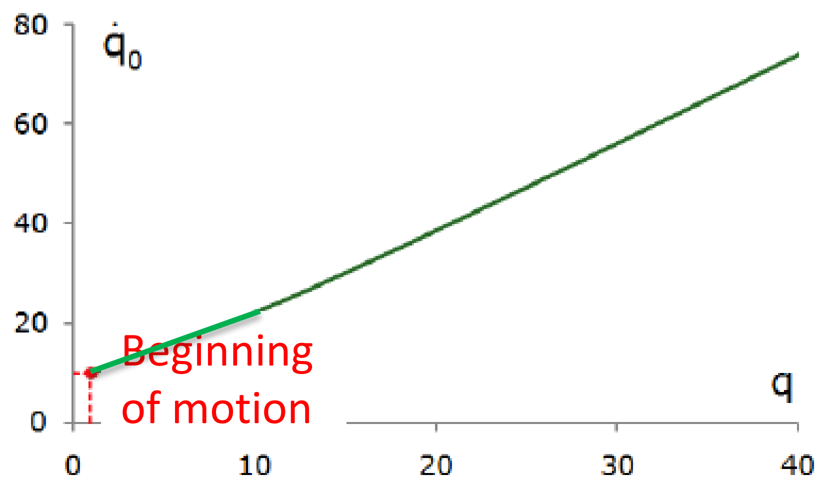
unbounded vibration exponentially

increasing amplitude

$$\boxed{\begin{aligned}\omega &= 1 \cdot i \\ q_0 &= 1 \\ \dot{q}_0 &= 10\end{aligned}}$$

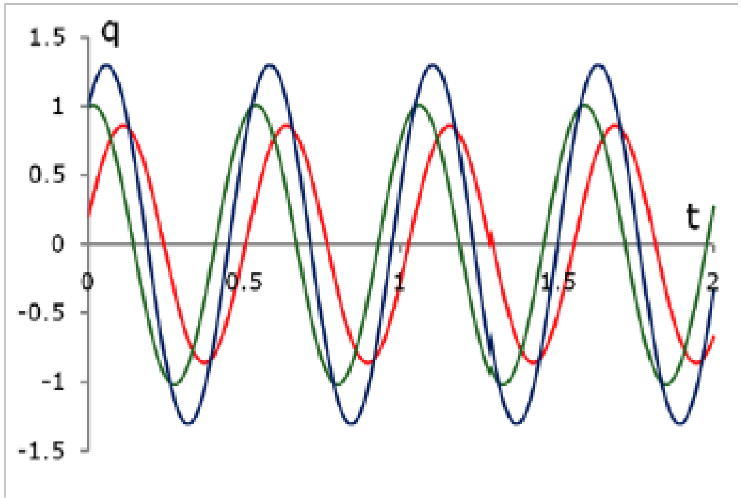


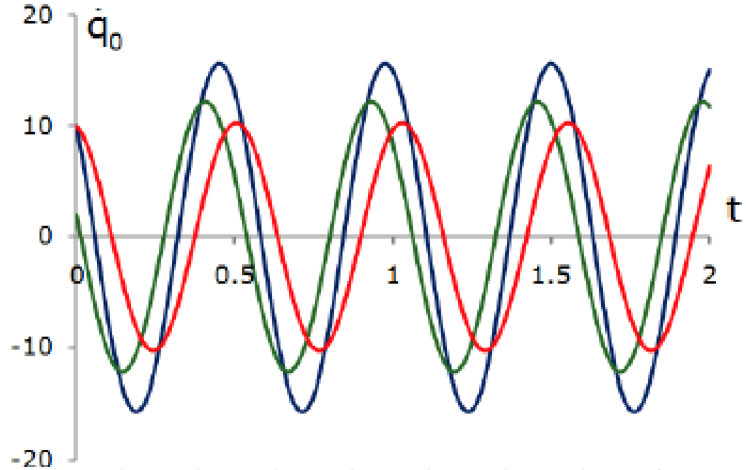




Influence of Initial Conditions for Single-Degree of Freedom Systems

The magnitude of initial perturbation affects the amplitude of vibration but not the type of response



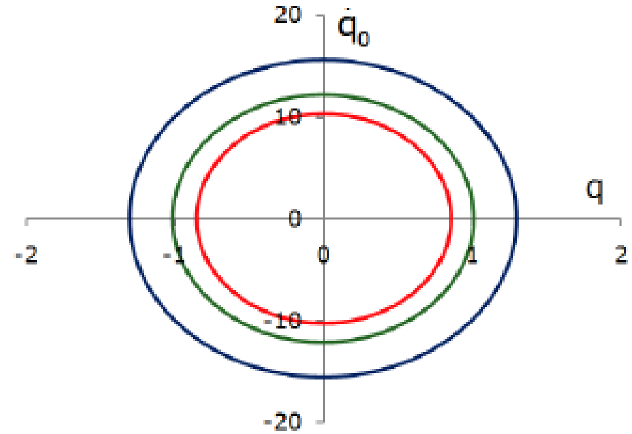


$q^0=1,\ \dot{q}^0=10$

$\boxed{\omega = 12}$

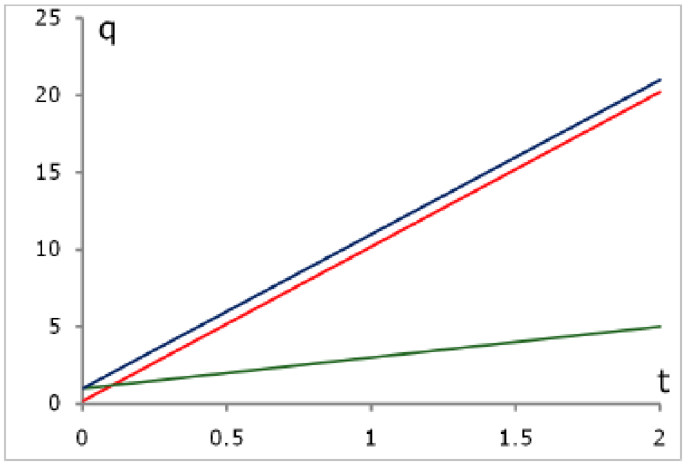
$q^0=1,\ \dot{q}^0=2$

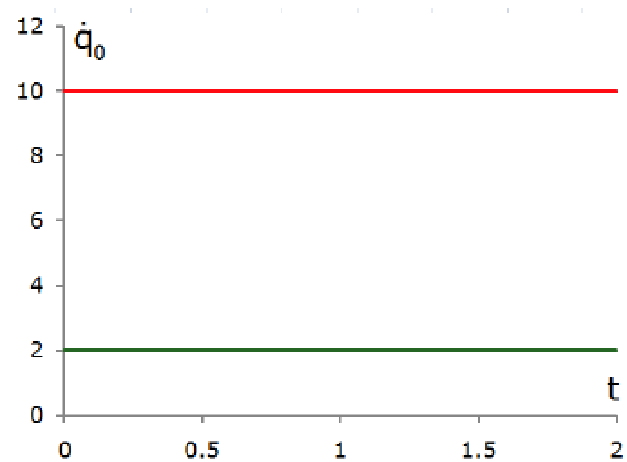
$q^0=0.2,\ \dot{q}^0=10$

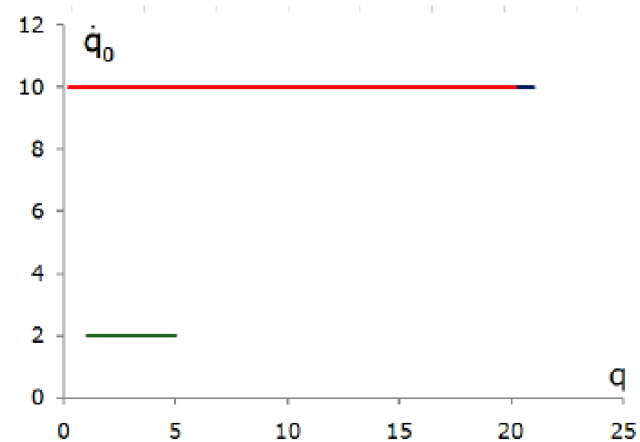


Influence of Initial Conditions for Single-Degree of Freedom Systems

The magnitude of initial perturbation affects the amplitude of vibration but not the type of response







$\boxed{\omega = 0}$

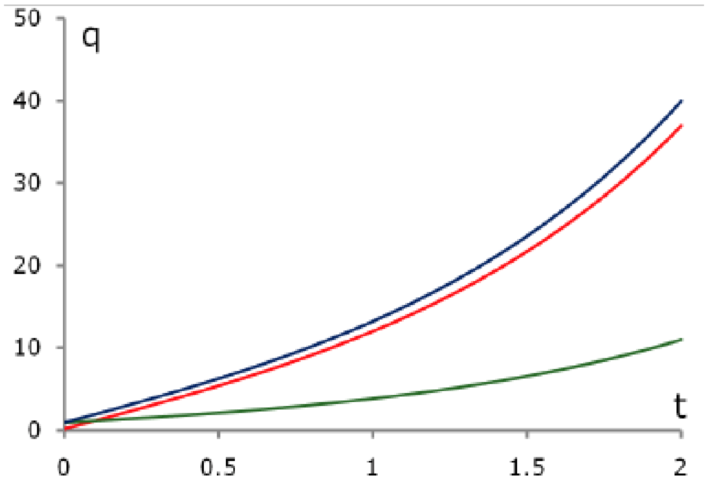
$q^0=1,\ \dot{q}^0 = 10$

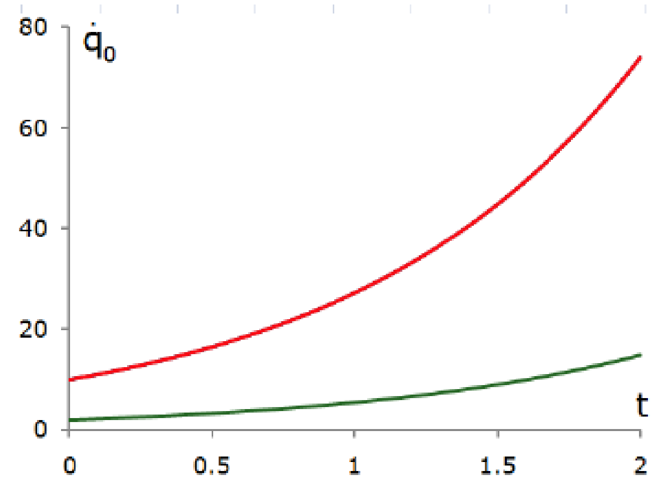
$q^0=1,\ \dot{q}^0 = 2$

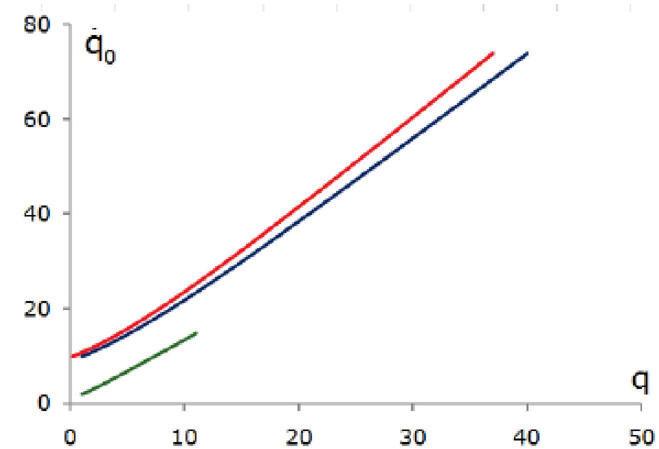
$q^0=0.2,\ \dot{q}^0 = 10$

Influence of Initial Conditions for Single-Degree of Freedom Systems

The magnitude of initial perturbation affects the amplitude of vibration but not the type of response







$q^0=1,\ \dot{q}^0=10$

$\boxed{\omega = 1 \cdot i}$ $\quad q^0=1,\ \dot{q}^0=2$

$q^0=0.2,\ \dot{q}^0=10$

Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Damped Vibration

✧ Suppose you have a damped SDF system with mass *m* and stiffness k, its equation of motion is as follows:

$$m \cdot \frac{\partial^2 q}{\partial t^2} + c \cdot \frac{\partial q}{\partial t} + k \cdot q = 0$$

✧ Assume that the solution will be: $q = A \cdot \mathrm{e}^{\rho t}$

✧ By substitution,

$$mA\rho^2\mathrm{e}^{\rho t} + cA\rho\mathrm{e}^{\rho t} + \; kC\mathrm{e}^{\rho t} = 0 \Longrightarrow \left(\rho^2 + \frac{c}{m}\rho + \frac{k}{m}\right)\mathrm{A} \cdot \mathrm{e}^{\rho t} = 0$$

✧ Characteristic Equation,

$$\rho^2 + \frac{c}{m}\rho + \frac{k}{m} = 0$$

Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Damped Vibration

$$\rho^2 + \frac{c}{m}\rho + \frac{k}{m} = 0$$

$$\Delta = \left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right) = \left(\frac{c}{m}\right)^2 - 4\omega^2$$

