CIVIL 369: “Structural Stability”

School of Architecture, Civil & Environmental Engineering
Civil Engineering Institute
Resilient Steel Structures Laboratory (RESSLab)

Lateral Torsional Buckling of Open Cross-Sections

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Objectives of Today's Lecture

To introduce:

✧ Need for lateral stability of members

✧ Famous failures due to lateral torsional buckling

✧ Torsion, warping

✧ Fundamental equations for lateral torsional buckling

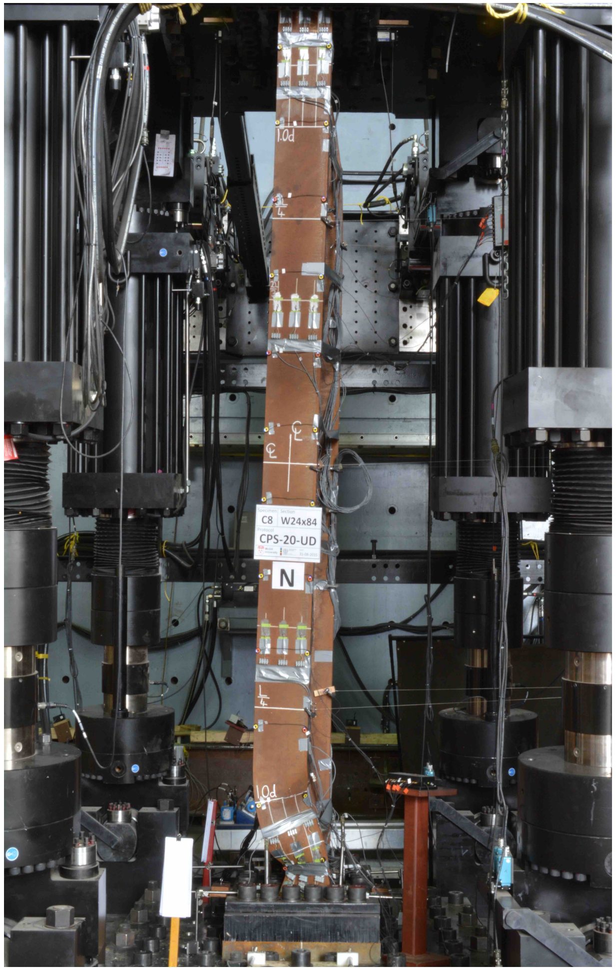
✧ Effects of end restraints

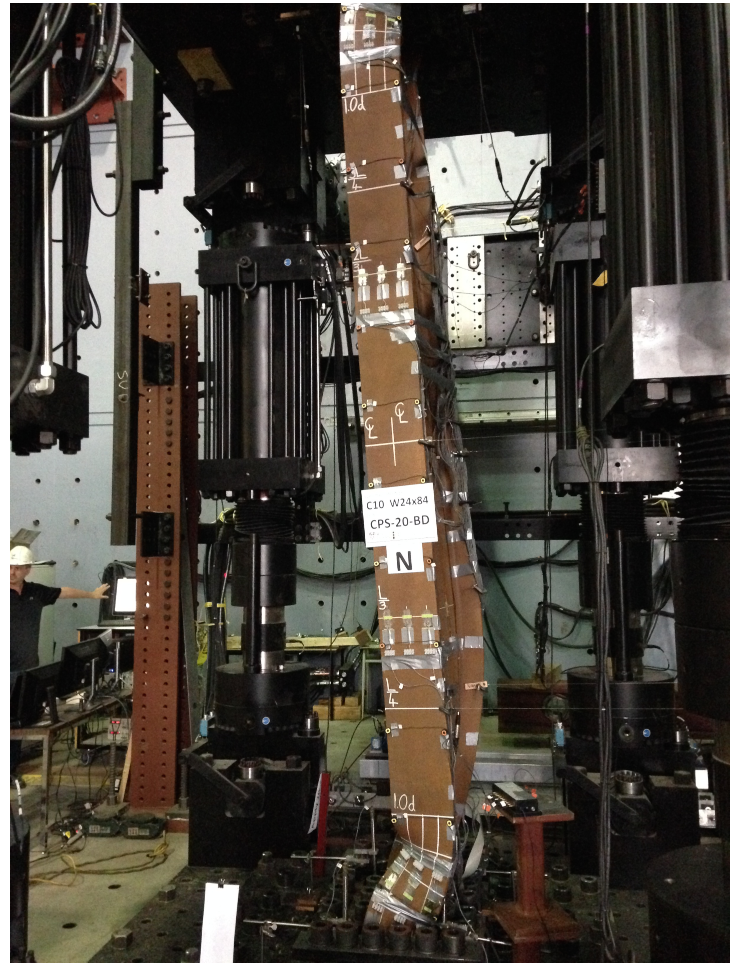
✧ Effects of loading conditions and point of load application

✧ Singly symmetric cross-sections

Lateral Stability
-Lateral Torsional Beam Buckling due to Fire

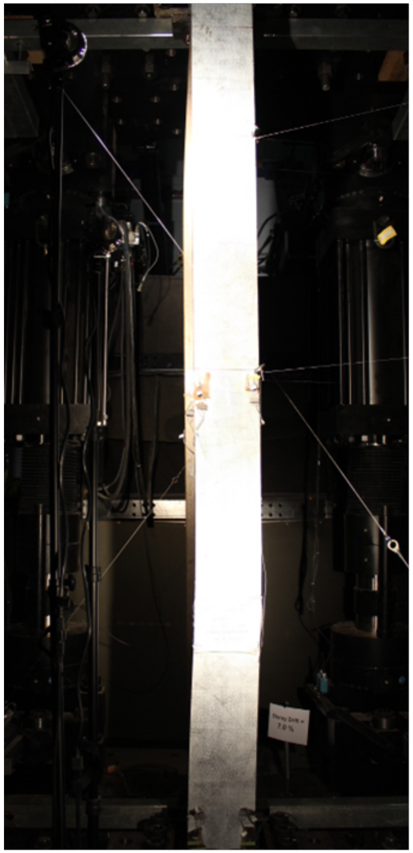
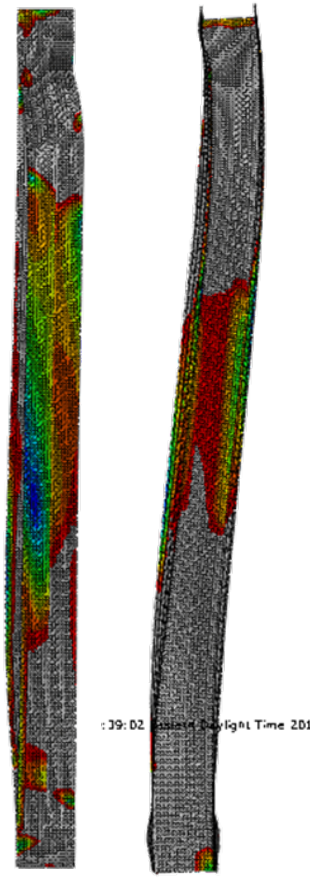
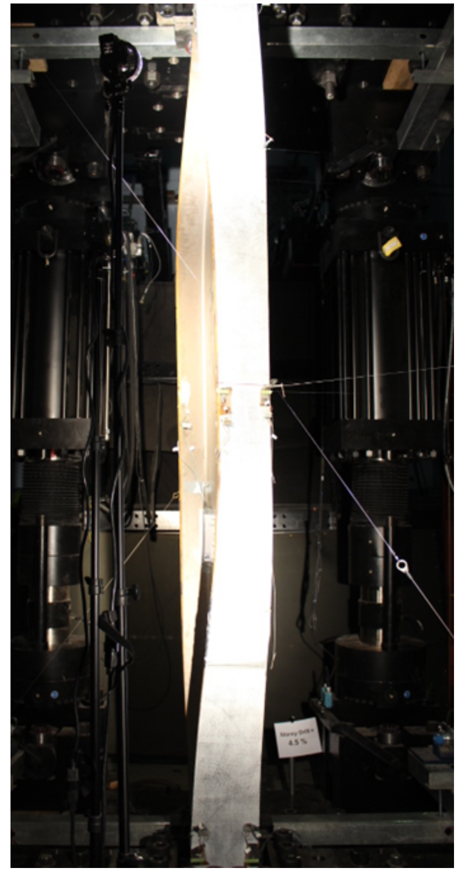
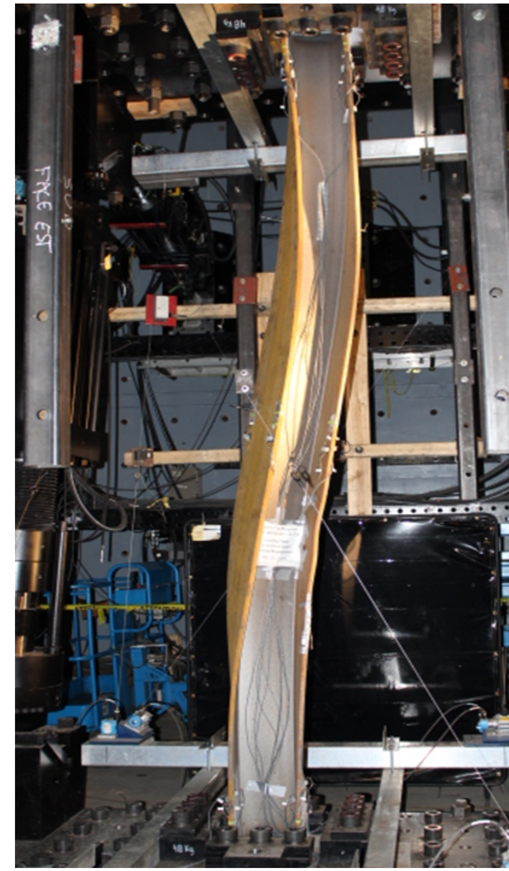
Lateral Stability
-Steel Columns: Lateral Torsional Buckling





(Elkady and Lignos, 2015)

Lateral Stability
-Steel Columns: Lateral Torsional Buckling

(Images courtesy of Prof. Tremblay)