✧ If $c > c_{cr} = 2m\omega$

$$\rho_{1,2} = -\zeta\omega \pm \omega\sqrt{1-\zeta^2}$$

✧ In which,

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega} \text{ (damping ratio)}$$

✧ Assume,

$$\omega_D = \omega\sqrt{1-\zeta^2}$$

Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Damped Vibration

✧ General solution,

$$q(t) = e_n^{-\zeta\omega t}(C_1 \cdot sin\omega_D t + C_2 \cdot cos\omega_D t)$$

$$\dot{q}(t) = -\zeta\omega e_n^{-\zeta\omega t}(C_1 \cdot sin\omega_D t + C_2 \cdot cos\omega_D t) + e_n^{-\zeta\omega t}(\omega_D C_1 \cdot cos\omega_D t - \omega_D C_2 \cdot sin\omega_D t)$$

$$q(0) = q_0 \Rightarrow C_1 = q_0 \qquad \dot{q}(0) = \dot{q}_0 \Rightarrow C_2 = \frac{\dot{q}_0 - q_0\omega\zeta}{\omega_D}$$

q_0 Initial displacement

$\dot{q}_0$ Initial velocity

✧ Bounded vibration

$$\max\{q(t)\} = e^{-\zeta\omega t}\sqrt{(q_0)^2 + \left(\frac{\dot{q}_0 + \omega\zeta q_0}{\omega_D}\right)^2}$$

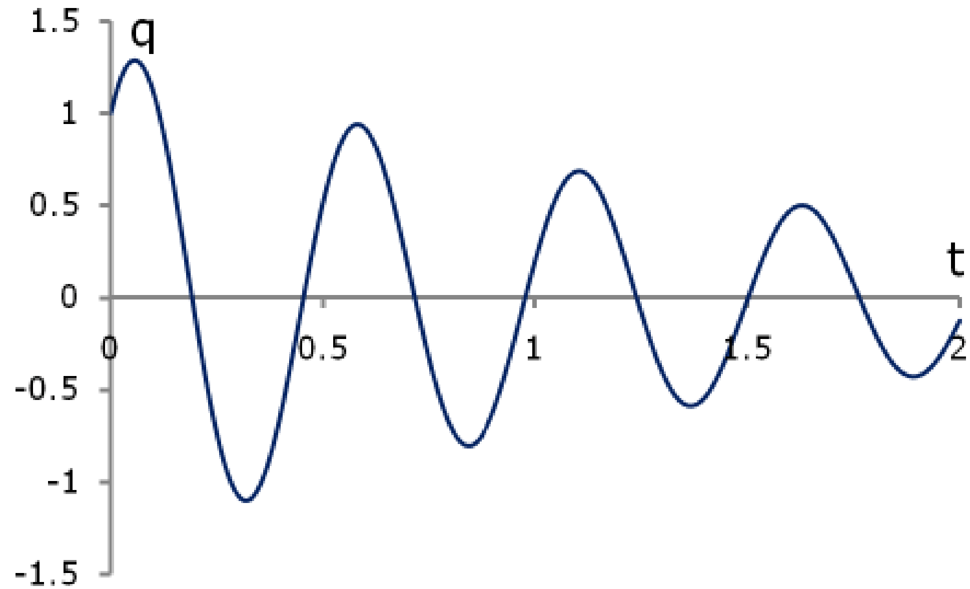
$$\max\{\dot{q}(t)\} = e^{-\zeta\omega t}\sqrt{\left(q_0\omega_D + \frac{\zeta\omega\dot{q}_0 + (\zeta\omega)^2 q_0}{\omega_D}\right)^2 + (\dot{q}_0)^2}$$

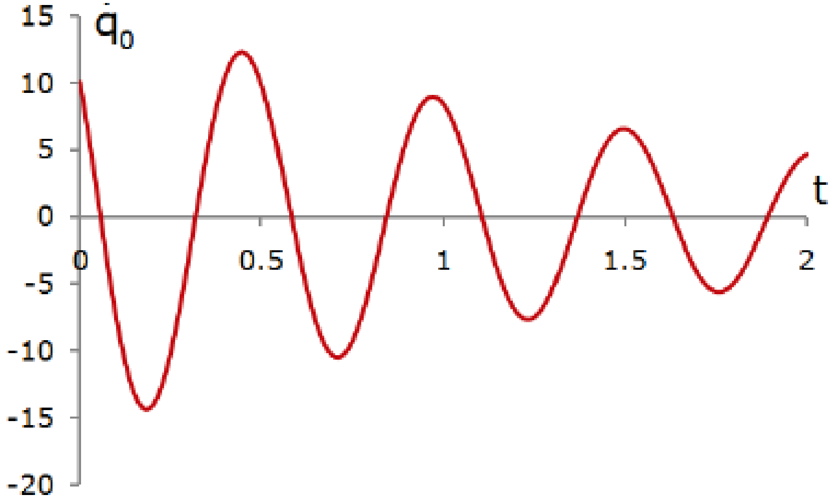
Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Damped Vibration

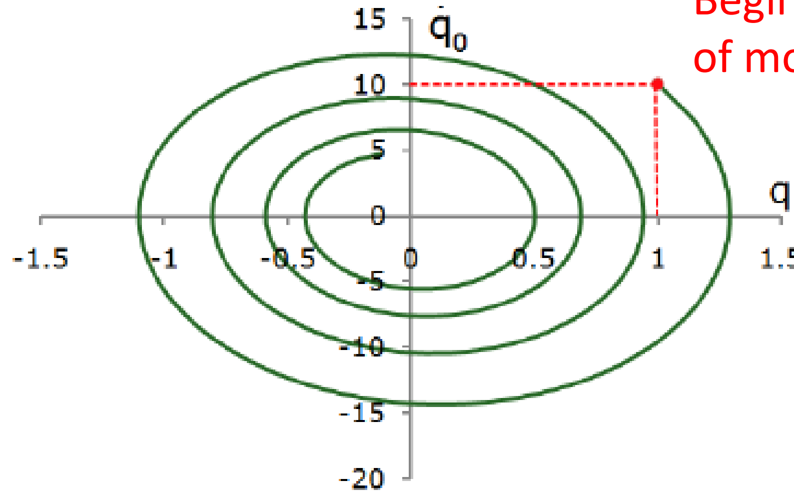
Positive eigen frequency

Bounded vibration

$$\boxed{\begin{aligned} \omega &= 12 \\ q_0 &= 1 \\ \dot{q}_0 &= 10 \\ \zeta &= 5\% \end{aligned}}$$







Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Damped Vibration

✧ If $\dfrac{k}{m} = \omega^2 = 0$

$$q(t) = C_1 + C_2 \cdot e^{-\frac{c}{m}t}$$

$$\dot{q}(t) = -C_2 \cdot \frac{c}{m} \cdot e^{-\frac{c}{m}t}$$

$$q(0) = q_0 \Rightarrow C_1 = q_0 + \frac{\dot{q}_0 m}{c} \qquad \dot{q}(0) = \dot{q}_0 \Rightarrow C_2 = \frac{-\dot{q}_0 m}{c}$$

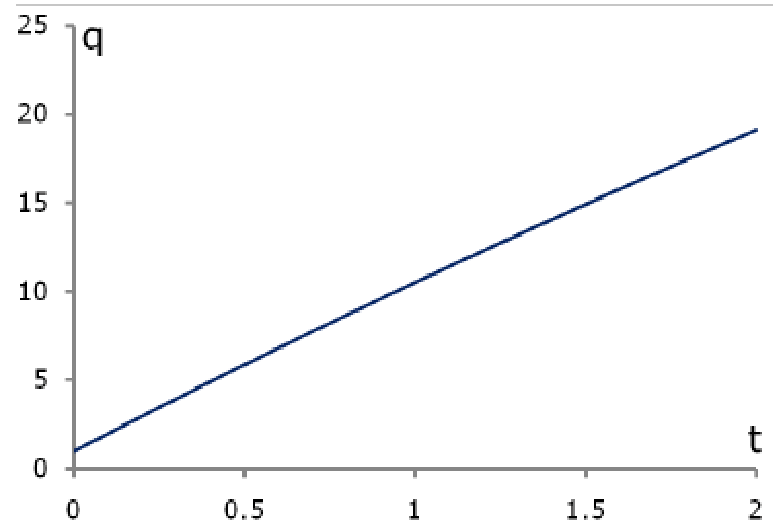
q_0 Initial displacement

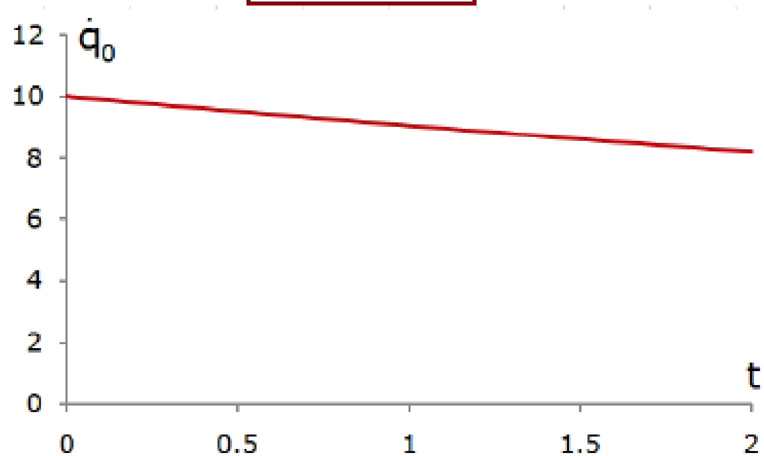
$\dot{q}_0$ Initial velocity

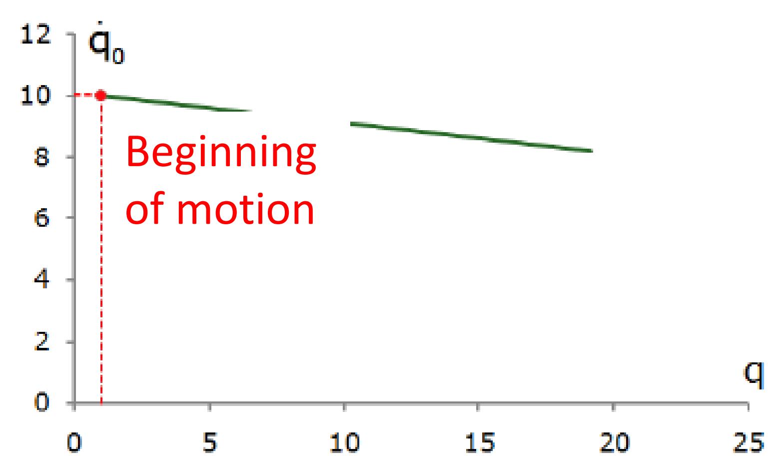
Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Damped Vibration

Zero eigen frequency unbounded vibration with linearly increasing amplitude

$$\boxed{\begin{aligned} \omega &= 0 \\ q_0 &= 1 \\ \dot{q}_0 &= 10 \\ \zeta &= 5\% \end{aligned}}$$







Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Damped Vibration

✧ If $\dfrac{k}{m} = \omega^2 < 0$ The eigen frequency of the system is imaginary, $\rho=\pm\omega$

$$\frac{\partial^2 q}{\partial t^2} + \frac{c}{m} \cdot \frac{\partial q}{\partial t} + (i\omega)^2 \cdot q = 0$$

$$q(t) = C_1 e^{\left(-\zeta\omega+\omega\sqrt{\zeta^2+1}\right)t} + C_2 \cdot e^{\left(-\zeta\omega-\omega\sqrt{\zeta^2+1}\right)t}$$

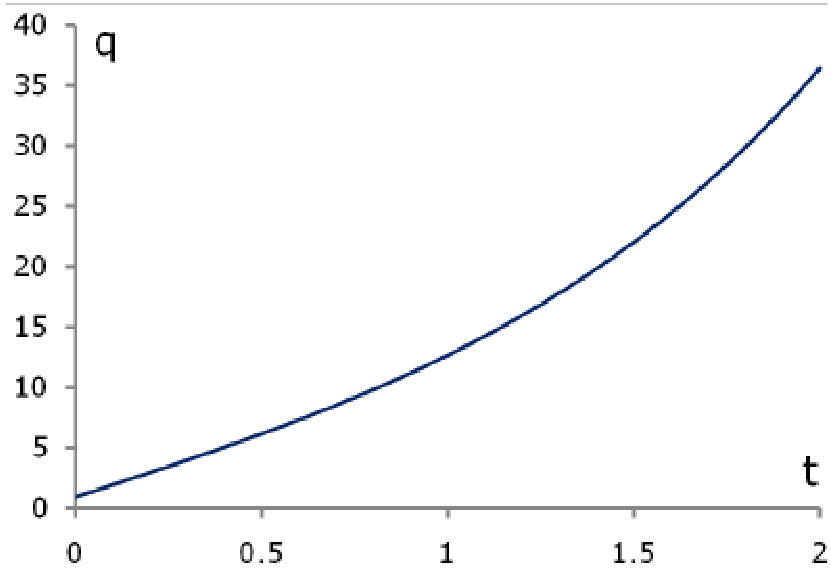
$$\dot{q}(t) = C_1\left(-\zeta\omega + \omega\sqrt{\zeta^2+1}\right)e^{\left(-\zeta\omega+\omega\sqrt{\zeta^2+1}\right)t}$$
$$-C_2\left(+\zeta\omega + \omega\sqrt{\zeta^2+1}\right)e^{\left(-\zeta\omega-\omega\sqrt{\zeta^2+1}\right)t}$$

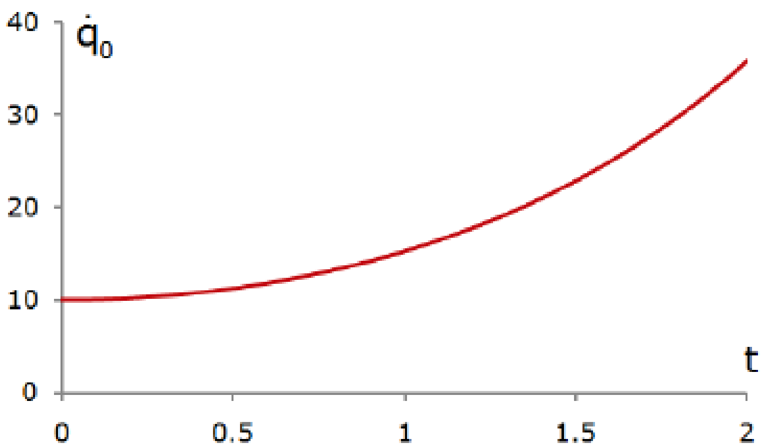
$\dot{q}_0$ Initial velocity $\quad q_0$ Initial displacement

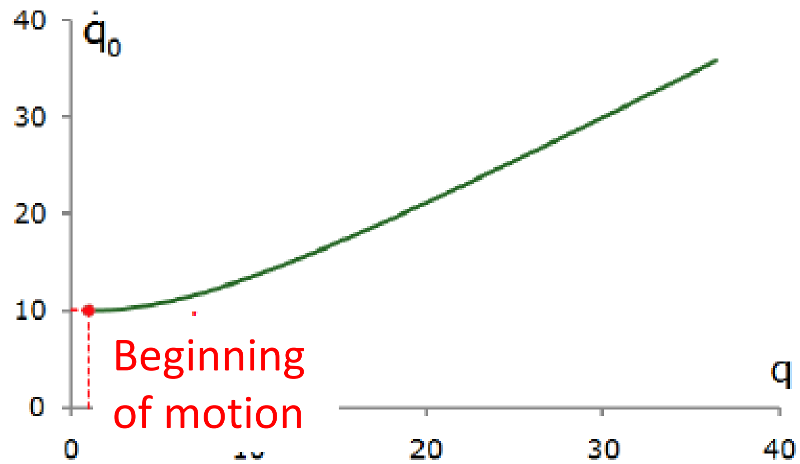
Mathematical Representation of Dynamic Approach
-Single Degree of Freedom Systems – Damped Vibration

Imaginary eigen frequency
unbounded vibration with
exponentially increasing amplitude

$$\boxed{\begin{aligned} \omega &= 1 \cdot i \\ q_0 &= 1 \\ \dot{q}_0 &= 10 \\ \zeta &= 5\% \end{aligned}}$$

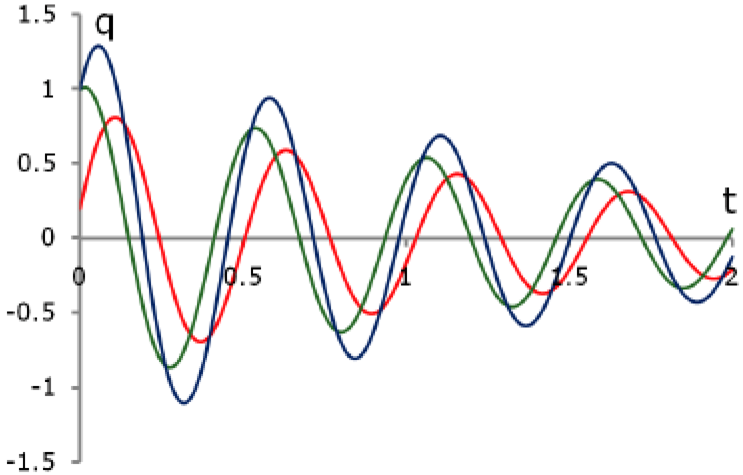


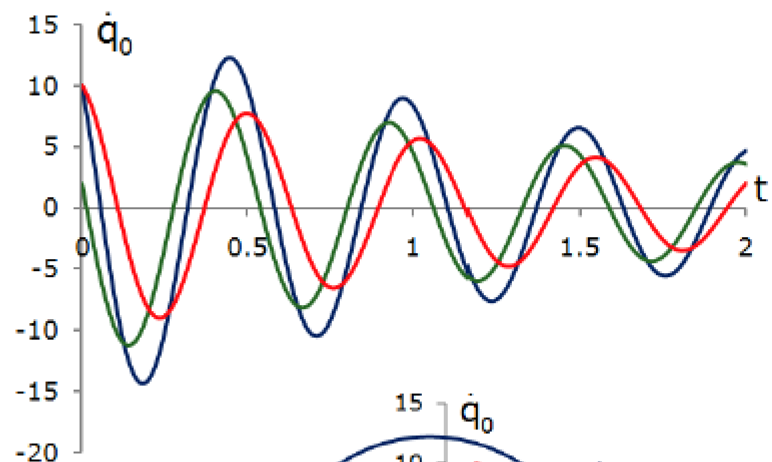


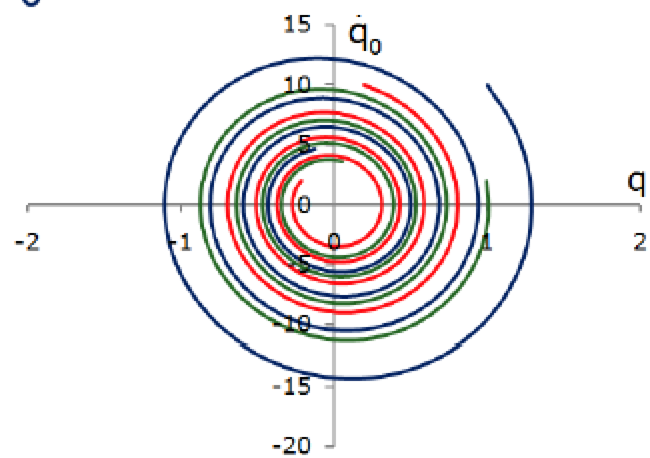


Influence of Initial Conditions for Single-Degree of Freedom Systems

The magnitude of initial perturbation affects the amplitude of vibration but not the type of response