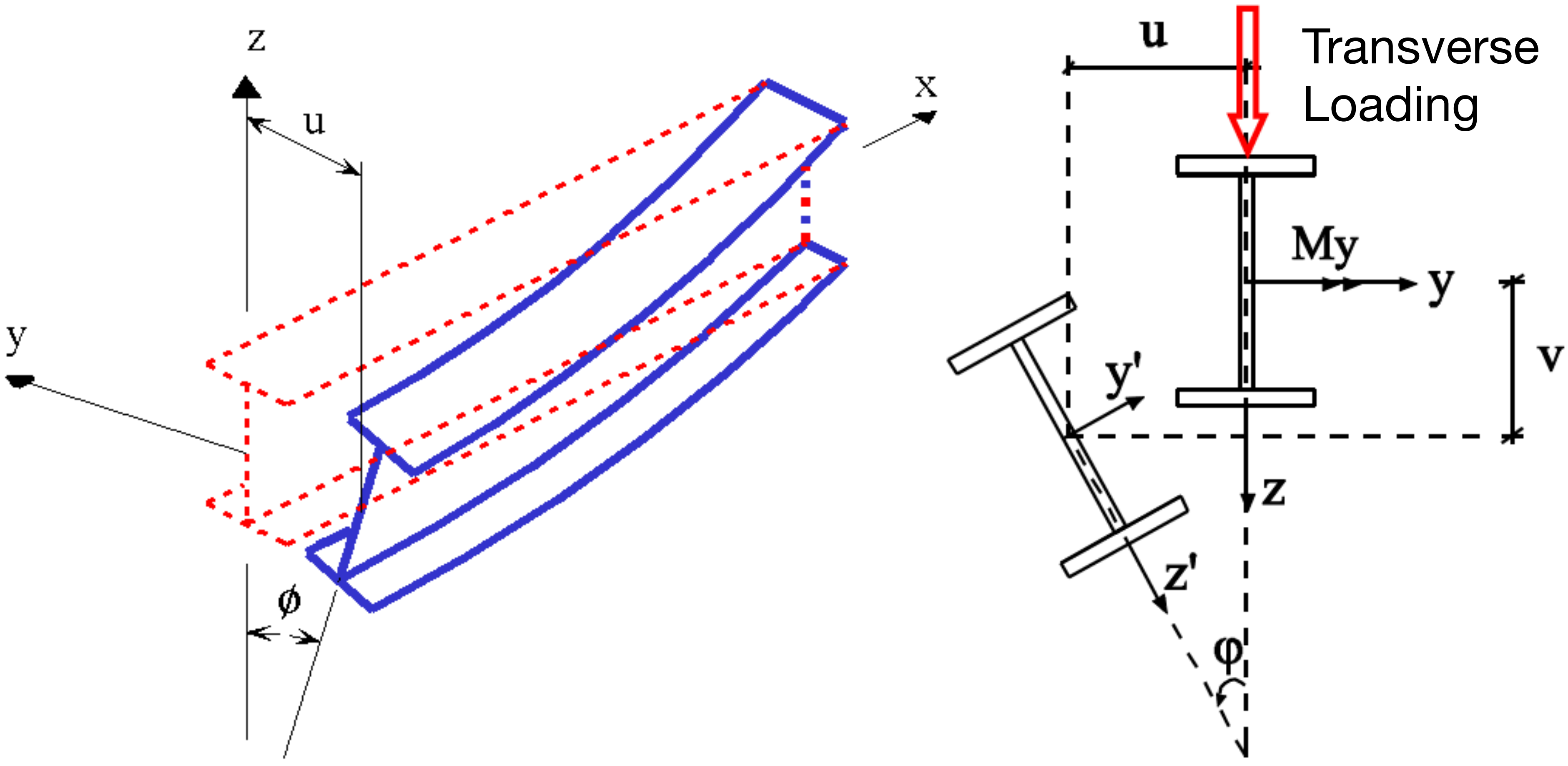
Lateral Stability

-Lateral Torsional Beam Buckling – Bridge Girders

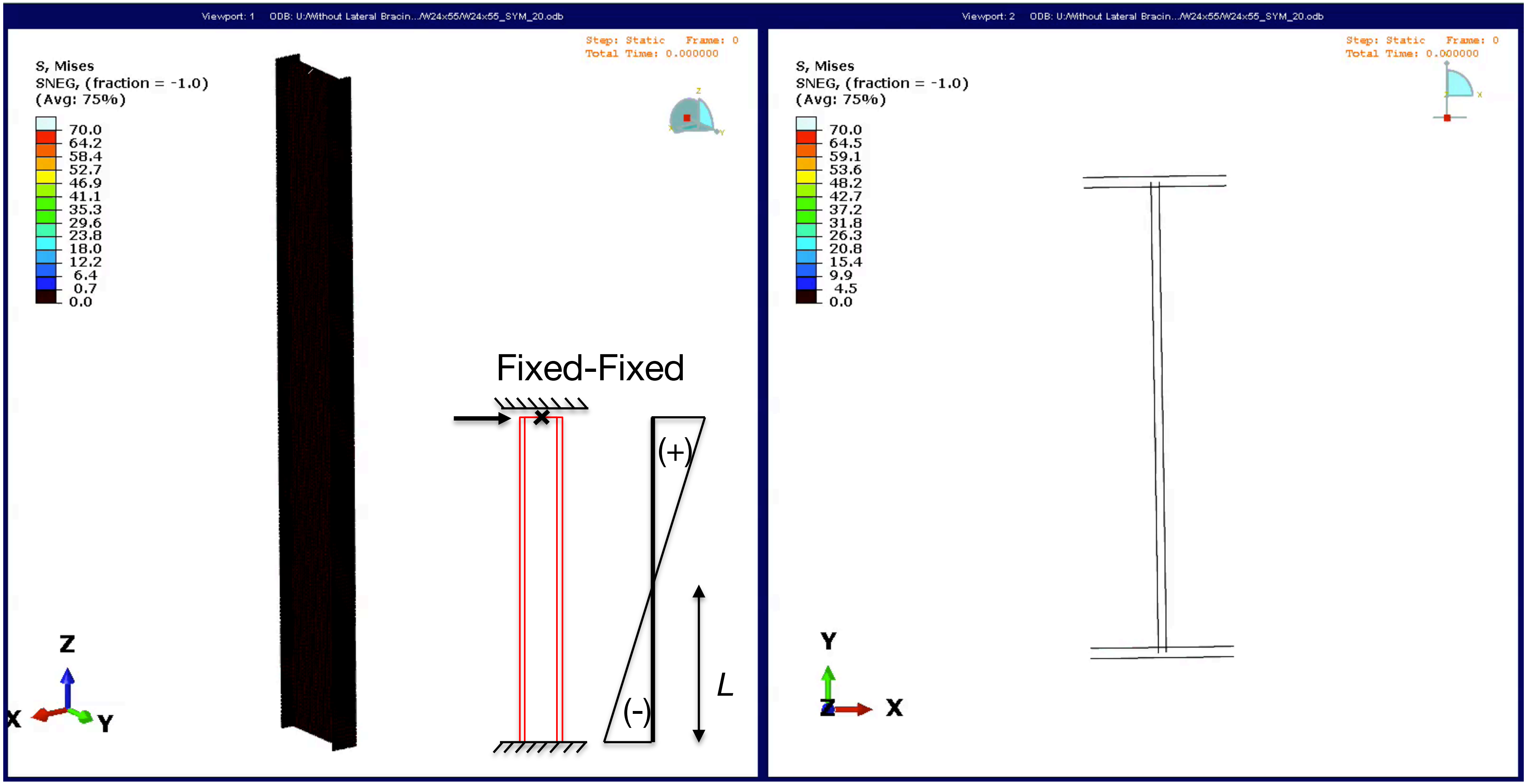
Lateral Stability
-Lateral Torsional Beam Buckling – Bridge Girders

Edmonton Bridge, Canada

Lateral Torsional Buckling

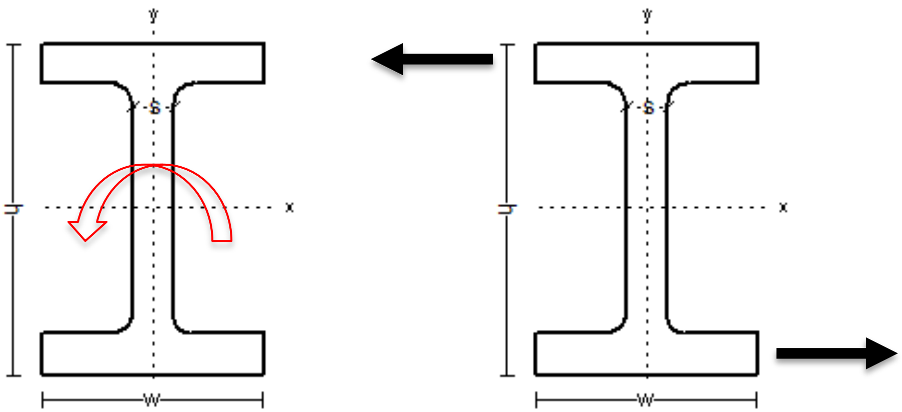


Lateral Torsional Buckling -Illustration

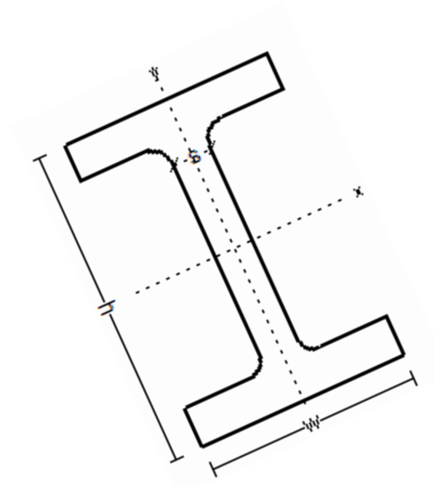


Lateral Torsional Buckling

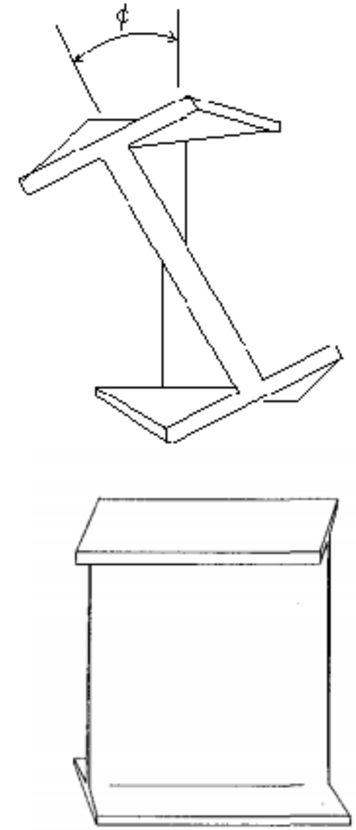
-Dependence on Torsional and Warping Properties



Saint-Venant Torsion



Warping

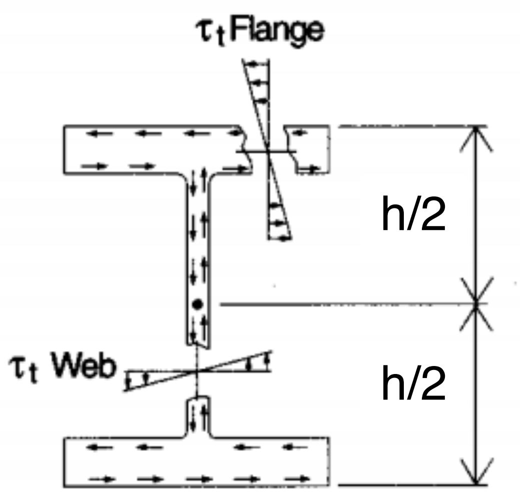


Total torsion *T* is in part taken by Saint-Venant Torsion and Warping

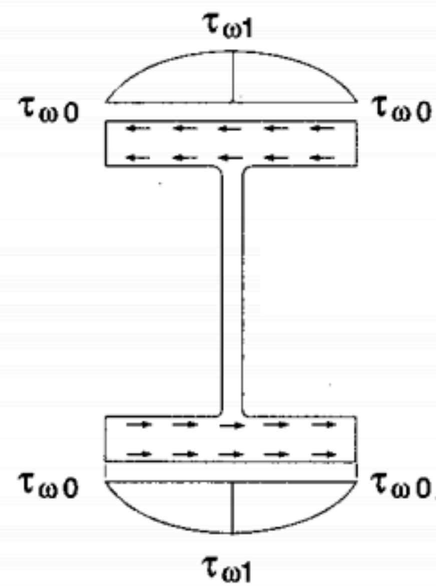
$$T = T_t + T_\omega$$

Lateral Torsional Buckling

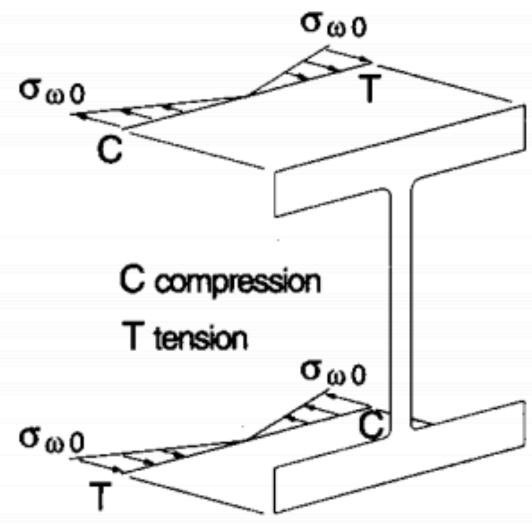
-Stress Distribution due to Torsion and Warping



Shear stress due to pure torsion



Shear stress due to warping



normal stress due to warping

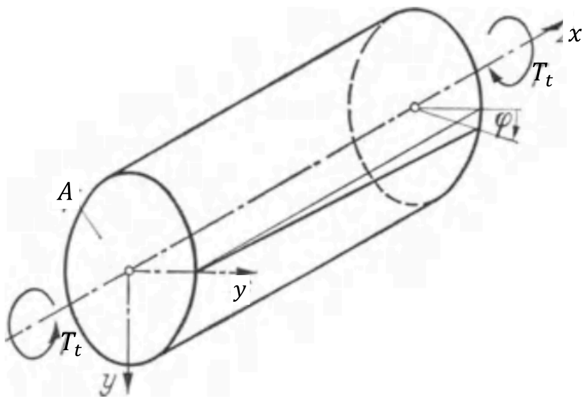
Properties Critical For Torsion and Warping

Uniform torsion – definitions

Consider x the longitudinal axis of the member and φ the cross-section rotation.

Uniform torsion is defined by being proportional to the change in cross-section rotation along its length, that is,

$$T_t = GK\frac{\partial \varphi}{\partial x} = GK\varphi' \qquad [N.m]$$



[adapted from Kollbrunner & Basler, 1969]

where $G[N.m^{-2}]$is the shear modulus and $K[m^4]$is a proportionality constant, also known as the torsion constant.

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Properties Critical For Torsion and Warping

Warping constant – general definition

Consider $w[m]$the displacement along x and $v[m]$the displacement along arc s. Then,

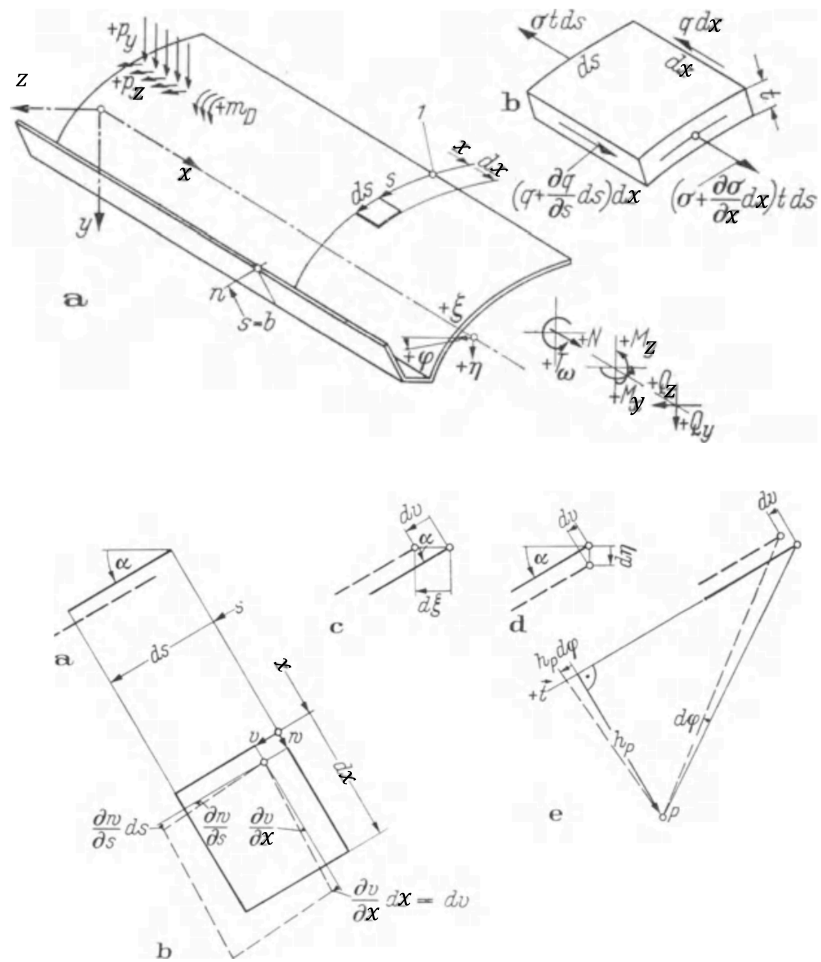
$$\frac{\partial v}{\partial x} = \frac{d\varphi}{dx} h_p = \varphi' h_p$$

Assuming negligible distortion, $\gamma = 0$, then,

$$\frac{\partial w}{\partial s} = -\frac{\partial v}{\partial x}$$

Making $w(s)[m]$, equal to

$$w(s) = -\varphi' \int h_p(s) ds + w_0$$



[adapted from Kollbrunner & Basler, 1969]

Properties Critical For Torsion and Warping

Warping constant – general definition

Consider $\Omega[m^2]$as the non-normalized sectorial coordinate:

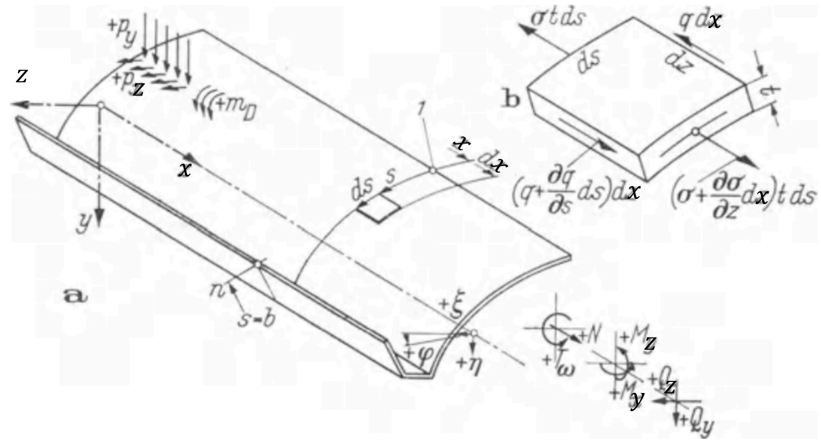
$$d\Omega(d\omega) = h_p\, ds \Longrightarrow \Omega(s) = \int h_p(s)$$