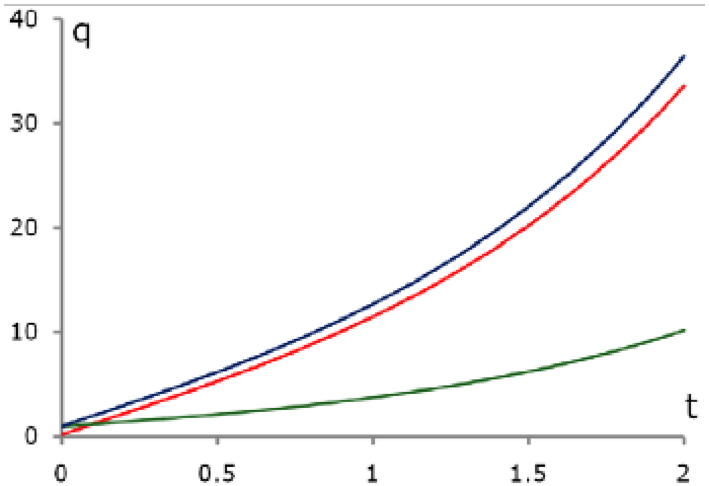
$q^0=1,\ \dot{q}^0=10$

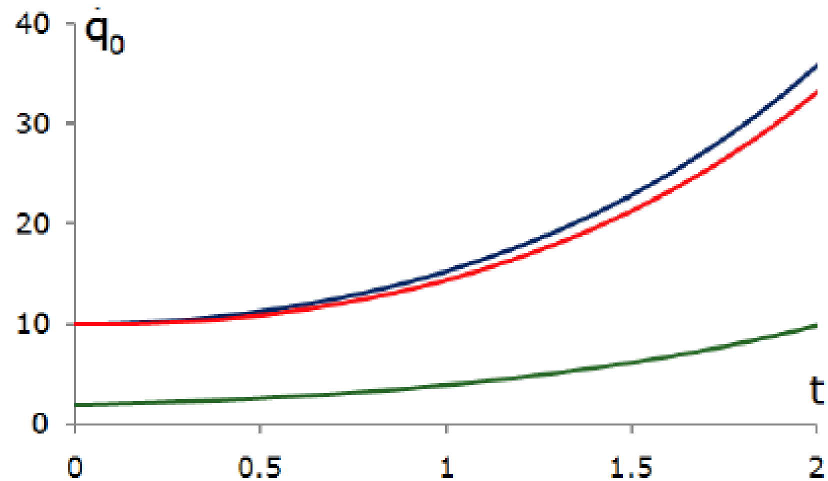
$\boxed{\omega = 12}$

$q^0=1,\ \dot{q}^0=2$

$q^0=0.2,\ \dot{q}^0=10$

Influence of Initial Conditions for Single-Degree of Freedom Systems



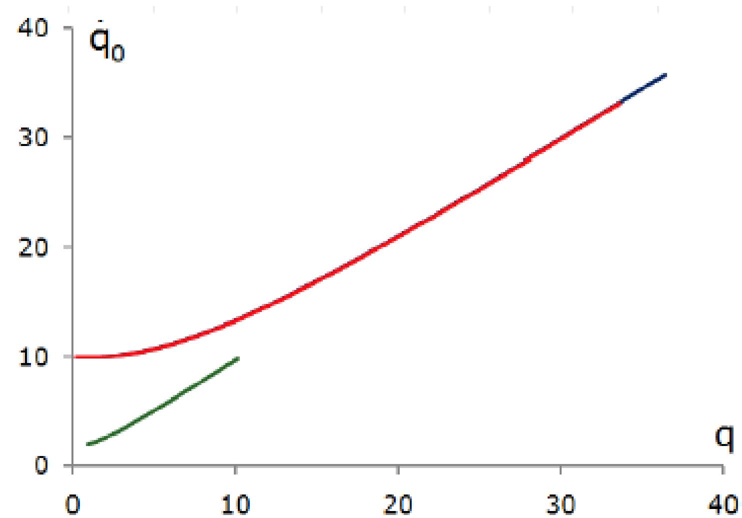


$$\boxed{\omega = 1 \cdot i}$$

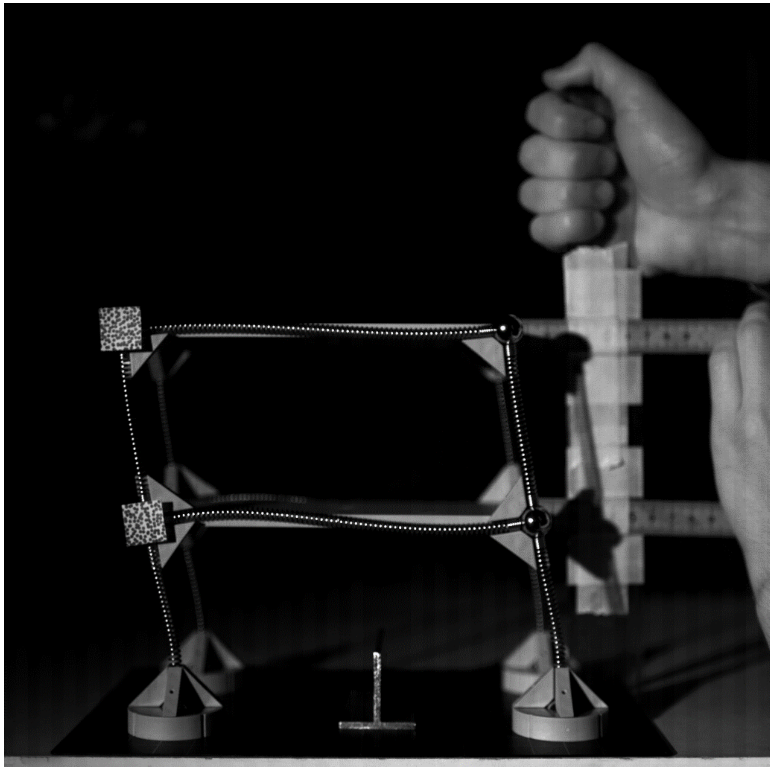
$q^0 = 1,\ \dot{q}^0 = 10$

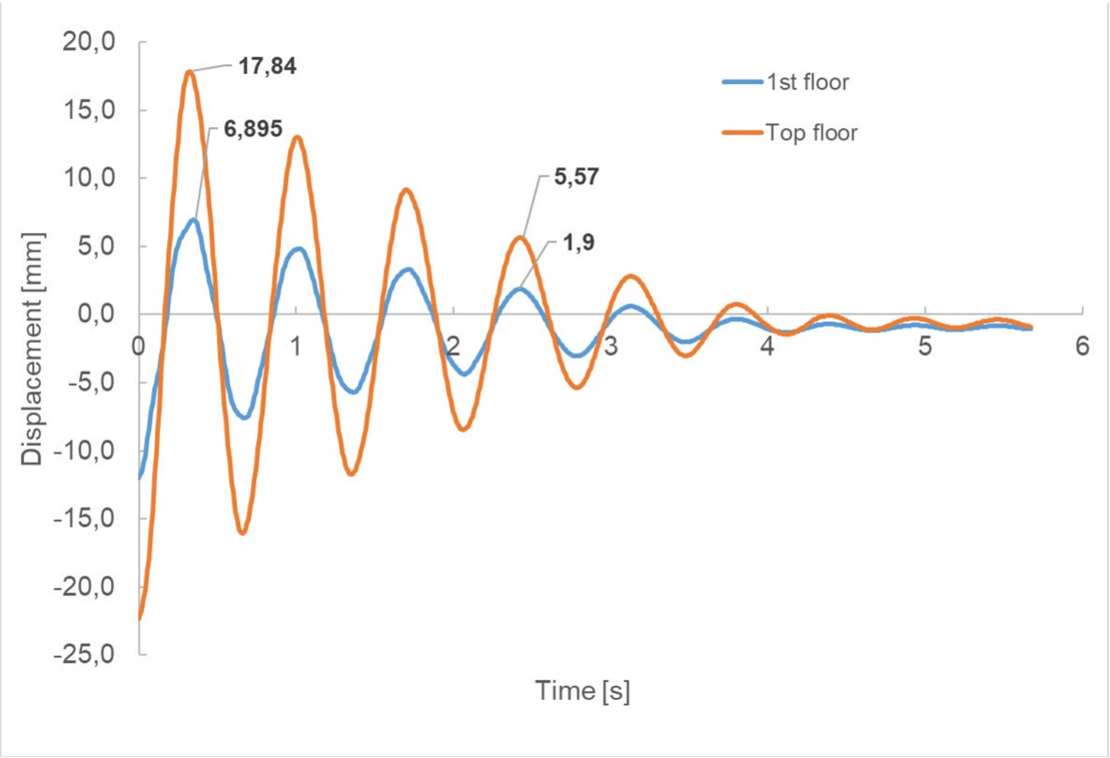
$q^0 = 1,\ \dot{q}^0 = 2$

$q^0 = 0.2,\ \dot{q}^0 = 10$



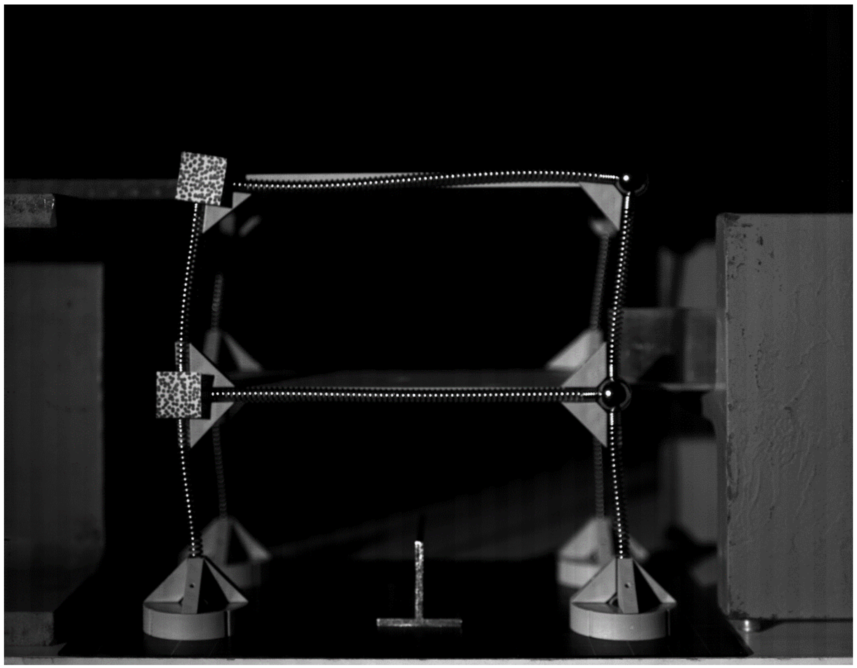
Dynamic Stability – RESSLab Applications
-Damped Free Vibration (Stable)

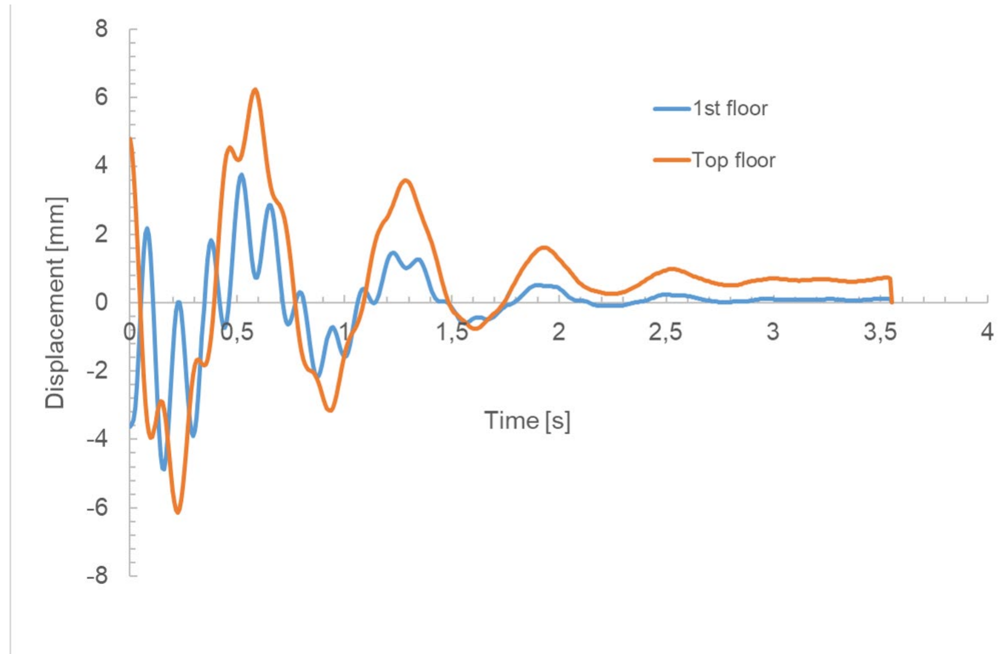




Source: (Master Students Hugo Ribet, Yan Zhang, RESSLab 2017)

Dynamic Stability – RESSLab Applications
-Damped Free Vibration (Stable) – Different Initial Conditions

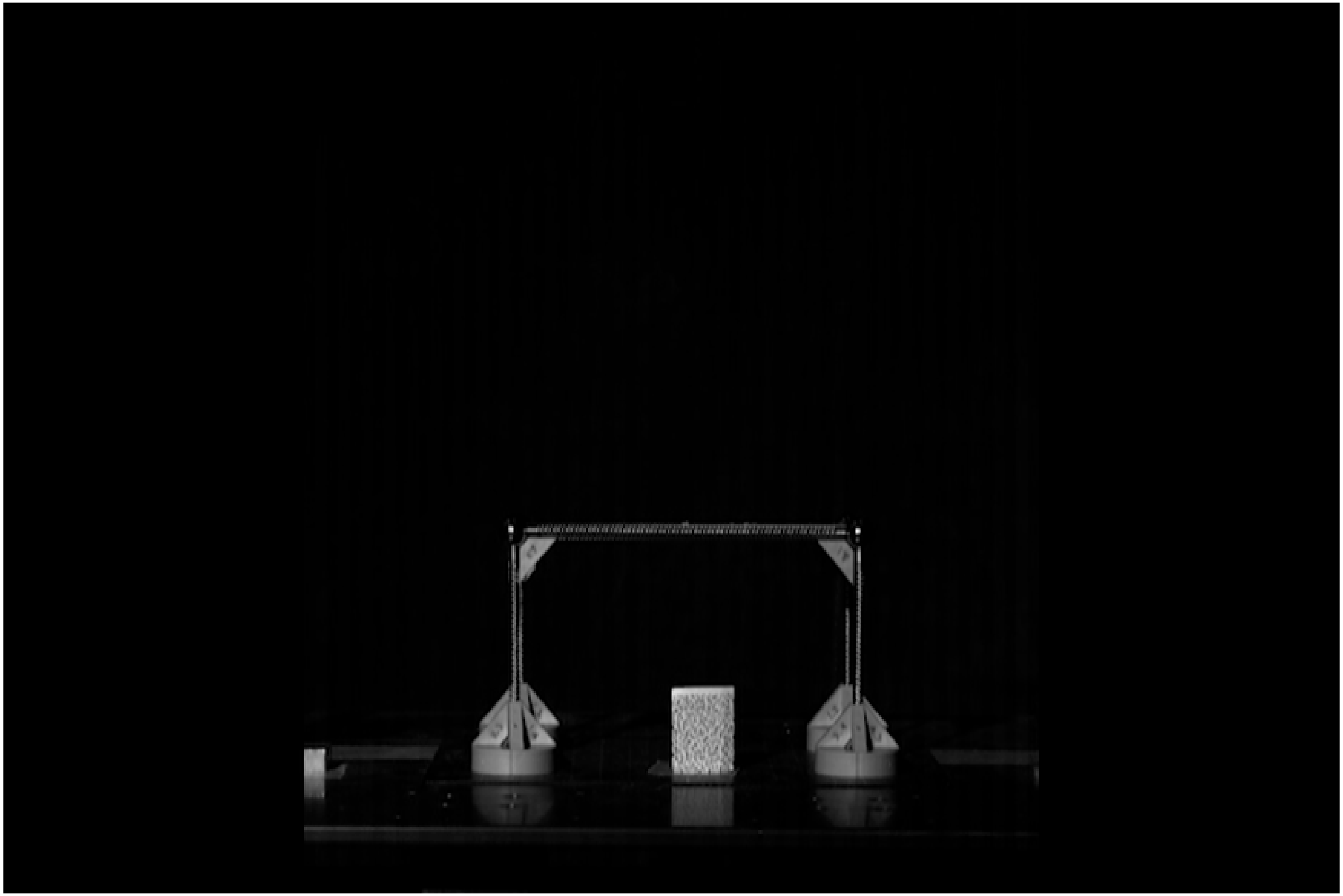




Source: (Master Students Hugo Ribet, Yan Zhang, RESSLab 2017)

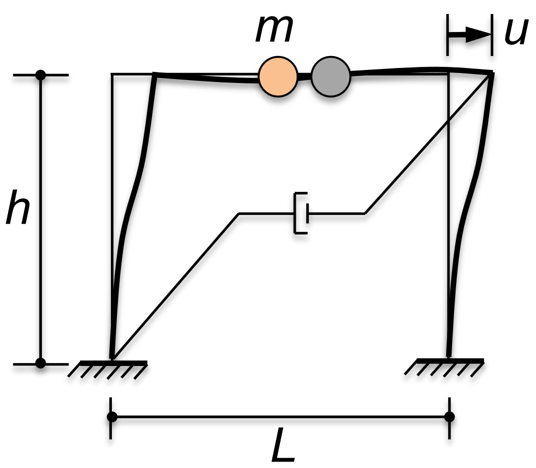
Dynamic Stability – RESSLab Applications

-Damped Forced Vibration (Stable) – Zero Initial Conditions



Source: (Master Student, Gerard Guell, RESSLab 2017)

Dynamic Stability – RESSLab Applications
-Damped Forced Vibration (Stable) – Zero Initial Conditions



Based on Newton's second law:

$$f_I + f_D + f_S(u, \dot{u}) = p(t) \Longrightarrow$$

$$m\ddot{u} + c\dot{u} + f_S(u, \dot{u}) = -m\ddot{u}_g(t)$$

Initial Conditions:

$$u = u(0),\ \dot{u} = \dot{u}(0)$$

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g(t)$$

Dynamic Stability – RESSLab Applications
-Damped Forced Vibration (Stable) – Zero Initial Conditions

- ***Newmark Method:*** *constant or linear variation of acceleration between time steps i and i+1 and solving equation for time step i+1 (implicit method).*

$$m\ddot{u} + c\dot{u} + ku = p_o \;\; where \;\; p_o = -m\ddot{u}_g$$

✧ Initial Calculations

$$\ddot{u}_0 = \frac{p_o - c\dot{u}_o - ku_o}{m}$$

$$\hat{k} = k + \frac{\gamma}{\beta * \Delta t}c + \frac{m}{\beta(\Delta t^2)}$$

$$a = \frac{m}{\beta * \Delta t} + \frac{\gamma}{\beta}c \quad and \quad b = \frac{m}{2\beta} + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)c$$

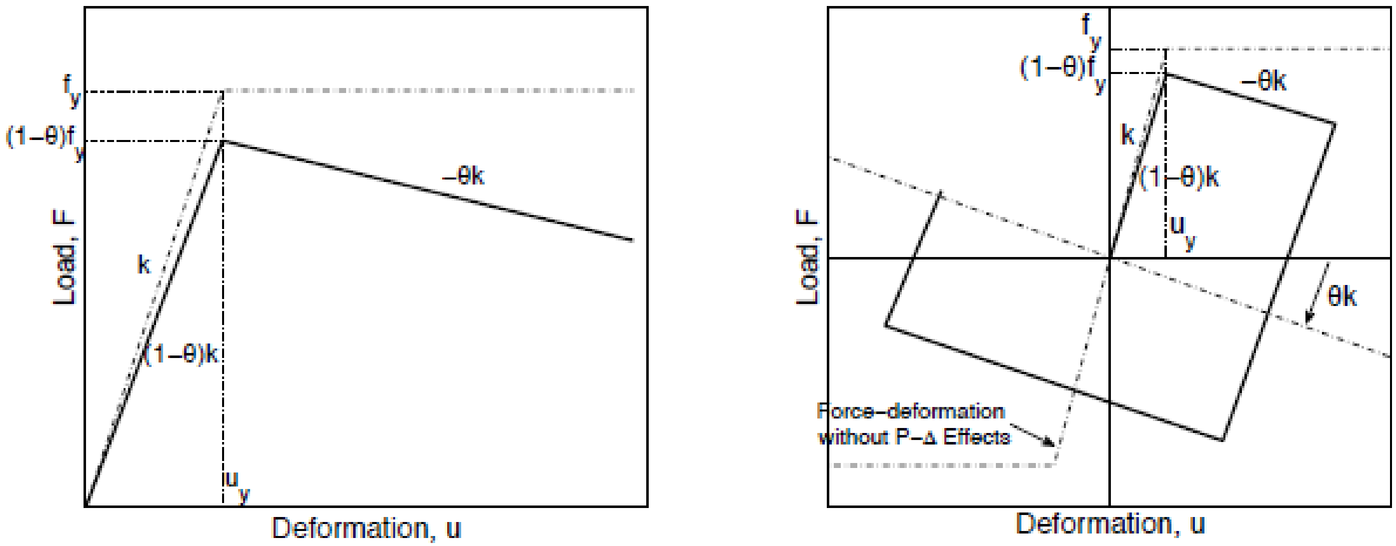
✧ Calculations for each time step

$$\Delta\hat{p}_i = \Delta p_i + a\dot{u}_i + b\ddot{u}_i, \Delta u_i = \frac{\Delta\hat{p}_i}{\hat{k}}$$

$$\Delta\dot{u}_i = \frac{\gamma}{\beta * \Delta t}\Delta u_i - \frac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i, \Delta\ddot{u}_i = \frac{\Delta u_i}{\beta(\Delta t^2)} - \frac{\dot{u}_i}{\beta * \Delta t} - \frac{\ddot{u}_i}{2\beta}$$

Dynamic Stability – RESSLab Applications
-Damped Forced Vibration (Stable) – Zero Initial Conditions

- P-Δ effects are quantified using the stability coefficient, θ
- Stability coefficient is associated with rotation of original force-deformation relationship.

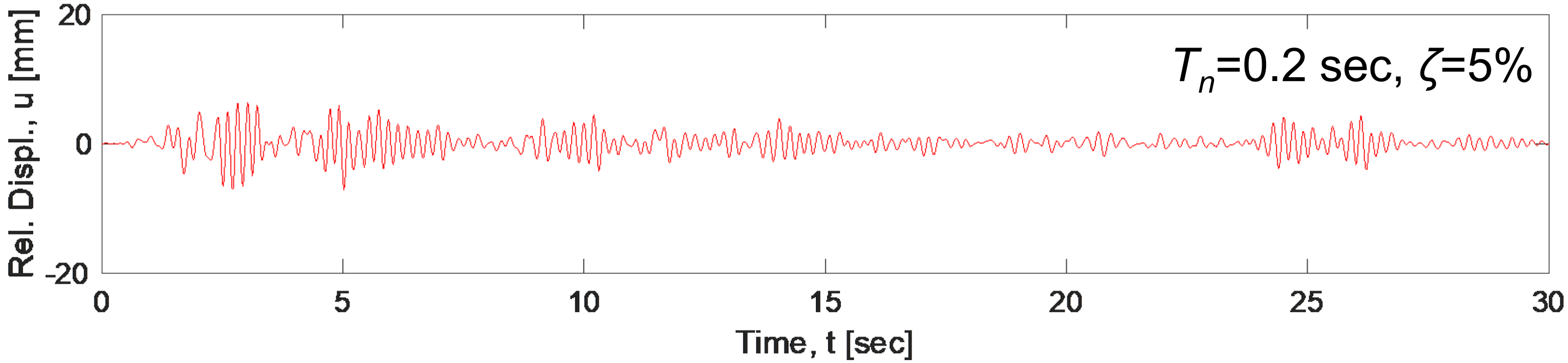


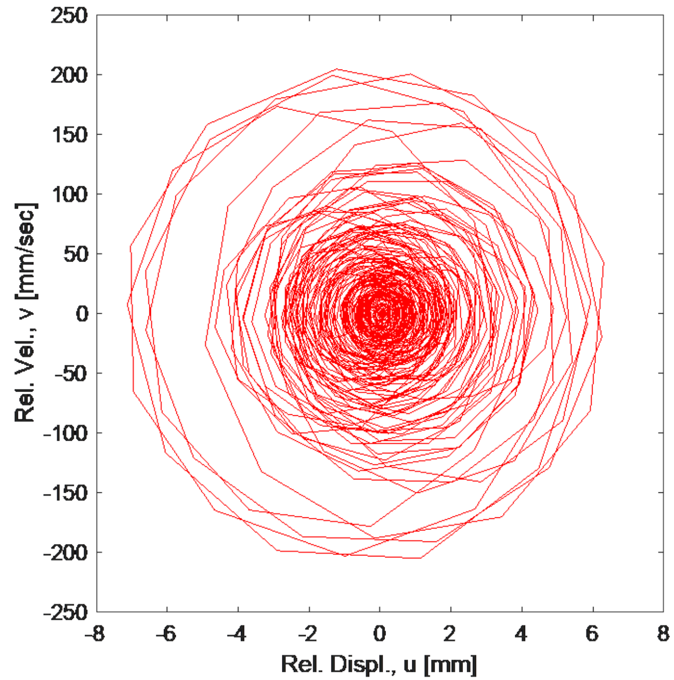
(a) Monotonic response (b) hysteretic response

- Integration in Newmark's algorithm: $\hat{k} = (1-\theta)\cdot k + \frac{\gamma}{\beta * \Delta t} c + \frac{m}{\beta(\Delta t^2)}$

Dynamic Stability – RESSLab Applications

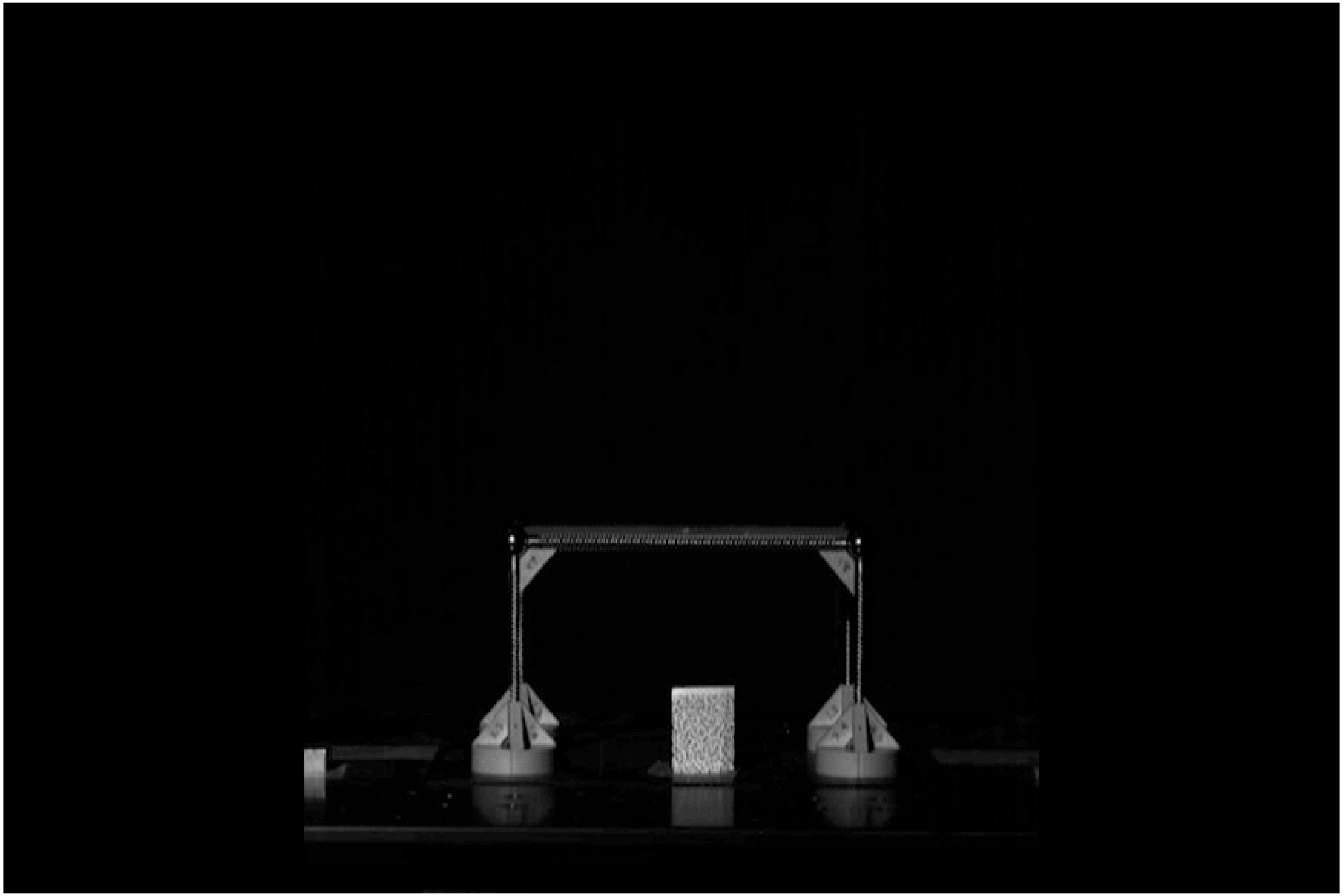
-Damped Forced Vibration (Stable) – Zero Initial Conditions





Dynamic Stability – RESSLab Applications

-Damped <u>Forced</u> Vibration (Unstable) – Zero Initial Conditions

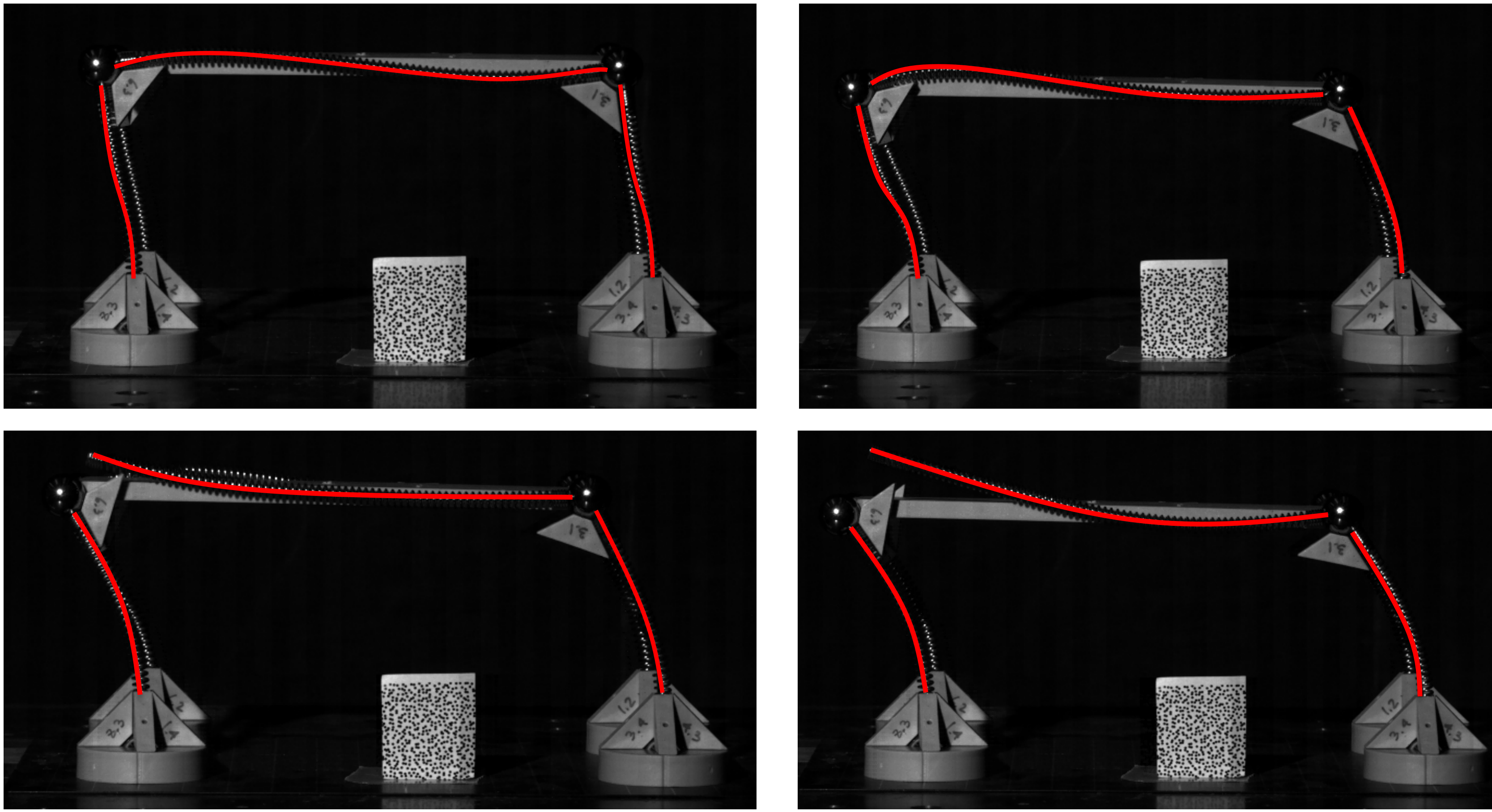


Source: (Master Student, Gerard Guell, RESSLab 2017)