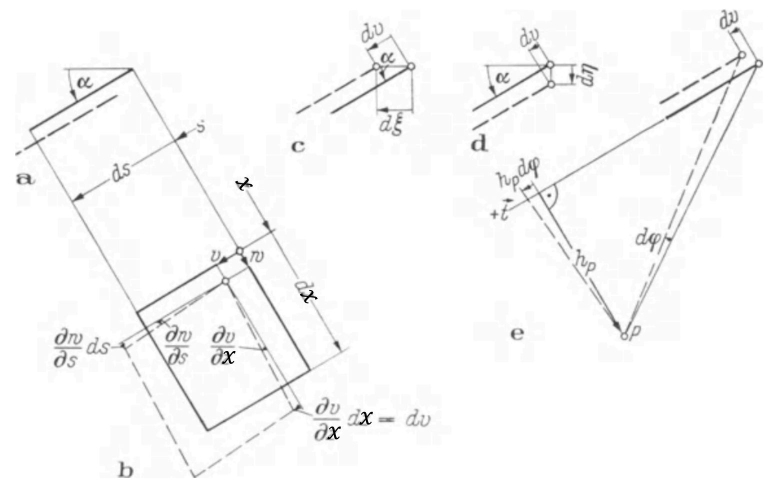
and also consider that

$$\sigma_x = \mathrm{E}\varepsilon_x = E\frac{dw}{dx}$$

which yields,

$$\sigma_x = -\varphi''\Omega + \mathrm{E}w_0'$$





[adapted from Kollbrunner & Basler, 1969]

Properties Critical For Torsion and Warping

Warping constant – general definition

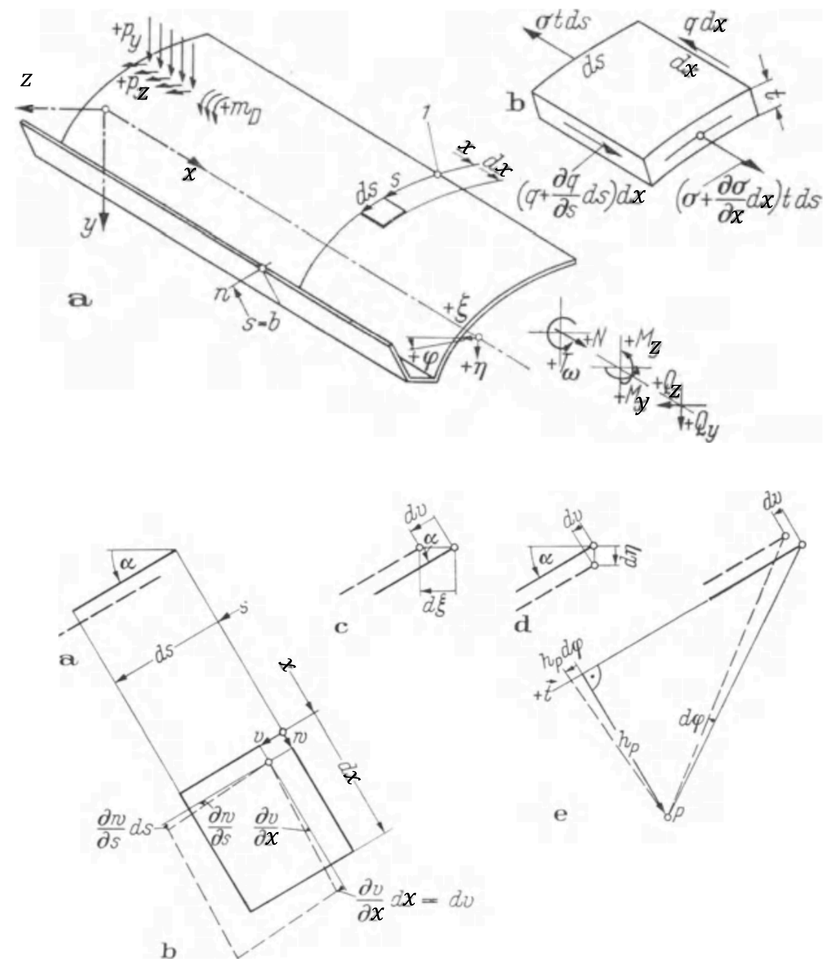
Given that for an applied torque there should be no axial force in our cross-section,

$$t\int \sigma_x \, ds = 0 \Longrightarrow$$

$$-E\varphi'' \int \Omega \, dA + \mathrm{E}w_0' \int \, dA = 0$$

which allows us to express the integration constant as,

$$w_0' = \varphi'' \frac{\int \Omega \, dA}{A}$$



[adapted from Kollbrunner & Basler, 1969]

Properties Critical For Torsion and Warping

Warping constant – general definition

Making the axial stress along the arc,

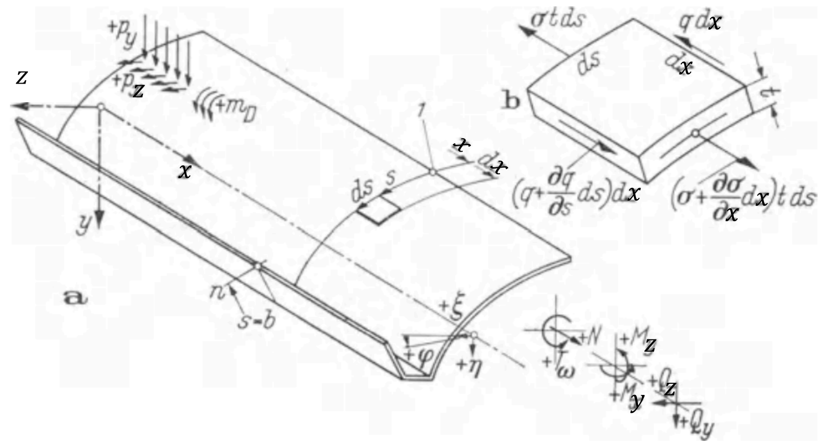
$$\sigma_x = -E\varphi''\left(\Omega - \frac{\int \Omega \, dA}{A}\right)$$

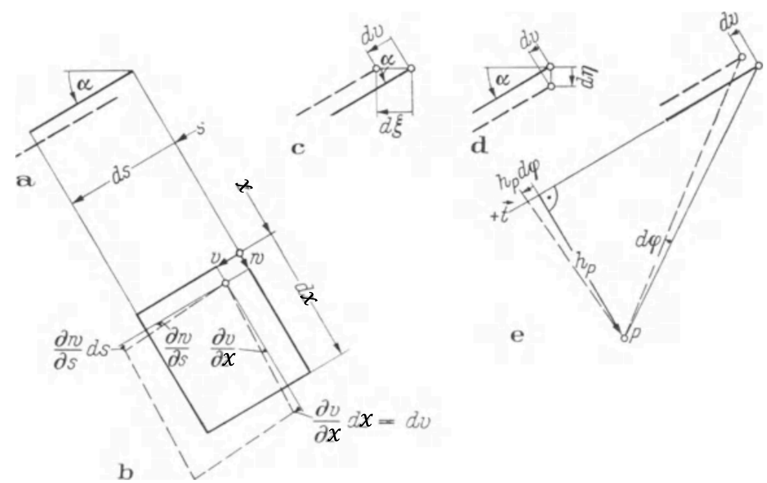
Defining the normalized sectorial coordinate ω ,

$$\omega = \Omega - \frac{\int \Omega \, dA}{A}$$

The axial stress is simply,

$$\sigma_x = -E\varphi''\omega$$





[adapted from Kollbrunner & Basler, 1969]

Properties Critical For Torsion and Warping

Warping constant – general definition

Lastly, the torsion due to warping about point P can be quantified as,

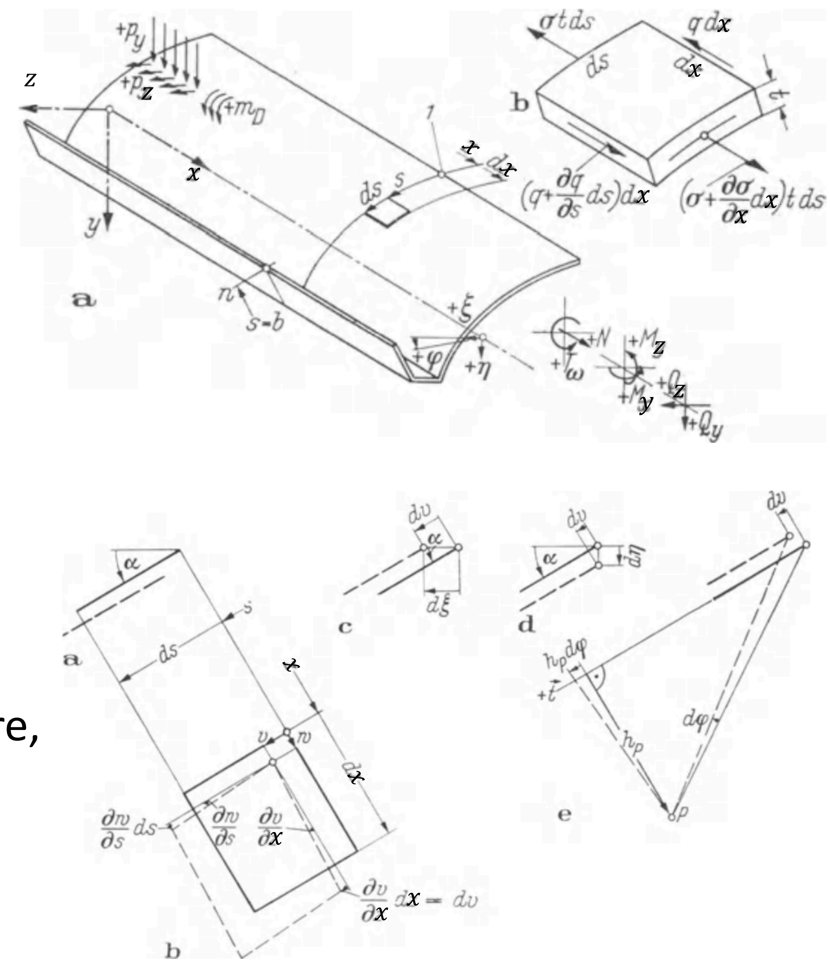
$$T_\omega = t\int \tau \cdot h_p ds = \int q \cdot h_p ds = \\ = \int q d\omega$$

integrating by parts,

$$T_\omega = \int q d\omega = q\omega\Big|_{s=0}^{s=b} - \int \frac{dq}{ds}\omega ds$$

which from equilibrium of the infinitesimal square,

$$T_\omega = \int \frac{d\sigma_x}{dx} t\omega ds$$



[adapted from Kollbrunner & Basler, 1969]

Properties Critical For Torsion and Warping

Warping constant – general definition

From slide 16,

$$\sigma_x = -E\varphi''\omega \Rightarrow \frac{d\sigma_x}{dx} = -E\varphi'''\omega$$

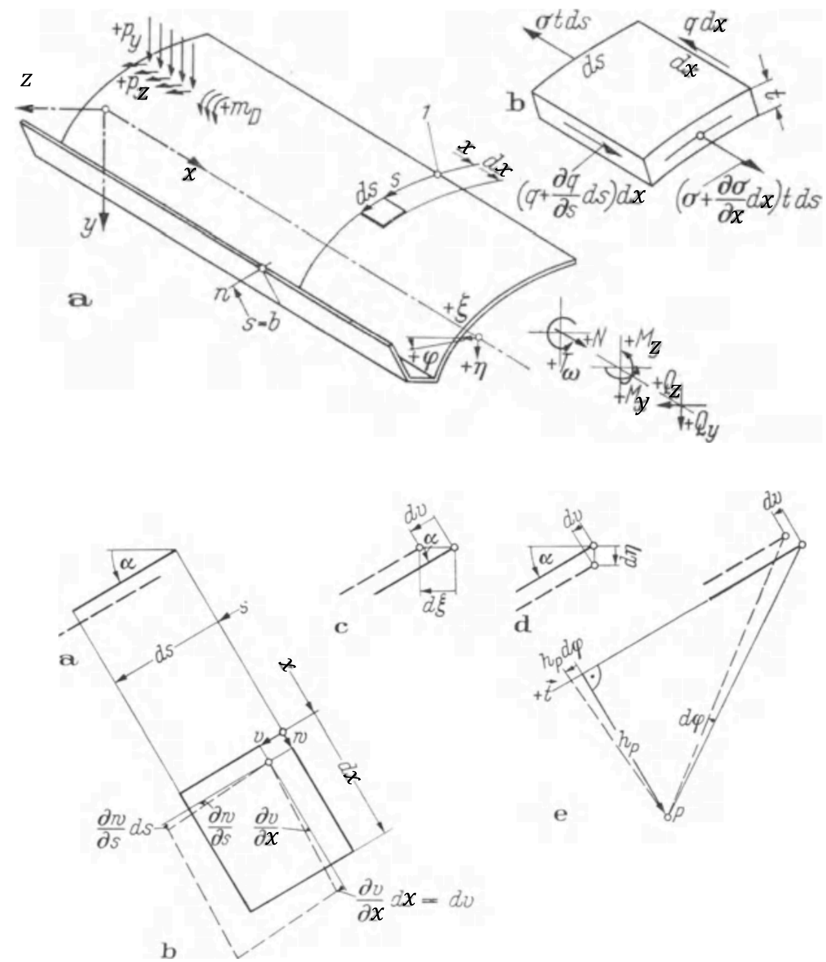
and so,

$$T_\omega = -\int E\varphi'''\omega^2 t ds =$$

$$= -E\varphi'''\int \omega^2 \mathrm{dA}$$

The warping constant is then defined as,

$$I_\omega = \int_A \omega(s)^2 dA$$



[adapted from Kollbrunner & Basler, 1969]

Properties Critical For Torsion and Warping

Warping constant

$$I_\omega = \int_A \omega(s)^2 dA \qquad [m^6]$$

Bi-moment

$$M_\omega = \int_A \sigma\omega\, dA = -EI_\omega \varphi'' \quad [N.m^2]$$

Warping moment

$$T_\omega = \frac{\partial M_\omega}{\partial x} = -EI_\omega \varphi''' \qquad [N.m]$$

Moment of inertia

$$I_y = \int_A y^2 dA \qquad [m^4]$$

Bending moment

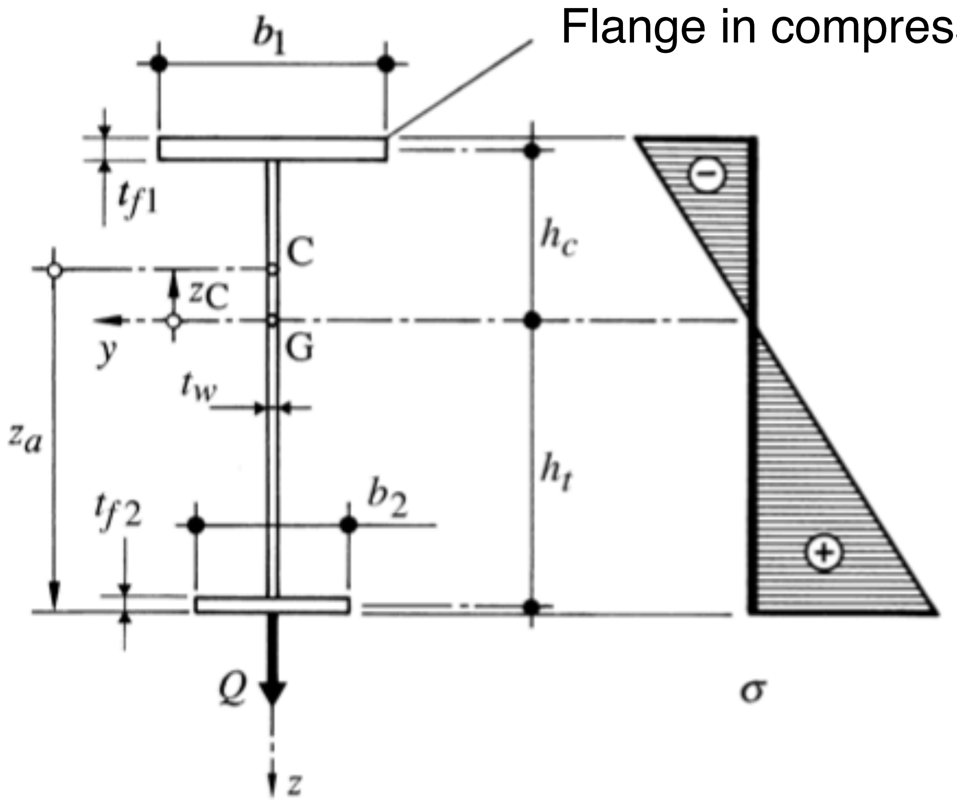
$$M_y = \int_A \sigma y\, dA = -EI_y v'' \quad [N.m]$$

Shear

$$Q_z = \frac{\partial M_y}{\partial x} = -EI_y v''' \qquad [N]$$

Analogy

Geometric Properties Critical For Torsion and Warping - Singly Symmetric cross-sections



C : Shear Centre

G : Centre of Gravity

Cross-Sectional Properties for Torsion/Warping

Sectorial Characteristic of Cross-Section
(β=0 for doubly symmetric cross-sections)

$$\beta = z_C + \frac{1}{2I_y}\left[h_t\left(\frac{b_2^3\, t_{f2}}{12} + b_2\, t_{f2}\, h_t^2 + \frac{h_t^3 t_w}{4}\right) - h_C\left(\frac{b_1^3\, t_{f1}}{12} + b_1\, t_{f1}\, h_c^2 + \frac{h_c^3\, t_w}{4}\right)\right]$$

Location of Shear Center

$$z_C = \frac{b_1^3\, h_c\, t_{f1} - b_2^3\, h_t\, t_{f2}}{b_1^3\, t_{f1} - b_2^3\, t_{f2}}$$

Warping Constant

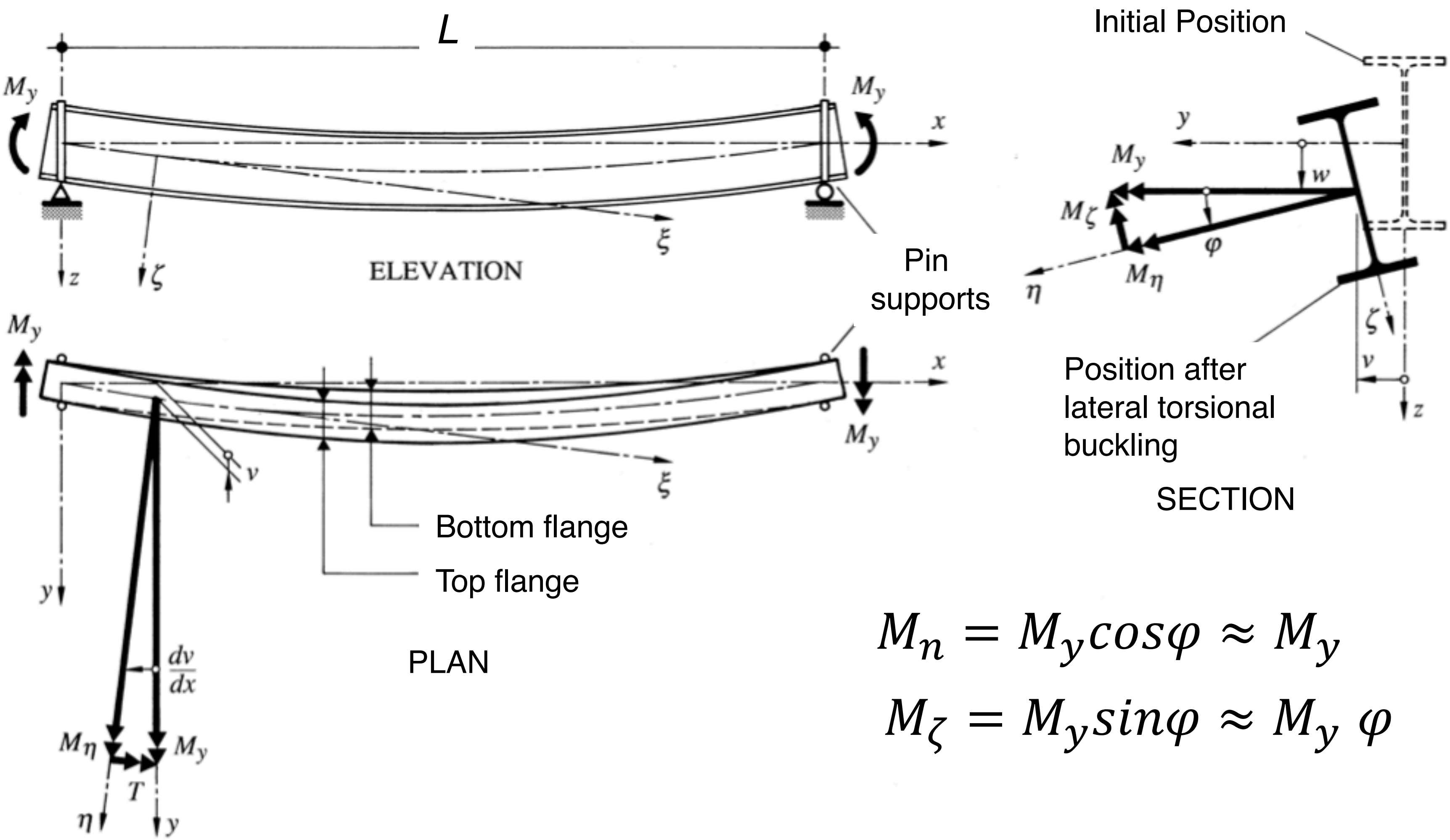
$$I_\omega = \frac{(h_c + h_t)^2\, b_1^3\, t_{f1}\, b_2^3\, t_{f2}}{12[b_1^3\, t_{f1} + b_2^3\, t_{f2}]}$$

Torsional Moment of Inertia

$$K = \frac{1}{3}[b_1\, t_{f1}^3 + b_2\, t_{f2}^3 + (h_c + h_t)t_w^3]$$

Lateral Torsional Buckling

-Reference Case: Simple Supported Beam Subjected to Uniform Bending



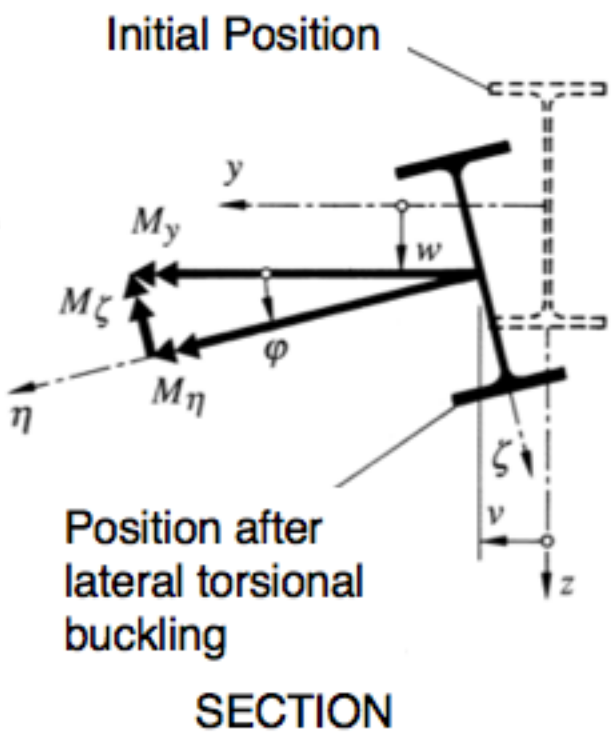
$$M_n = M_y cos\varphi \approx M_y$$

$$M_\zeta = M_y sin\varphi \approx M_y\,\varphi$$

Lateral Torsional Buckling
-Reference Case: Beam Subjected to Uniform Bending

Equilibrium requires that these moments are equal to the internal moments, resulting in the following differential equation for bending about *y* axis,

$$M_y = -EI_y \frac{d^2w}{dx^2}$$



Flexure due to bending with respect to the ζ axis,

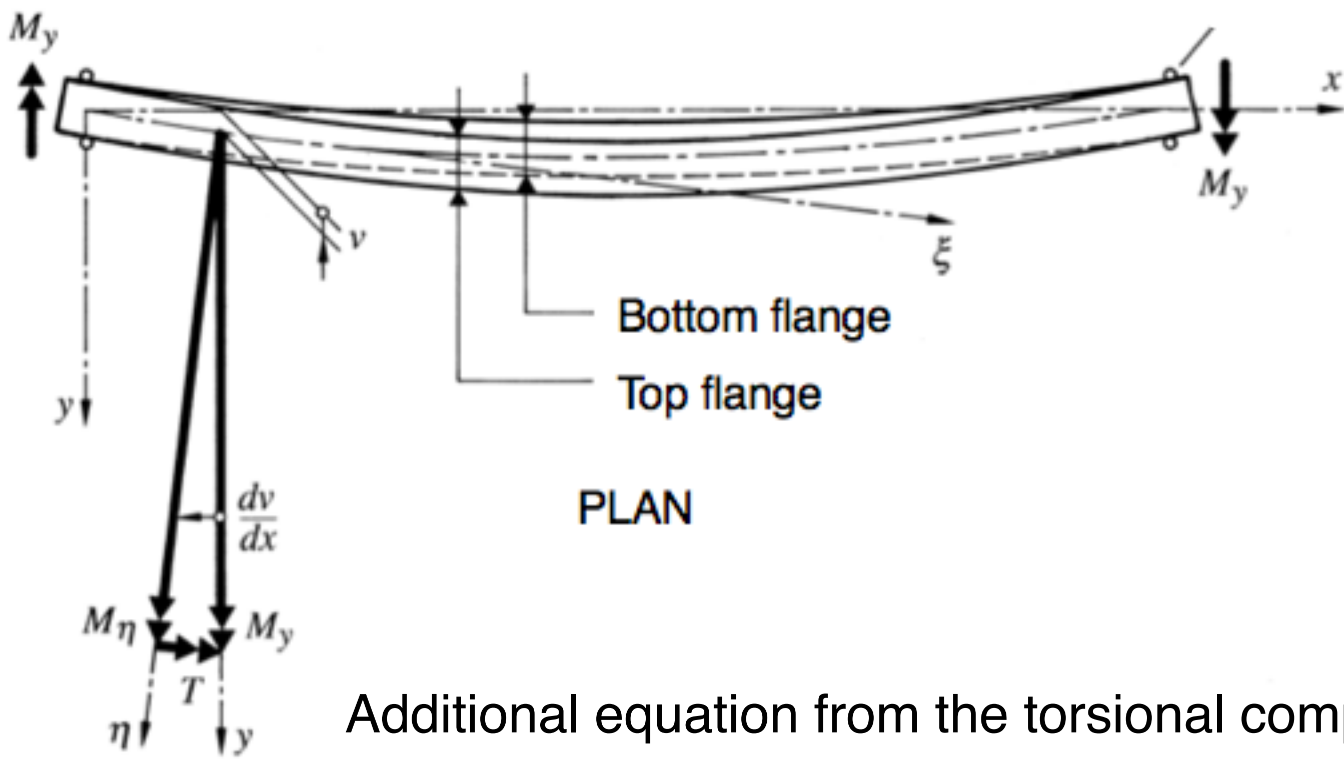
$$M_\zeta = -EI_z \frac{d^2v}{dx^2}$$

$$M_\zeta = M_y sin\varphi \approx M_y\ \varphi$$

$$M_y\varphi + EI_z \frac{d^2v}{dx^2} = 0$$

Lateral Torsional Buckling

-Reference Case: Beam Subjected to Uniform Bending



Additional equation from the torsional component T

$$T = M_y \sin\left(\frac{dv}{dx}\right) \approx M_y \cdot \frac{dv}{dx}$$

Lateral Torsional Buckling

-Reference Case: Beam Subjected to Uniform Bending

The internal moment of torsion consists of a warping and a uniform torsion component. *G* is the shear modulus, I_ω is the warping constant, and *K* is the St. Venant's torsion constant.

$$EI_\omega \varphi''' - GK\varphi' + M_y \frac{dv}{dx} = 0$$

The two differential equations involving the lateral-torsional displacements *v* and φ are then equal to,

$$M_y \varphi + EI_z \frac{d^2 v}{dx^2} = 0$$

$$EI_\omega \varphi''' - GK\varphi' + M_y \frac{dv}{dx} = 0$$

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Lateral Torsional Buckling

-Reference Case: Beam Subjected to Uniform Bending

Therefore,

$$\frac{d^2v}{dx^2} = -\frac{M_y\varphi}{EI_z}$$

Differentiation of the second equation with respect to *x* and substitution results in the following equation,

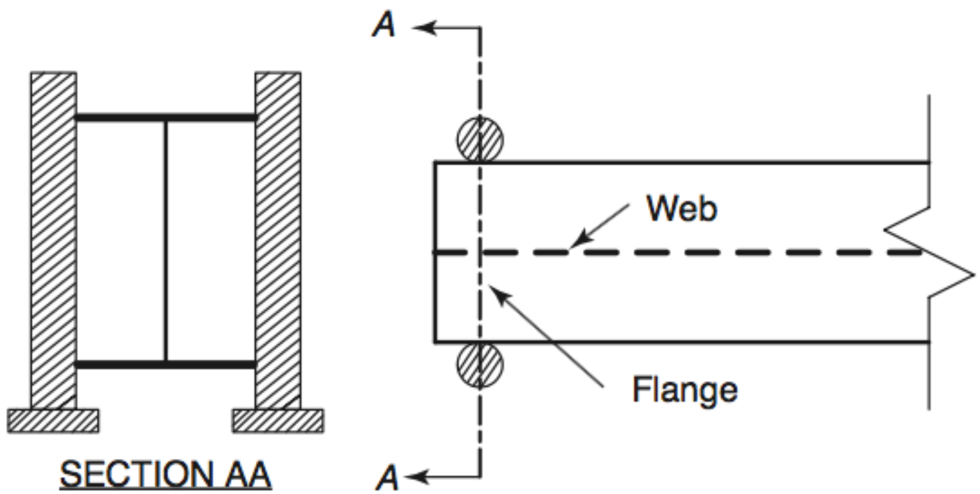
$$EI_\omega \frac{d^4\varphi}{dx^4} - GK\frac{d^2\varphi}{dx^2} - \frac{M_y^2\varphi}{EI_z} = 0$$

$$\frac{d^4\varphi}{dx^4} - \frac{GK}{EI_\omega}\frac{d^2\varphi}{dx^2} - \frac{M_y^2\varphi}{E^2I_zI_\omega} = 0$$

$$\left(\lambda_1 = \frac{GK}{EI_\omega}, \lambda_2 = \frac{M_y^2}{E^2I_zI_\omega}\right) \qquad \frac{d^4\varphi}{dx^4} - \lambda_1\frac{d^2\varphi}{dx^2} - \lambda_2\varphi = 0$$