Discussion

- Dynamic Response: Prediction– Stiffness Loss during excitation

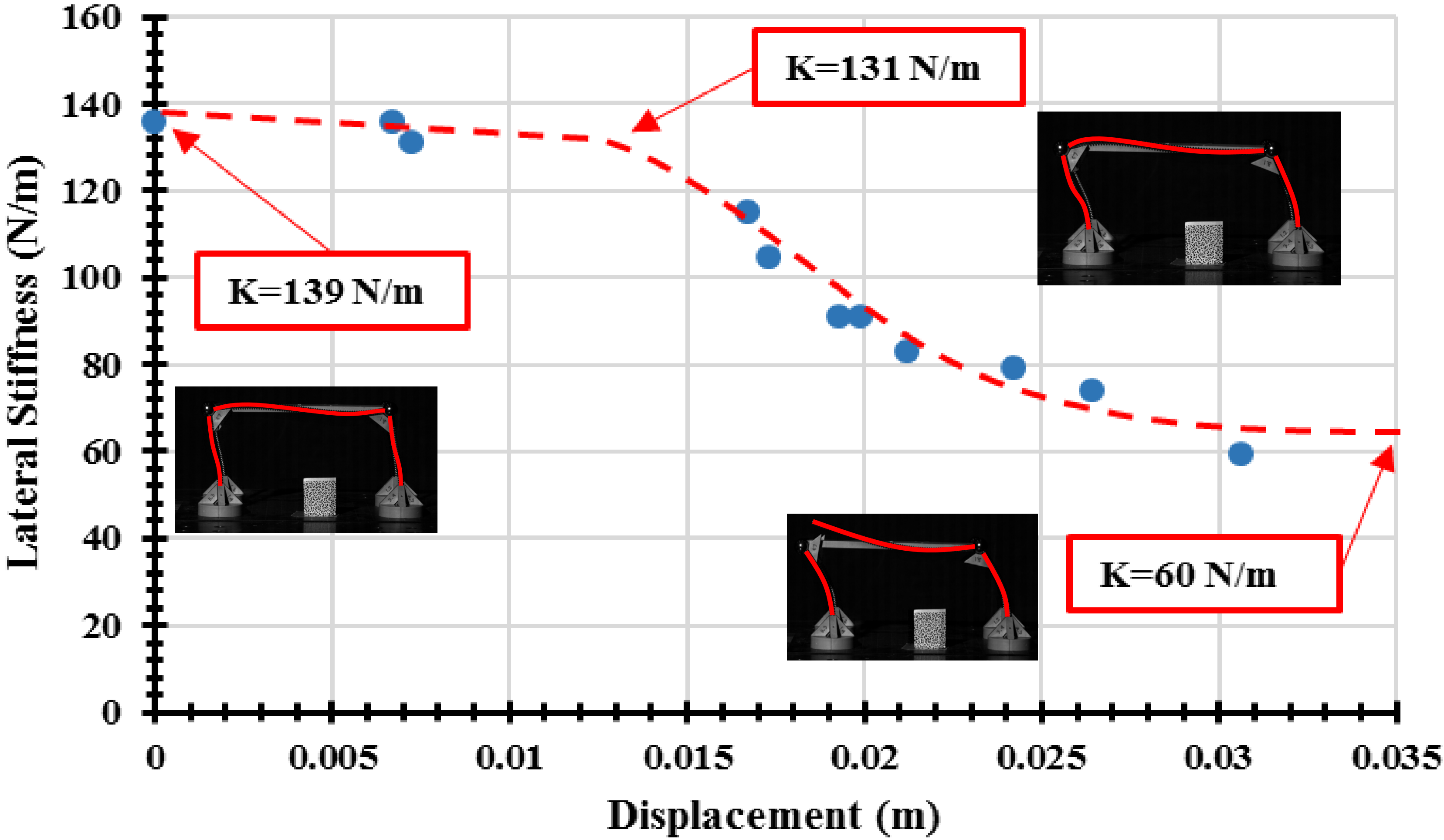
✧ As frame deforms, magnetic components detach and stiffness decreases



Source: (Master Student, Gerard Guell, RESSLab 2017)

Discussion

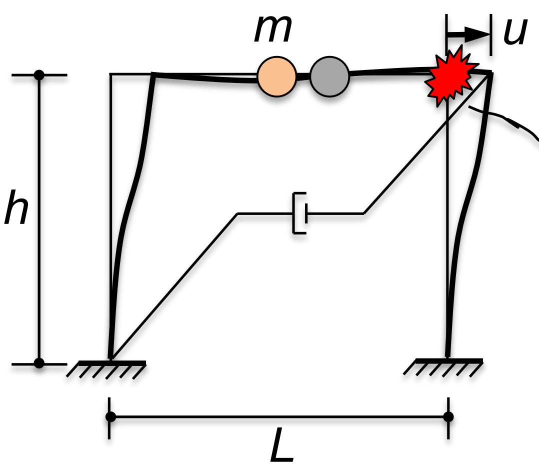
- Dynamic Response: Prediction– Stiffness Loss during excitation



Source: (Master Student, Gerard Guell, RESSLab 2017)

Dynamic Stability – RESSLab Applications

-Damped Forced Vibration (Stable) – Zero Initial Conditions

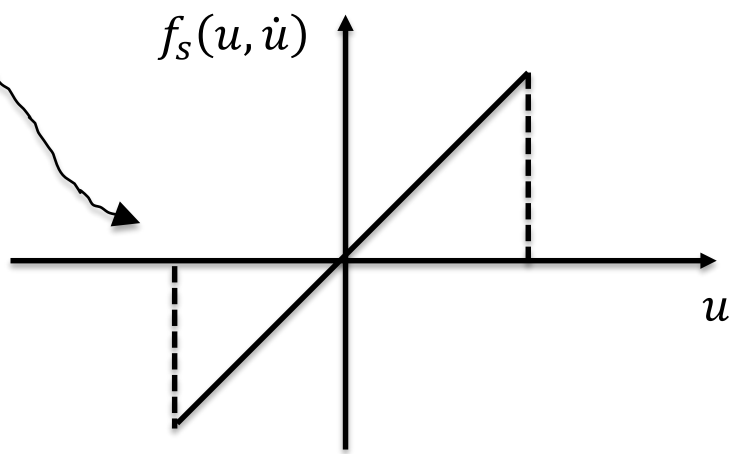


Based on Newton's second law:

$$f_I + f_D + f_s(u,\dot{u}) = p(t) \Longrightarrow$$

$$m\ddot{u} + c\dot{u} + f_s(u,\dot{u}) = -m\ddot{u}_g(t)$$

Nonlinear Model



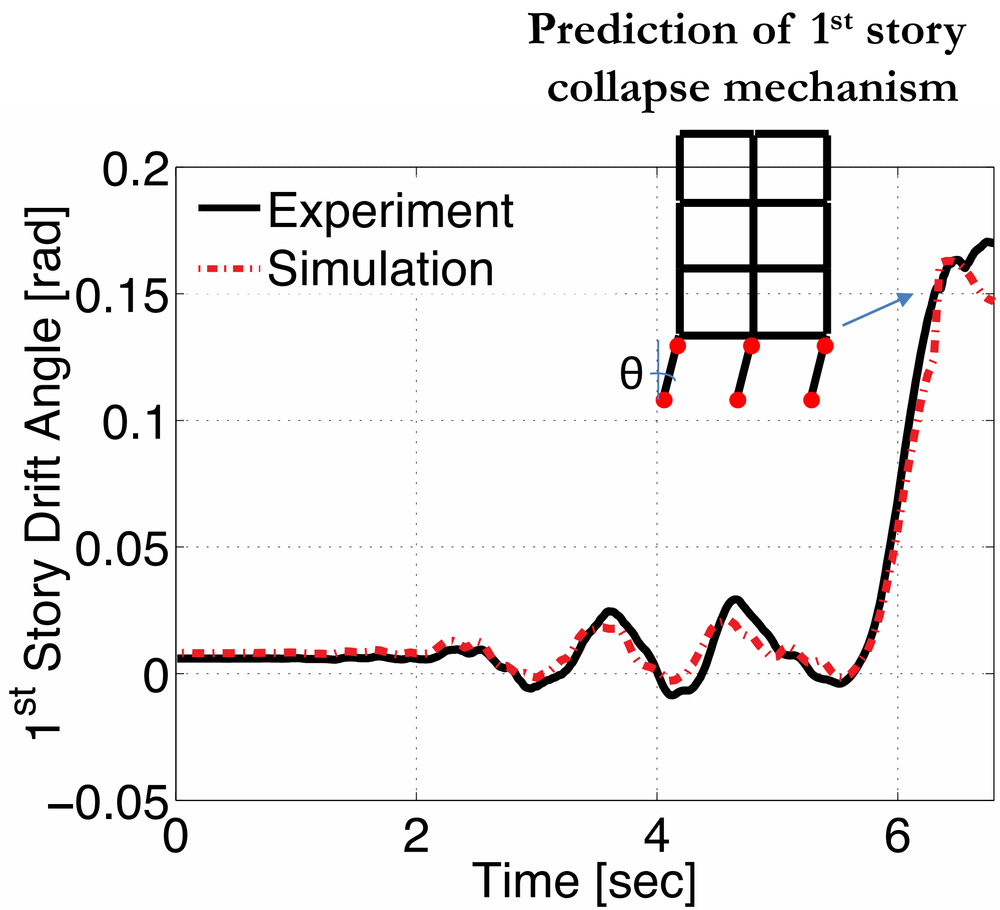
Dynamic Stability – Collapse of Structures

-Damped Forced Vibration (Unstable) – Zero Initial Conditions

Source: E-Defense 2007, Lignos et al. (2013)

Dynamic Stability – Collapse of Structures

-Damped Forced Vibration (Unstable) – Zero Initial Conditions



(Source: Lignos et al. 2013)*

Newton’s Second Law for Rotation
-Small Revision from Structural Mechanics

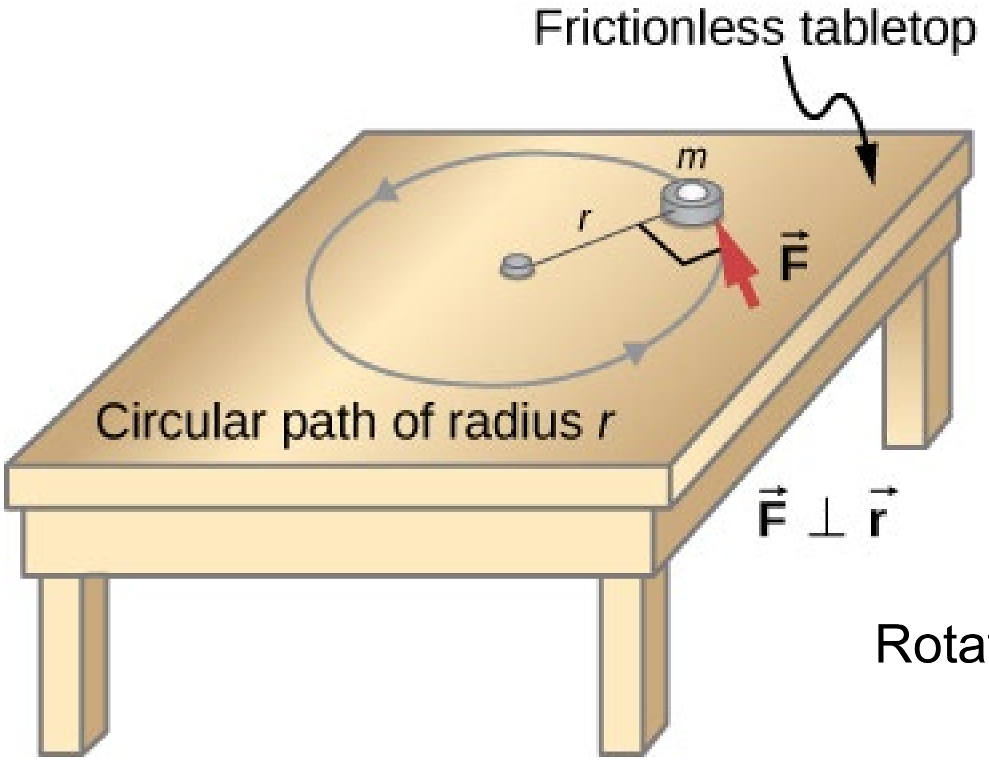


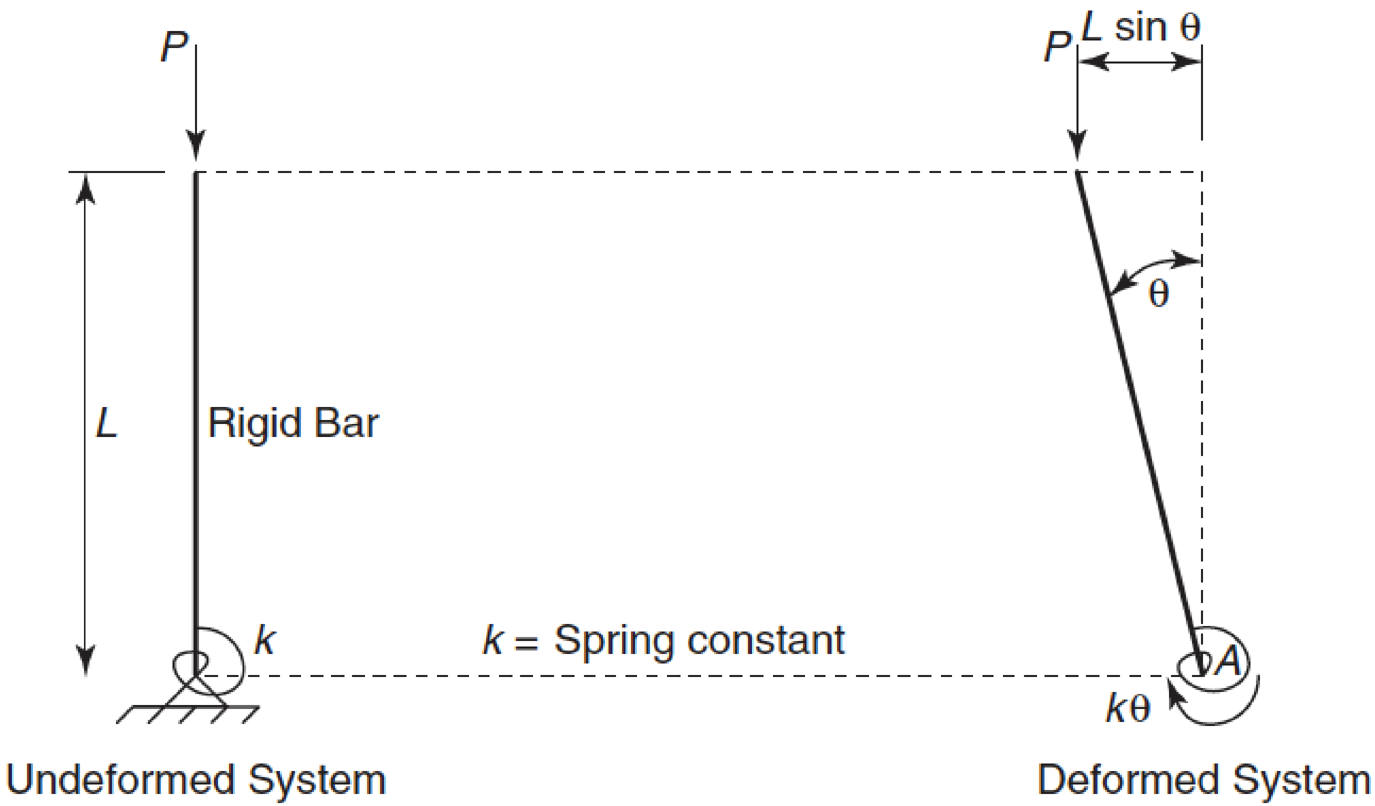
Image Source: University Physics Volume 1

Rotational moment of Inertia

$$I = mr^2$$

$$M = rF \qquad rF = mr^2 \frac{d^2\theta}{dt^2} \Longrightarrow rF = I \frac{d^2\theta}{dt^2}$$

Moving Bridge and its Mathematical Model



Newton's second law

$$I\frac{d^2\theta}{dt^2} = M$$

Equilibrium of moments with respect to A

$$M = PLsin\theta - k\theta$$

Image Source: Galambos and Surovek 2008

Moving Bridge and its Mathematical Model

Newton's second law

$$I\frac{d^2\theta}{dt^2} = M$$

Equilibrium of moments with respect to base

$$M = PLsin\theta - k\theta$$

Therefore,

$$I\frac{d^2\theta}{dt^2} - (PLsin\theta - k\theta) = 0$$

Example 1: Moving Bridge

For an equilibrium point $\boldsymbol{\theta_o}$ (i.e., initial perturbation)

$$I\frac{d^2\theta_0}{dt^2} - (PLsin\theta_0 - k\theta_0) = 0 \quad (1)$$

Applying a small perturbation $\boldsymbol{\theta}^*$:

$$I\frac{d^2(\theta_0+\theta^*)}{dt^2} - [PLsin(\theta_0+\theta^*) - k(\theta_0+\theta^*)] = 0$$

Taylor Series: $\sin(\theta_0 + \theta^*) = sin\theta_0 + \theta^* cos\theta_0 + \mathrm{O}[{\theta^*}^2]$

$$I\frac{d^2(\theta_0+\theta^*)}{dt^2} - [PL(sin\theta_0 + \theta^* cos\theta_0) - k(\theta_0+\theta^*)] = 0 \quad (2)$$

Example 1: Moving Bridge

Subtracting Equations (1) from (2)

$$I\frac{d^2\theta^*}{dt^2} - \theta^*(PLcos\theta_0 - k) = 0$$

We need to investigate the sign of the term:

$$A = \frac{k - PLcos\theta_0}{I}$$

Example 1: Moving Bridge

On the main equilibrium path: $\theta_0 = 0$

Therefore, $A = \dfrac{k - PL}{I}$

If: $P = \dfrac{k}{L}$ then zero eigen frequency $P_{cr} = \dfrac{k}{L}$

If: $P < P_{cr} = \dfrac{k}{L}$ then $A = \dfrac{k - PL}{I} > 0 \Longrightarrow$ Stability (bounded vibration)

If: $P > P_{cr} = \dfrac{k}{L}$ then $A = \dfrac{k - PL}{I} < 0 \Longrightarrow$ instability (unbounded vibration)

Example 1: Moving Bridge

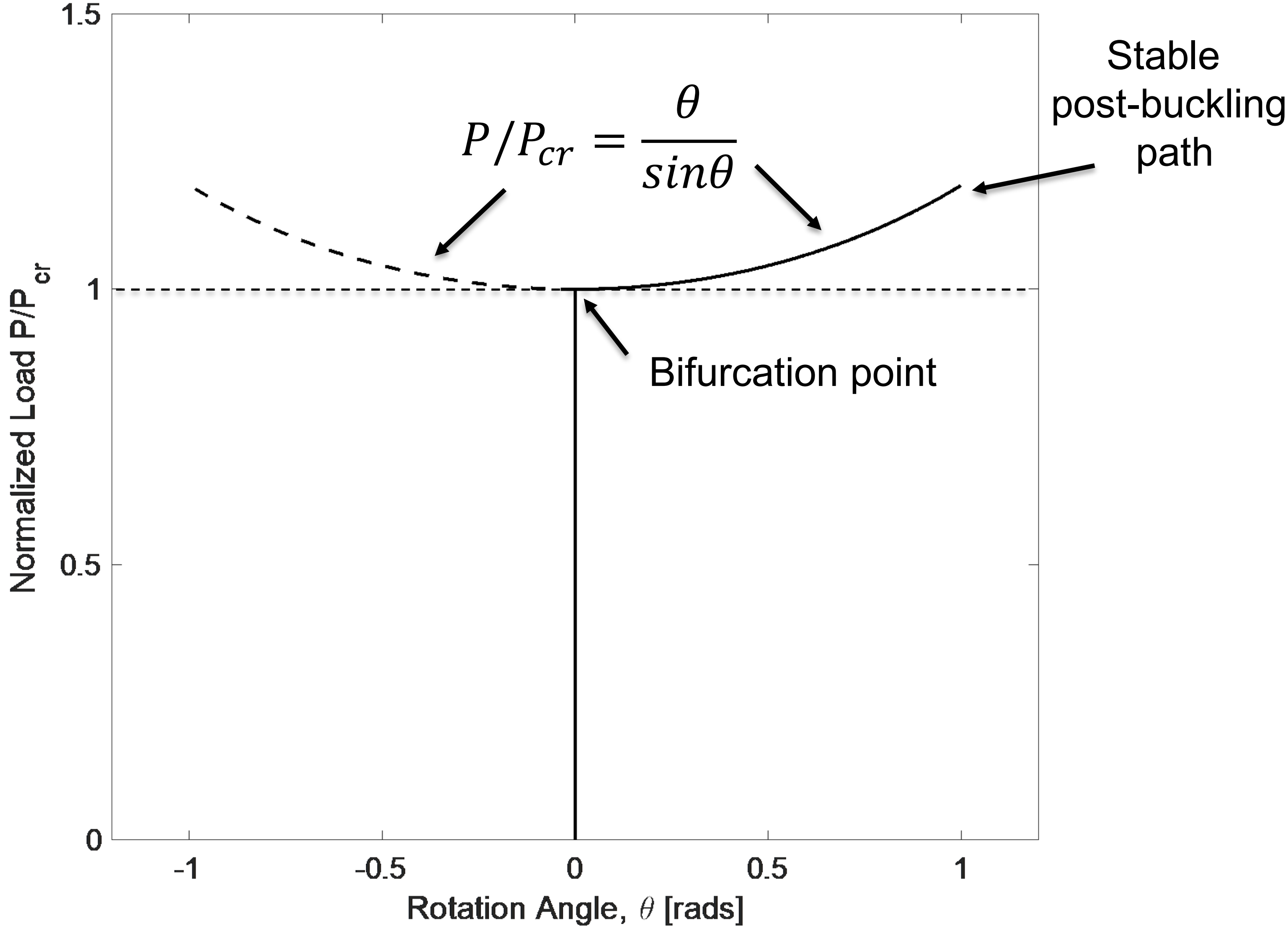
On the secondary equilibrium path: $P = \frac{k}{L} \frac{\theta_0}{\sin \theta_0}$

Therefore,

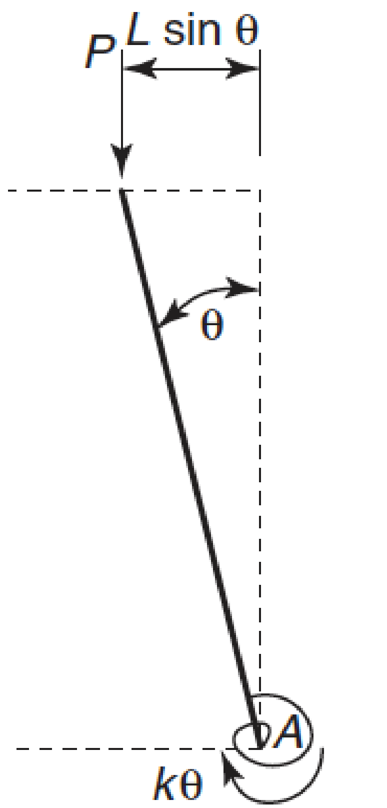
$$A = \frac{k}{I}\left(1 - \frac{\theta_0}{tan\theta_0}\right) > 0$$

Stability (bounded vibration)