Lateral Torsional Buckling
-Reference Case: Beam Subjected to Uniform Bending

Boundary conditions,



The lateral deflection and the angle of twist equals zero at each end,

$$v(0) = v(L) = \varphi(0) = \varphi(L) = 0$$

There is no moment about the z-axis at both ends and they are free to warp,

$$v''(0) = v''(L) = \varphi''(0) = \varphi''(L) = 0$$

Lateral Torsional Buckling
-Reference Case: Beam Subjected to Uniform Bending

$$\frac{d^4\varphi}{dx^4} - \lambda_1 \frac{d^2\varphi}{dx^2} - \lambda_2\varphi = 0$$

Roots of the differential equation are,

$$\varphi[r^4 - \lambda_1 r^2 - \lambda_2] = 0$$

Where,

$$r = [a_1, -a_1, ia_2, -ia_2]^T$$

Where,

$$a_1 = \sqrt{\frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}} \qquad a_2 = \sqrt{\frac{-\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}}$$

Lateral Torsional Buckling
-Reference Case: Beam Subjected to Uniform Bending

The expression for the angle of twist can be written as follows,

$$\phi = A_1 e^{a_1 x} + A_2 e^{-a_1 x} + A_3 e^{i a_2 x} + A_4 e^{-i a_2 x} \quad (i = \sqrt{-1})$$

or

$$\phi = C_1 cosh a_1 x + C_2 sinh a_1 x + C_3 sin a_2 x + C_4 \cos a_2 x$$

$A_1, A_2, A_3, A_4 \ and \ C_1, C_2, C_3, C_4$: Constants of integration
(depend on the boundary conditions of the beam)

Lateral Torsional Buckling
-Reference Case: Beams Subjected to Uniform Bending

Buckling determinant (after applying the boundary conditions of the problem)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ \alpha_1^2 & 0 & 0 & -\alpha_2^2 \\ \cosh\alpha_1 L & \sinh\alpha_1 L & \sin\alpha_2 L & \cos\alpha_2 L \\ \alpha_1^2\cosh\alpha_1 L & \alpha_1^2\sinh\alpha_1 L & -\alpha_2^2\sin\alpha_2 L & -\alpha_2^2\cos\alpha_2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = 0$$

$$\underbrace{(a_1^2 + a_2^2) sinh a_1 L}_{\text{Cannot be zero}} \cdot sin a_2 L = 0$$

Therefore, $sin a_2 L = 0$

$$a_2 L = n\pi, n = 1,2,3, \ldots .$$

Lateral Torsional Buckling

-Reference Case: Beam Subjected to Uniform Bending

$$a_2 L = n\pi, n = 1,2,3, \dots.$$

$$a_2 = \sqrt{\frac{-\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}}$$

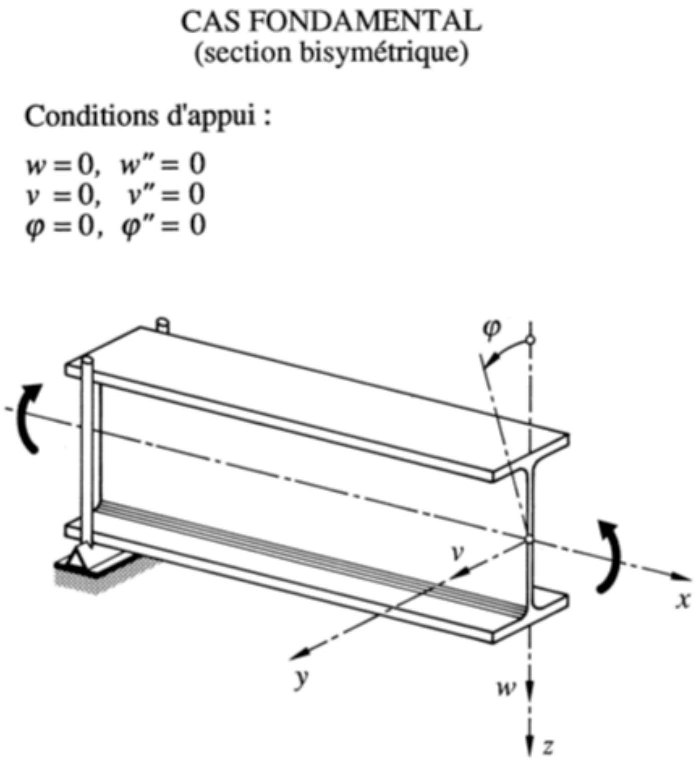
$$\lambda_2 = \frac{n^2\pi^2}{L^2}\left[\lambda_1 + \frac{n^2\pi^2}{L^2}\right] \qquad \left(\lambda_1 = \frac{GK}{EI_\omega}, \lambda_2 = \frac{M_o^2}{E^2 I_z I_\omega}\right)$$

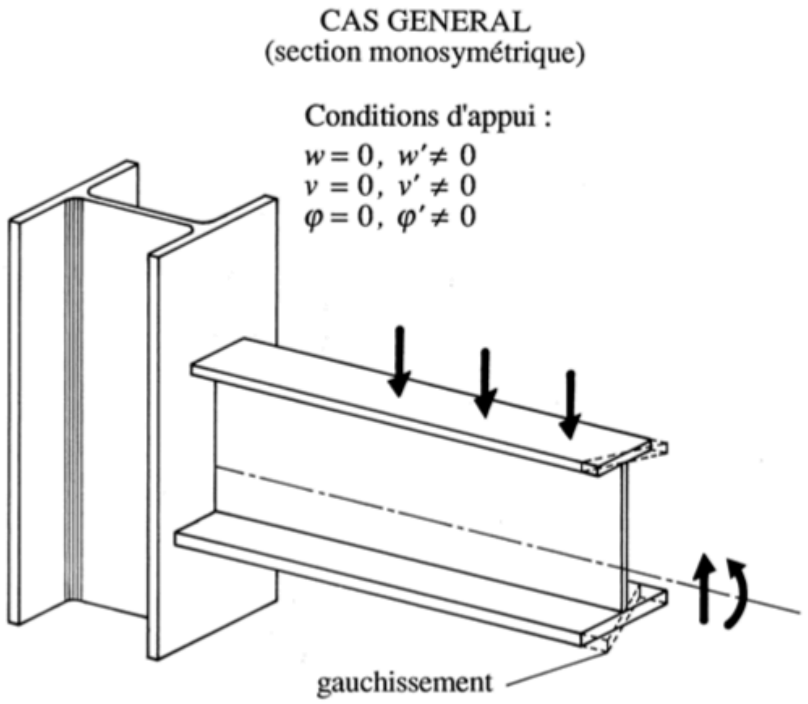
$$M_{y,cr} = \frac{\pi}{L}\sqrt{EI_z GK}\sqrt{1 + \frac{\pi^2 EI_\omega}{GKL^2}}$$

Lateral Torsional Buckling

Effect of Loading and Boundary Conditions

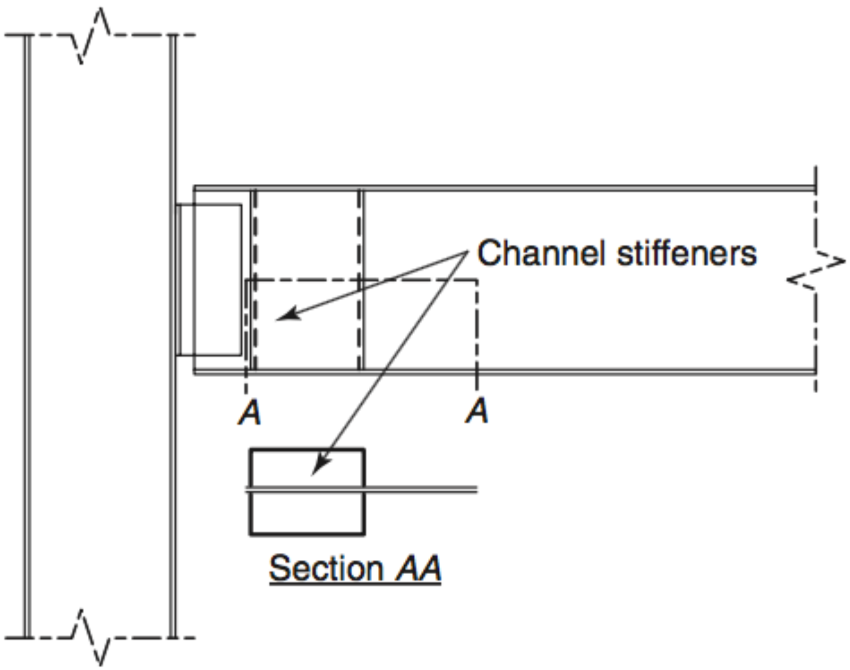
The special case of a simple supported beam is practically never found in reality. There is often bending at beam ends and torsion is often restrained. The external loads are not just uniform and the section may be asymmetrical.





Lateral Torsional Buckling
-Effect of Boundary Conditions: Warping Prevention

The beam ends are prevented from freely warping by a thick end plate or by a channel stiffener.



$$v(0) = v(L) = v''(0) = v''(L) = 0$$
$$\varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = 0$$

(Source: Ojalvo and Chambers 1977)

Lateral Torsional Buckling
-Effect of Boundary Conditions: Warping Prevention

In this case the buckling determinant is as follows,

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & \alpha_1 & \alpha_2 & 0 \\ \cosh\alpha_1 L & \sinh\alpha_1 L & \sin\alpha_2 L & \cos\alpha_2 L \\ \alpha_1\sinh\alpha_1 & \alpha_1\cosh\alpha_1 L & \alpha_2\cos\alpha_2 L & -\alpha_2\sin\alpha_2 L \end{vmatrix} = 0$$

$$cosha_1 L \cdot cosha_2 L - 1 + \left(\frac{a_2^2 - a_1^2}{2a_1 a_2}\right) sinha_1 L \cdot sina_2 L = 0$$

The solution to this problem is not easy even though a closed-form analytical solution can be obtained. Therefore, Newton-Raphson iteration (or bisection method) is typically used. For most other boundary conditions obtaining a closed-form solution is either difficult or impossible. In design we use an approach similar to the effective length method (classical buckling).

Lateral Torsional Buckling

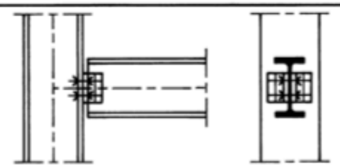
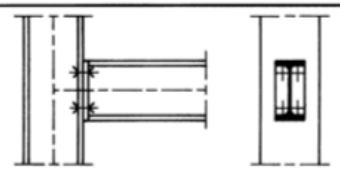
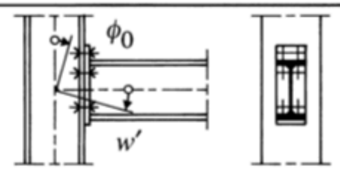
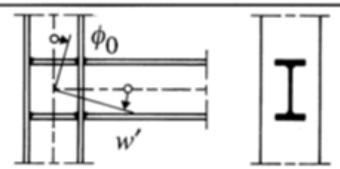
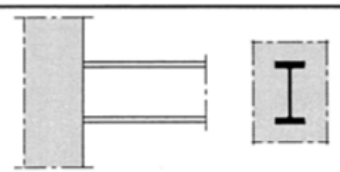
-Effect of Boundary Conditions – "Effective Length Factors"

$$M_{y,cr} = \frac{\pi}{k_v L}\sqrt{EI_z GI_\omega}\sqrt{1 + \frac{\pi^2 EK}{GI_\omega\left(k_\varphi L^2\right)}}$$

I_z - is the second moment of area about the minor axis *z-z*
I_ω - is the torsional constant of the cross-section
K - is the warping constant of the cross-section
L - is the beam length between lateral restraint points
k_v – effective length factor of end rotation on plane
k_φ - effective length factor for member end warping

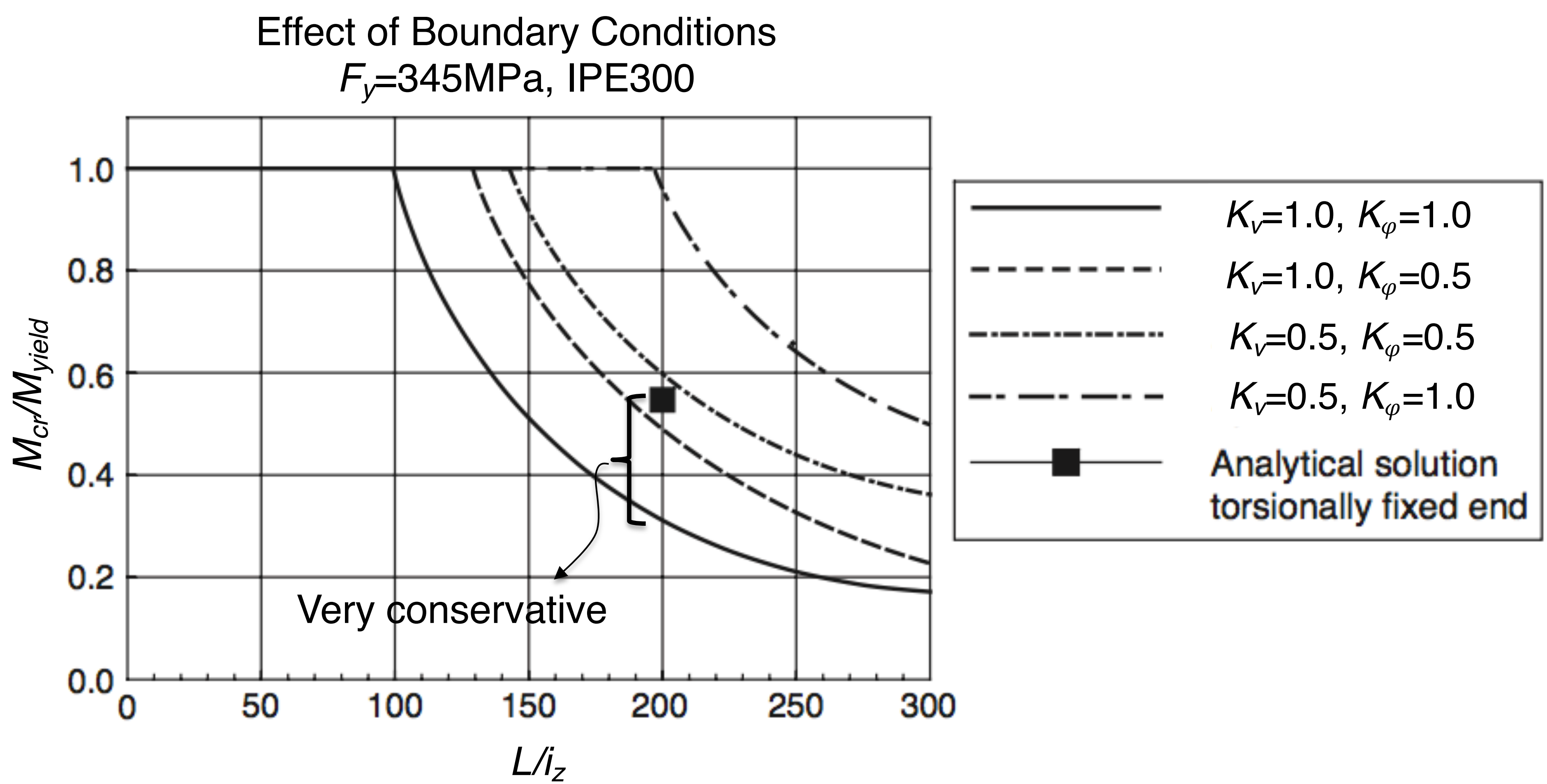
Lateral Torsional Buckling

-Effect of Boundary Conditions on M_{cr}

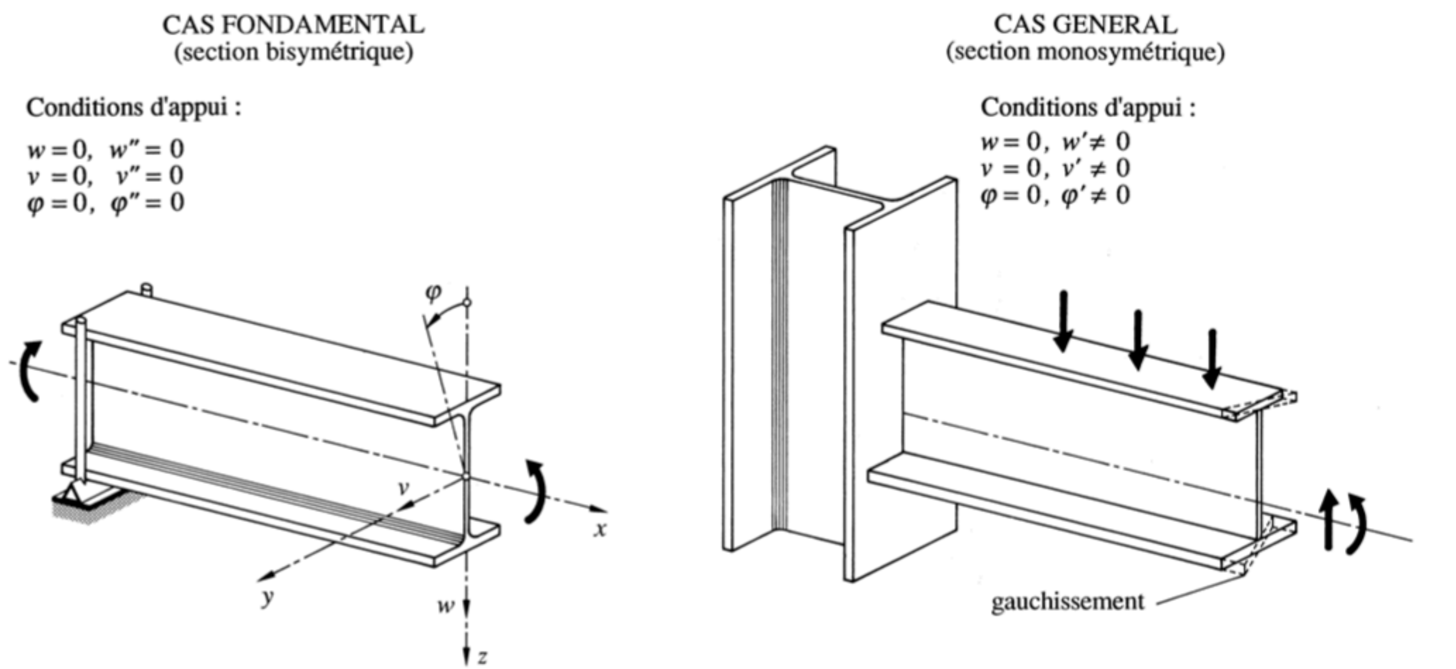
Schémas des appuis	Conditions aux limites	k_v et k_φ	Remarques
	$w = w_0$ $w'' = 0$ $v = 0$ $v'' = 0$ $\varphi = 0$ $\varphi'' = 0$	$k_v = 1.0$ $k_\varphi = 1.0$	Raidisseurs ou appuis latéraux au niveau de la membrure comprimée nécessaires afin d'empêcher la rotation de la poutre aux appuis. w_0 : déplacement vertical de l'appui.
	$w = 0$ $w'' = 0$ $v = 0$ $v'' = 0$ $\varphi = 0$ $\varphi'' = 0$	$k_v = 1.0$ $k_\varphi = 1.0$	Liaison articulée dans laquelle les cornières doivent être suffisantes pour empêcher la rotation de la poutre aux appuis.
	$w = 0$ $w'' = 0$ $v = 0$ $v'' = 0$ $\varphi = 0$ $\varphi'' = 0$	$k_v = 1.0$ $k_\varphi = 1.0$	Liaison semi-rigide peu résistante en flexion et avec gauchissement non empêché des ailes. On néglige la rigidité à la torsion du poteau.
	$w = 0$ $w' = \phi_0$ $v = 0$ $v'' = 0$ $\varphi = 0$ $\varphi'' = 0$	$k_v = 1.0$ $k_\varphi < 1.0$	Liaison semi-rigide résistante en flexion et avec gauchissement empêché des ailes ($k_\varphi < 1.0$). On néglige la rigidité à la torsion du poteau. ϕ_0: inclinaison du poteau.
	$w = 0$ $w' = \phi_0$ $v = 0$ $v'' = 0$ $\varphi = 0$ $\varphi' = 0$	$k_v = 1.0$ $k_\varphi = 0.5$	Encastrement parfait de la poutre à la torsion, gauchissement empêché par les raidisseurs. On néglige la rigidité à la torsion du poteau. ϕ_0: inclinaison du poteau.
	$w = 0$ $w' = 0$ $v = 0$ $v' = 0$ $\varphi = 0$ $\varphi' = 0$	$k_v = 0.5$ $k_\varphi = 0.5$	Encastrement parfait de la poutre à la flexion et à la torsion, gauchissement empêché par l'appui que l'on considère comme étant rigide à la torsion.

Lateral Torsional Buckling

-Effect of Boundary Conditions on M_{cr}



Lateral Torsional Buckling
-Effect of Loading Conditions on M_{cr}

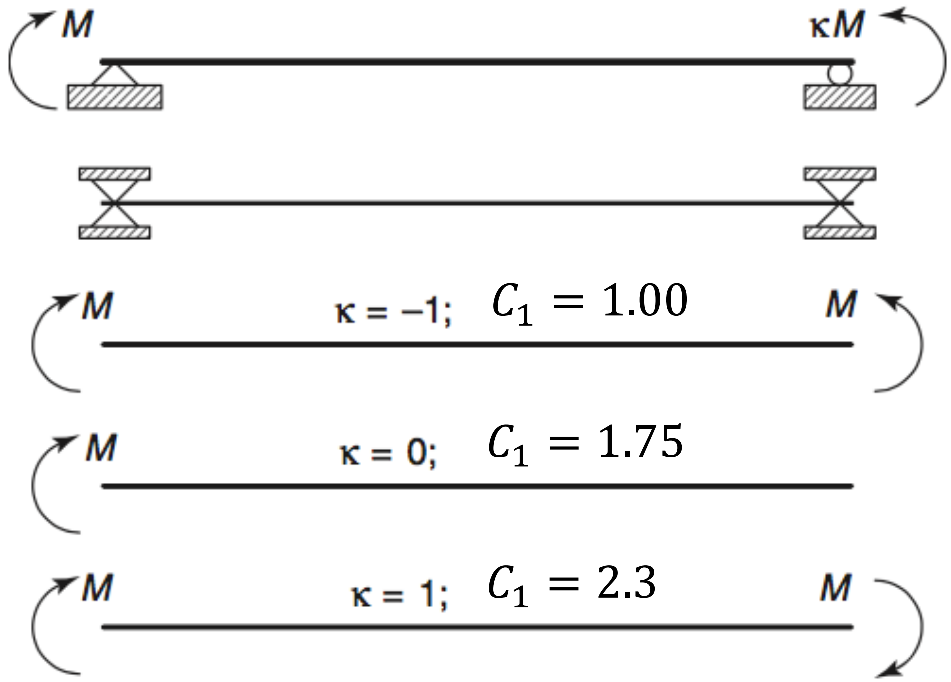


Salvadori (1955) proposed the use of a C_b multiplier for design code applications

$$M_{y,cr} = \frac{C_1 \pi}{k_v L}\sqrt{EI_z GI_\omega}\sqrt{1+\frac{\pi^2 EK}{GI_\omega\left(k_\varphi L^2\right)}}$$

Lateral Torsional Buckling

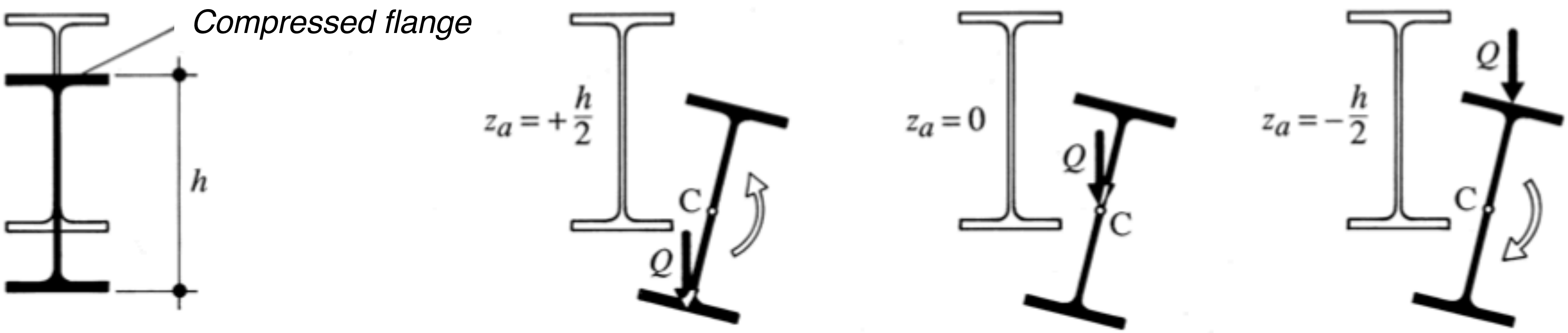
-Effect of Loading Conditions on M_{cr}



Salvadori (1955) formula, $C_1 = 1.75 + 1.05\kappa + 0.3\kappa^2 \leq 2.3$

Lateral Torsional Buckling
-Effect of the Location of the Load

Without Lateral Torsional Buckling *With Lateral Torsional Buckling*



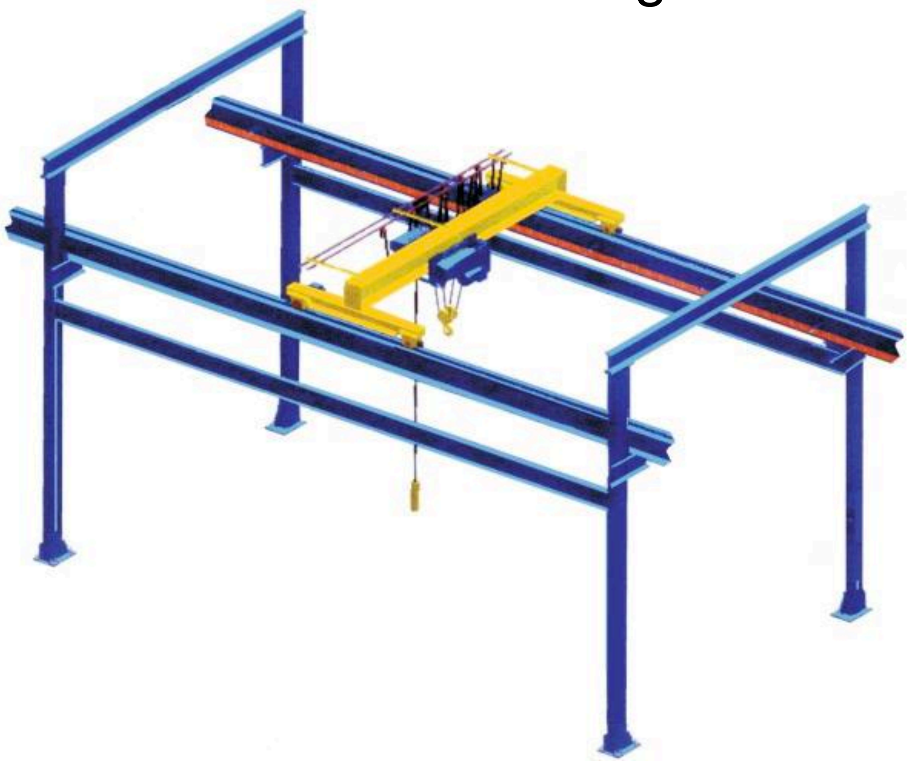
Secondary moment (helps stabilization)

No secondary moment

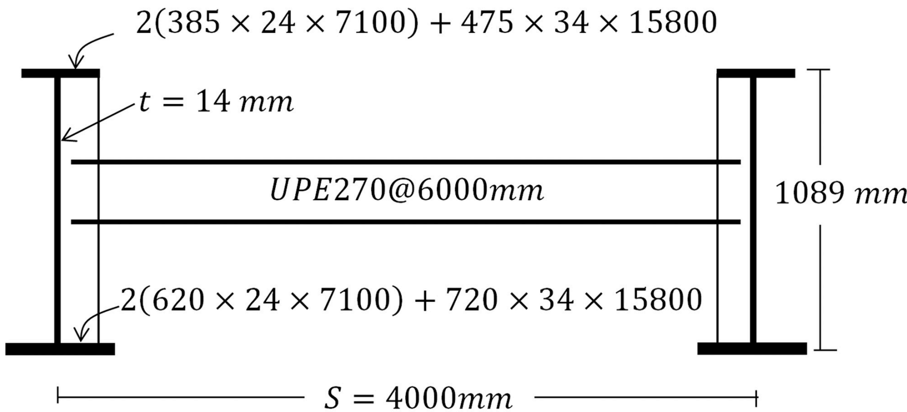
Secondary moment (prevents stabilization)