Example 1: - Nonlinear Theory



Example 1: Fixed Rod with Vertical Load – Numerical Assessment



Section: IPE300

Length of rod: $L = 3m$

Spring stiffness, $k = 300kNm/rad$

Expected critical load

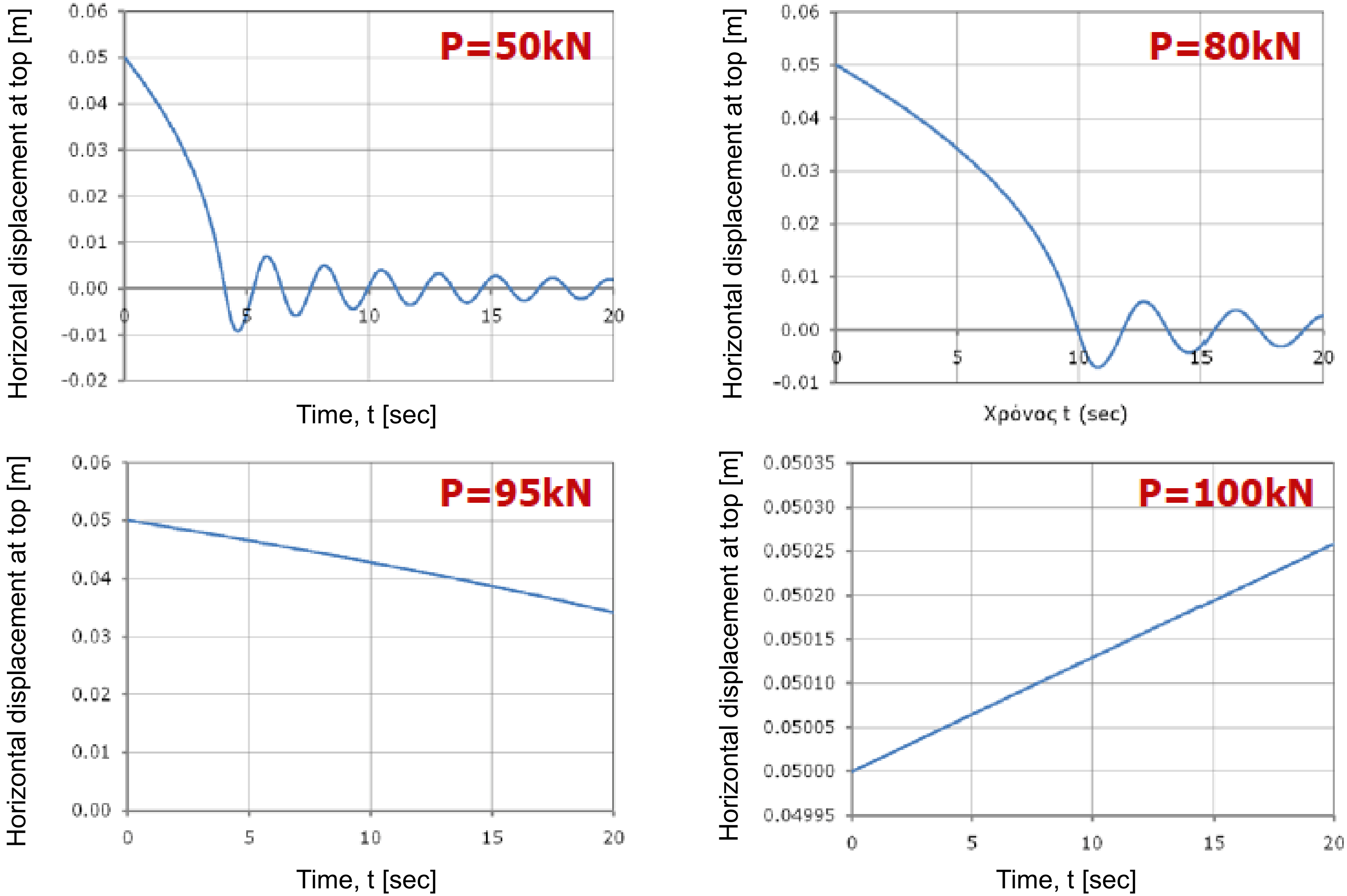
$$P_{cr} = k/L = 100kN$$

Initial offset $0.05m$ from the top

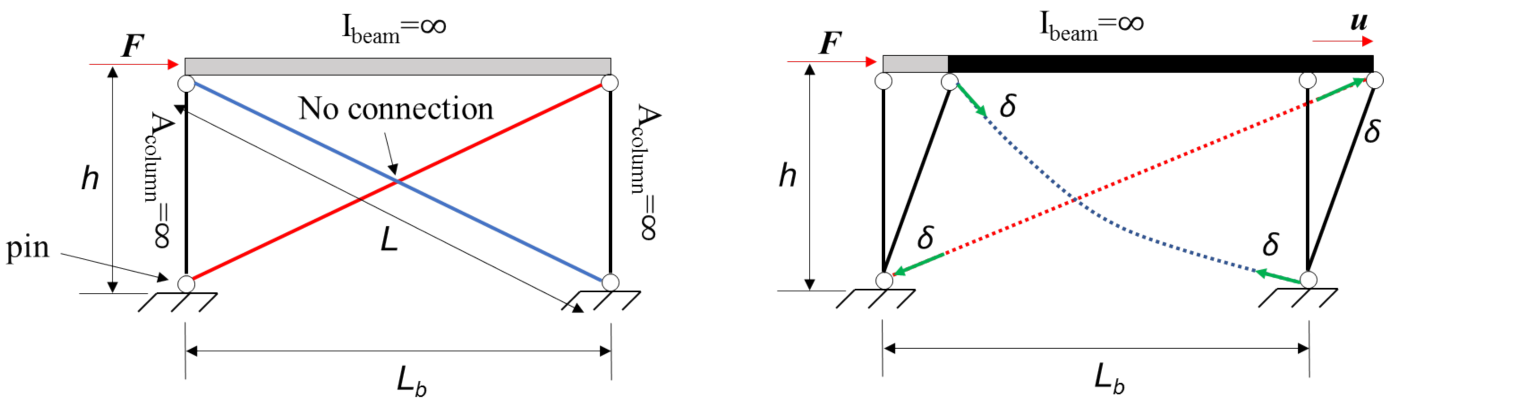
Initial velocity $0m/sec$

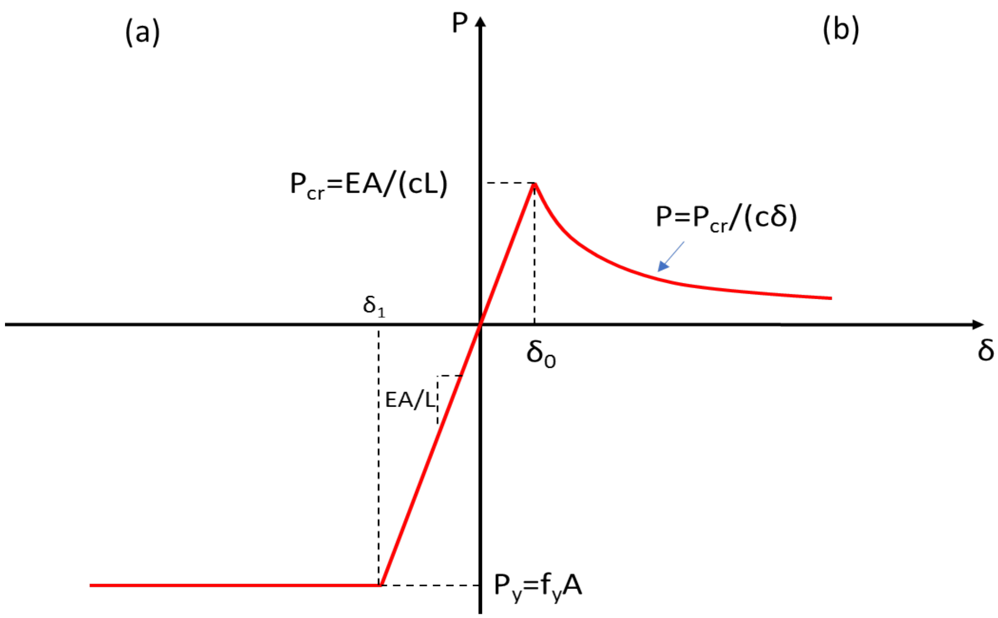
Damping ratio $\zeta = 1\%$

Example 1: Fixed Rod with Vertical Load – Numerical Assessment



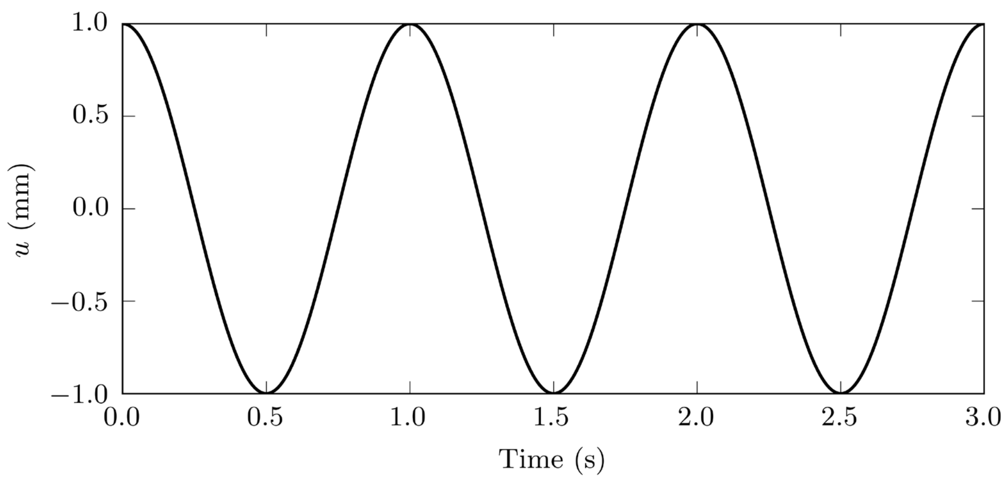
Example 2: Stability of Braced Frame under Lateral Loading

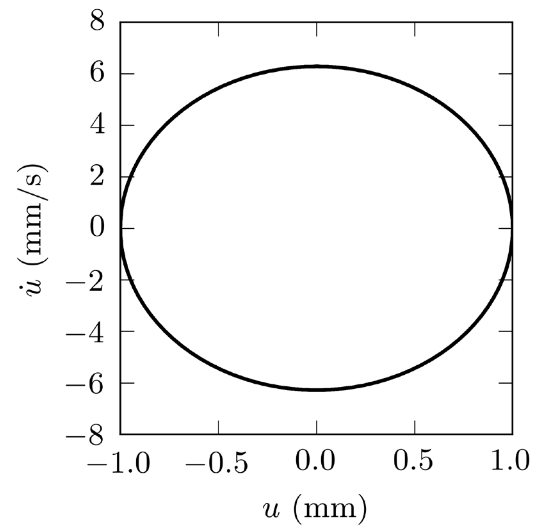




Example 2: Stability of Braced Frame under Lateral Loading

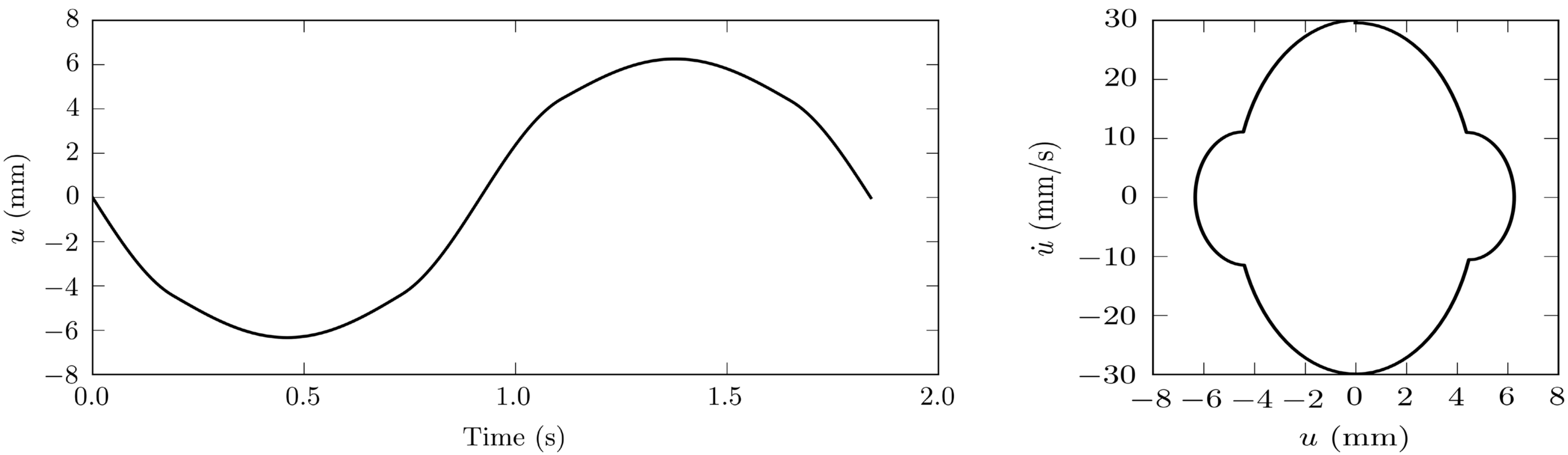
Initial conditions, $u_o = $1mm, $\dot{u}_o$=0mm/s





Example 2: Stability of Braced Frame under Lateral Loading

Initial conditions, $u_o = 0$mm, $\dot{u}_o$=30mm/s



Example 2: Stability of Braced Frame under Lateral Loading

Initial conditions, $u_o = 0$mm, $\dot{u}_o$=40mm/s