Lateral Torsional Buckling of Singly Symmetric Cross-Sections

Crane design

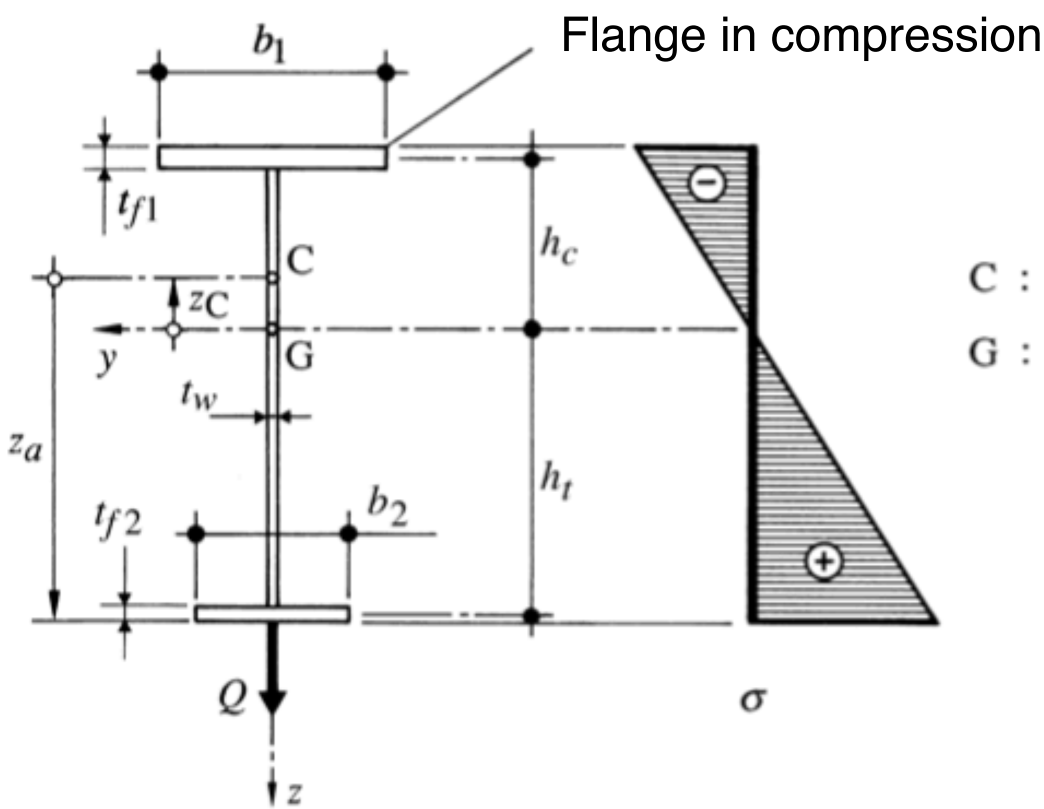


Bridge Design



The cross-section of Bridge 1530

Geometric Properties Critical For Pure Torsion and Warping



C : Shear Centre

G : Centre of Gravity

Cross-Sectional Properties for Torsion/Warping

Sectorial Characteristic of Cross-Section
(β=0 for doubly symmetric cross-sections)

$$\beta = z_C + \frac{1}{2I_y}\left[h_t\left(\frac{b_2^3\, t_{f2}}{12} + b_2\, t_{f2}\, h_t^2 + \frac{h_t^3 t_w}{4}\right) - h_C\left(\frac{b_1^3\, t_{f1}}{12} + b_1\, t_{f1}\, h_c^2 + \frac{h_c^3\, t_w}{4}\right)\right]$$

Location of Shear Center

$$z_C = \frac{b_1^3\, h_c\, t_{f1} - b_2^3\, h_t\, t_{f2}}{b_1^3\, t_{f1} - b_2^3\, t_{f2}}$$

Warping Constant

$$I_\omega = \frac{(h_c + h_t)^2\, b_1^3\, t_{f1}\, b_2^3\, t_{f2}}{12[b_1^3\, t_{f1} + b_2^3\, t_{f2}]}$$

Torsional Moment of Inertia

$$K = \frac{1}{3}[b_1\, t_{f1}^3 + b_2\, t_{f2}^3 + (h_c + h_t)t_w^3]$$

Lateral Torsional Buckling of Singly Symmetric Cross-Sections

The two differential equations involving the lateral-torsional displacements u and ϕ are equal to,

$$\frac{d^2 v}{dx^2} = -\frac{M_y \varphi}{EI_z}$$

$$EI_\omega \frac{d^3\varphi}{dx^3} - (GK + \underbrace{M_y\beta}) \varphi + M_y v = 0$$

Additional twisting moment caused by the normal stresses on each of the two differently warped surfaces on the differential elements *dz* along the *z*-axis of the beam

$$\frac{d^4\varphi}{dx^4} - \frac{GK + M_y\beta}{EI_\omega}\frac{d^2\varphi}{dx^2} - \frac{M_y^2\phi}{E^2 I_z I_\omega} = 0$$

$$\lambda_1 = \frac{GK + M_y\beta}{EI_\omega}, \lambda_2 = \frac{M_y^2}{E^2 I_z I_\omega}, \qquad a_2 = \sqrt{\frac{-\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}}$$

Lateral Torsional Buckling of Singly Symmetric Cross-Sections

The equations for a_2 and λ_2 are identical to the previous equations for the doubly symmetric wide-flange cross-section; only λ1 is different, since it includes the extra term $M_o\beta$,

$$a_2^2 L^2 = \pi^2 = \frac{1}{2}\left(-\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}\right)$$

$$\frac{M_y^2}{E^2 I_z I_\omega} = \frac{\pi^2}{L^2}\left[\frac{GK + M_y\beta}{EI_\omega} + \frac{\pi^2}{L^2}\right]$$ (Second order algebraic equation)

$$M_{y,cr} = \frac{\pi^2 E I_z \beta}{2L^2}\left[1 + \sqrt{\frac{I_\omega}{I_z}\left(\frac{GKL^2}{\pi^2 E I_\omega} + 1\right)}\right]$$

Lateral Torsional Buckling

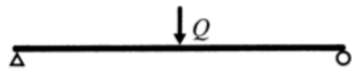
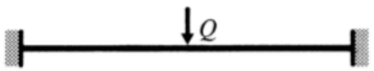
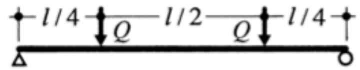
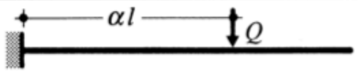
Effect of Boundary conditions, Loading & Loading Application, Cross-Section Shape

Clark and Hill proposed the following expression to take all issues into account:

$$M_{cr} = \frac{C_1 \pi^2 E I_z}{k_v k_\varphi L_b^2} [\sqrt{(C_2 z_a + C_3 \beta)^2 + \frac{I_\omega}{I_z}\left(\frac{G K k_\varphi^2 L_b^2}{\pi^2 E I_\omega} + 1\right)} + (C_2 z_a + C_3 \beta)$$

z_a – is the distance between the point of load application and the shear center
I_z - is the second moment of area about the minor axis z-z
I_ω - is the warping constant of the cross-section
K - is the torsional constant of the cross-section
L_b - is the beam length between lateral restraint points
k_v – effective length factor of end rotation in plane of loading (see slide 32)
k_φ - effective length factor for member end warping (see slide 32)
C_1, C_2 *and* C_3 - coefficients depending on the loading and end restraint conditions
β – Sectorial characteristic of a cross-section (β=0 for double-symmetric sections)

Lateral Torsional Buckling

Mode de chargement	$k_y = 1.0$			$k_y = 0.5$			$k_y = 2.0$		
	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
	1.00	①	1.00	1.00	①	1.14			
	1.32	①	0.99	1.51	①	2.27			
	1.88	①	0.94	2.15	①	2.15			
	2.70	①	0.68	3.09	①	1.55			
	2.75	①	0.00	3.15	①	0.00			
	1.13	0.46	0.53	0.97	0.30	0.98			
	1.28	1.56	0.75	0.71	0.65	1.07			
	1.36	0.55	1.73	1.07	0.43	3.06			
	1.56	1.27	2.64	0.94	0.71	4.80			
	1.05	0.43	1.12	1.01	0.41	1.89			
							$\frac{1.28}{\alpha}$	0.43	②
							2.05	0.83	②

Steel Beam Design for Lateral Torsional Buckling

-Lateral Torsional Buckling Resistance, Class 1 & 2 I-shape Members

General Condition:

$$\frac{M_{Ed}}{M_{b,Rd}} \leq 1$$

Class 1 & 2 Cross-Sections: Resistance in Lateral Torsional Buckling:

$$M_{b,Rd} = \frac{\chi_{LT} \cdot W_{pl,y} \cdot f_y}{\gamma_{M1}}$$

(γ_{M1} = 1.05)

Reduction factor for lateral torsional buckling:

$$\chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \bar{\lambda}_{LT}^2}} \leq 1.0$$

$$\bar{\lambda}_{LT} = \sqrt{\frac{W_{pl,y} \cdot f_y}{M_{cr}}}$$

EC3: $$\Phi_{LT} = 0.5 \cdot \left[1 + a_{LT} \cdot \left(\bar{\lambda}_{LT} - 0.2\right) + \bar{\lambda}_{LT}^2\right]$$

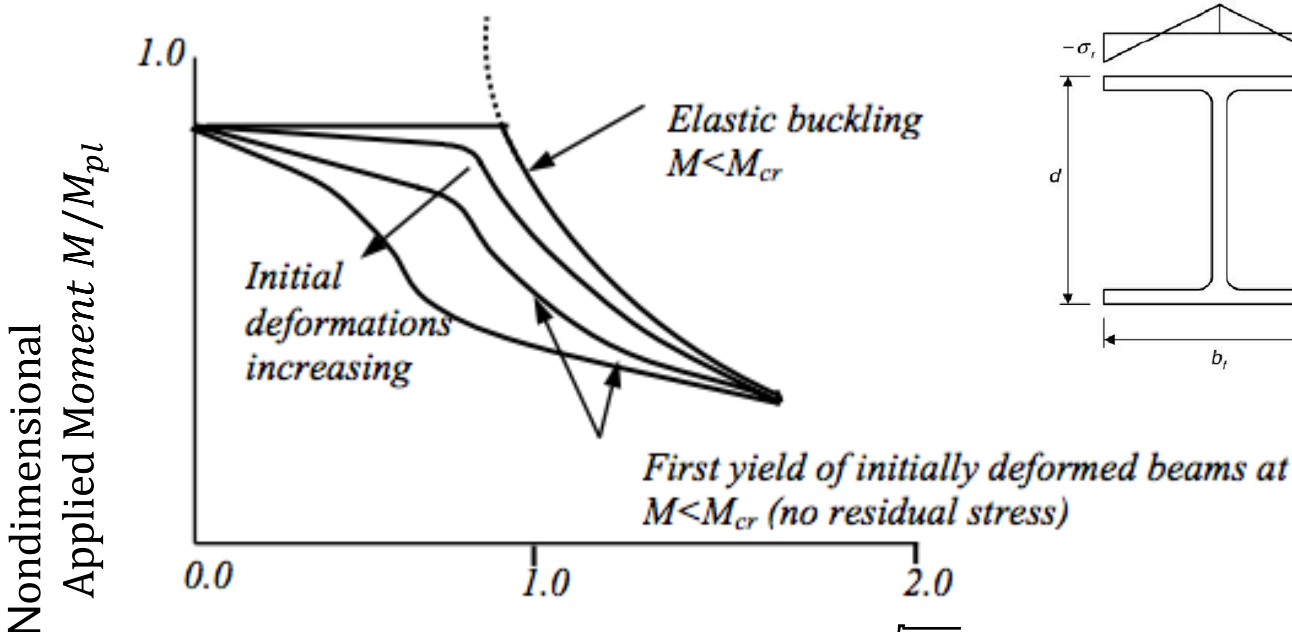
SIA 263: $$\Phi_{LT} = 0.5 \cdot \left[1 + a_{LT} \cdot \left(\bar{\lambda}_{LT} - 0.4\right) + \bar{\lambda}_{LT}^2\right]$$

Imperfection factor

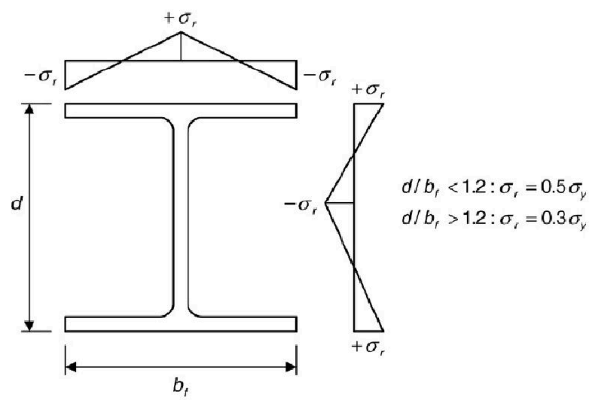
Lateral Torsional Buckling

-Effect of Imperfections on Lateral Torsional Buckling Curves

Residual stress distribution in the steel cross-section



Modified Slenderness Ratio $\lambda_{LT} = \sqrt{\frac{M_{pl}}{M_{cr}}}$



Lateral Torsional Buckling

-Effect of Imperfections on Lateral Torsional Buckling Curve (M_{cr})

Cross-Section	Limits	EN 1993-1-1	α_{LT}	SIA 263	α_{LT}
Rolled I- or H-sections	$h/b \leq 2$ $h/b > 2$	a b	*0.21* *0.34*	a	0.21
Welded I- or H-sections	$h/b \leq 2$ $h/b > 2$	c d	*0.49* *0.76*	c	0.49
Other sections	-	d	*0.76*	c	0.49

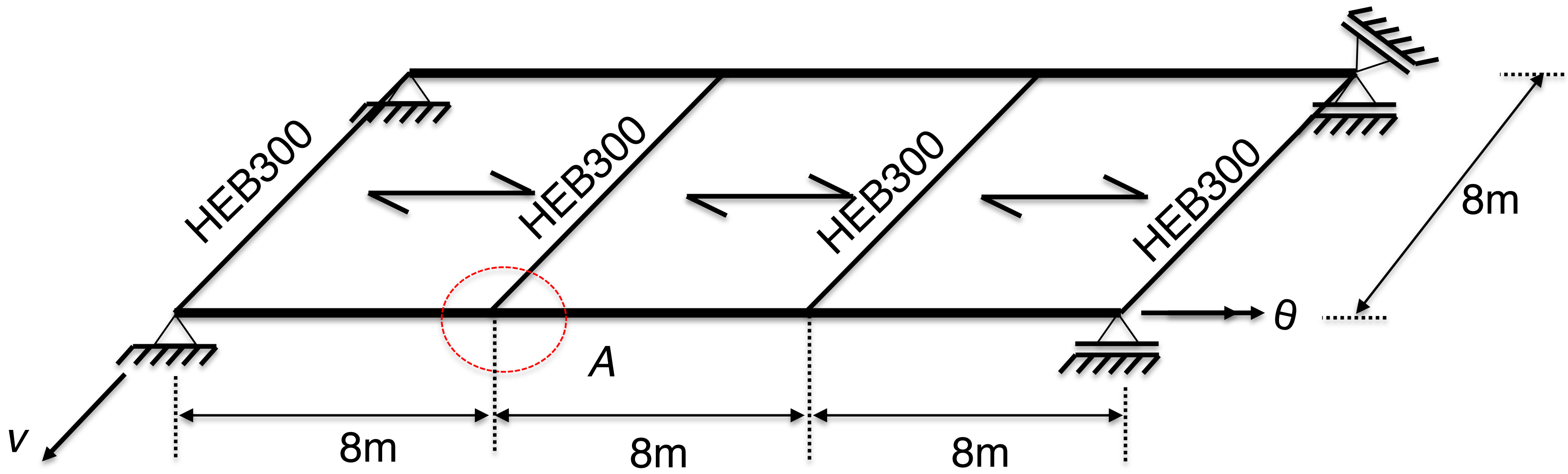
Other Aspects for Lateral Torsional Buckling

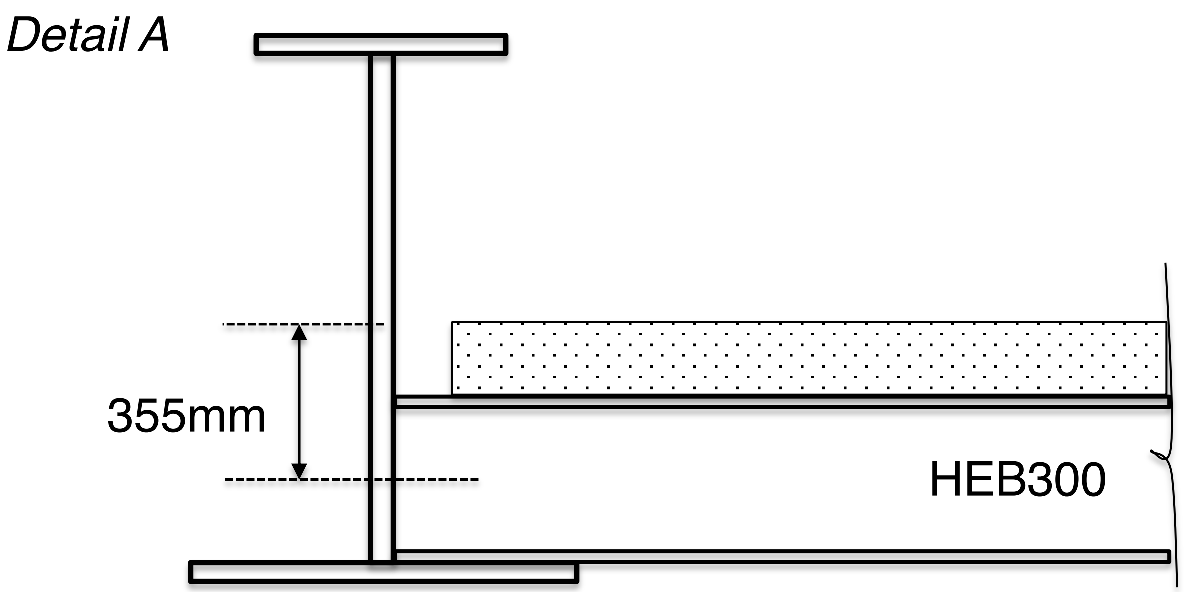
✧ Stability of bridge girders - Lateral torsional buckling by example (LTBeam software)

https://www.cticm.com/content/ltbeam-logiciel-calcul-moment-critique-deversement

✧ Critical stress due to lateral torsional buckling according to SIA 263 Design Guide

Stability of Bridge Girder – Example with LTBeam





Critical Stress Computations According to SIA 263

It is possible to use a simplified calculation method to determine the stress at which lateral torsional buckling may occur without having to consider the k_v, k_ϕ, C_1, C_2, C_3 factors provided that we are dealing with a simple supported beam, the section is doubly symmetric (β=0) and the loads act in the plane of symmetry (z_g=0)

$$M_{y,cr} = \sqrt{\frac{\pi^2 GKEI_z}{L^2} + \frac{\pi^4 EI_\omega EI_z}{L^4}}$$

Critical Stress Computations According to SIA 263

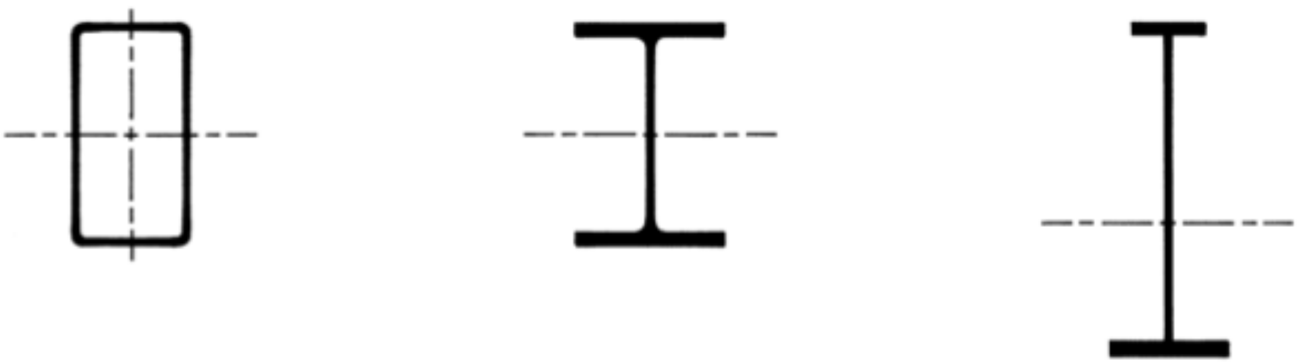
Critical stress for lateral torsional buckling, $\sigma_{cr,D}$

$$\sigma_{cr,D} = \frac{M_{y,cr}}{W_{y,el}} = \sqrt{\underbrace{\left(\frac{\pi}{W_{y,el}L}\sqrt{GKEI_z}\right)^2}_{\text{Stress due to uniform torsion only}} + \underbrace{\left(\frac{\pi^2 E}{L^2}\sqrt{\frac{I_\omega I_z}{W_{y,el}^2}}\right)^2}_{\text{Stress due to warping}}}$$

$$\sigma_{cr,D} = \sqrt{\sigma_{Dv}^2 + \sigma_{Dw}^2}$$

Critical Stress Computations According to SIA 263

Critical stress for lateral torsional buckling, $\sigma_{cr,D}$



Torsion uniforme

$$\sigma_{Dw} << \sigma_{Dv}$$

$$\sigma_{crD} \cong \sigma_{Dv}$$

Torsion mixte

$$\sigma_{crD} = \sqrt{\sigma_{Dv}^2 + \sigma_{Dw}^2}$$

Torsion non uniforme

$$\sigma_{Dv} << \sigma_{Dw}$$

$$\sigma_{crD} \cong \sigma_{Dw}$$

Critical Stress Computations According to SIA 263

Critical stress for lateral torsional buckling due to pure torsion, $\sigma_{cr,v}$

$$\sigma_{D,v} = \frac{n\pi}{W_{y,el}L}\sqrt{GKEI_z}$$

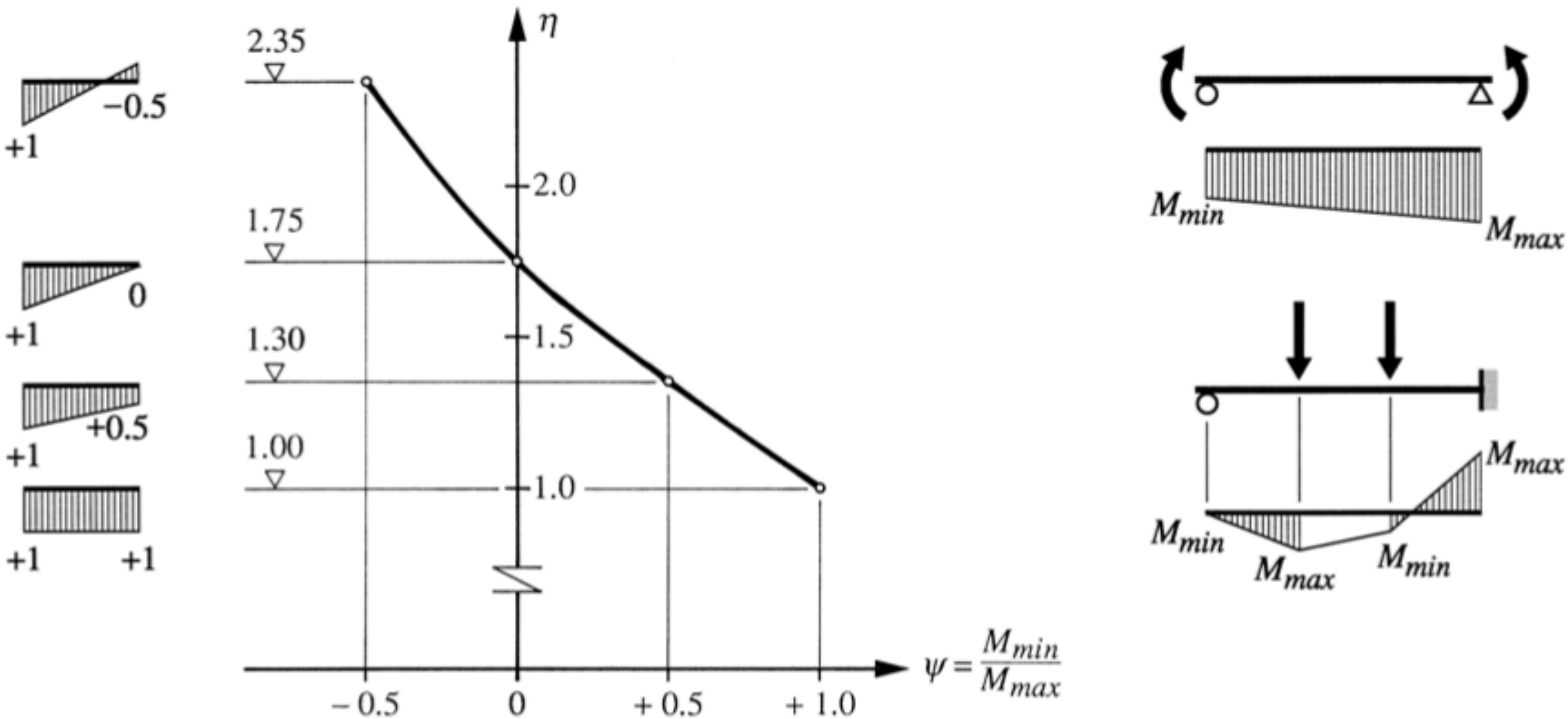
Modified Salvadori (1955) formula,

$$n = C_1 = 1.75 - 1.05\psi + 0.3\psi^2 \leq 2.3 \quad (-0.5 \leq \psi \leq 1.0)$$

$$\psi = M_{min}/M_{max}$$

Critical Stress Computations According to SIA 263

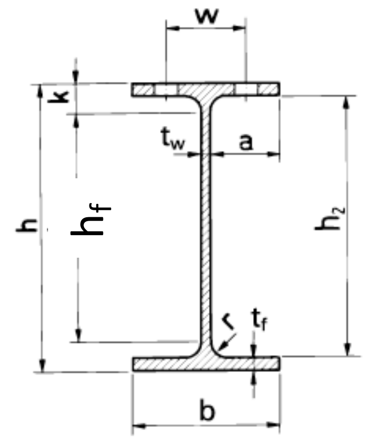
Critical stress for lateral torsional buckling due to pure torsion, $\sigma_{cr,v}$



EPFL Critical Stress Computations According to SIA 263

Critical stress for lateral torsional buckling due to pure warping, $\sigma_{cr,w}$

$$\sigma_{D,w} = n\frac{\pi^2 E}{L^2}\sqrt{\frac{I_\omega I_z}{W_{y,el}^2}}$$



For an I-shape section,

$$I_\omega = I_z\frac{h_f^2}{4}$$

$$I_y = 2bt_f\left(\frac{h_f}{2}\right)^2 + \frac{1}{12}t_w h_f^3 = \left(A_f + \frac{1}{6}A_w\right)\frac{h_f^2}{2}$$

$$W_{y,el} = \frac{I_y}{h/2} = \left(A_f + \frac{1}{6}A_w\right)\frac{h_f^2}{h}$$

$$A_f = bt_f$$

$$A_w = h_2 t_w$$

Critical Stress Computations According to SIA 263

Critical stress for lateral torsional buckling due to pure warping, $\sigma_{cr,w}$

$$\sigma_{D,w} = n\frac{\pi^2 E}{L^2}\left[\frac{I_z h}{\left(A_f + \frac{1}{6}A_w\right)2h_f}\right] = n\frac{\pi^2 E}{(L/i_D)^2}$$

Radius of gyration, i_D of compressed flange and 1/6 of web core (directly given in SZS C4.1 for sections with $h/h_2 = 1$),

$$i_D = \sqrt{\frac{I_z h}{\left(A_f + \frac{1}{6}A_w\right)2h_f}}$$

Calculation of Critical Moment for Lateral Torsional Buckling According to SIA 263

Critical Moment,

$$M_D = \chi_D M_R \qquad \left(M_R = \begin{cases} W_{y,el} f_y (for\ Class\ 3\ or\ 4) \\ W_{y,pl}, f_y\ \ (for\ Class\ 1\ or\ 2) \end{cases} \right)$$

Reduction factor for lateral torsional buckling:

$$\chi_D = \frac{1}{\Phi_D + \sqrt{\Phi_D^2 - \beta \bar{\lambda}_D^2}} \leq 1.0 \qquad \bar{\lambda}_D = \begin{cases} \sqrt{\dfrac{W_{pl,y} \cdot f_y}{\sigma_{cr,D} W_{y,el}}} \\ \sqrt{\dfrac{f_y}{\sigma_{cr,D}}} \end{cases}$$

Coefficient considering imperfections of the cross-section (residual stresses, variations on f_y, initial deformations)

$$\Phi_D = 0.5 \cdot \left[1 + a_D \cdot \left(\bar{\lambda}_D - \zeta\right) + \beta \bar{\lambda}_D^2\right]$$

a_D: imperfection factor (see table next slide)

$\beta = 1$ for bent beams of constant cross − section

$\zeta = 0.2$ when $\beta = 1.0$, else, $\zeta = 0.4$ when $\beta = 0.75$

Lateral Torsional Buckling – According to SIA 263

-Effect of Imperfections on Lateral Torsional Buckling Curve (M_D)

Cross-Section	Limits	EN 1993-1-1	α_{LT}	SIA 263	α_{LT}
Rolled I- or H-sections	h/b ≤ 2 h/b > 2	a b	*0.21* *0.34*	a	0.21
Welded I- or H-sections	h/b ≤ 2 h/b > 2	c d	*0.49* *0.76*	c	0.49
Other sections	-	d	*0.76*	c	0.49

Lateral Torsional Buckling – According to SIA 263

If one wants to consider the beneficial influence of a non-constant distribution of moments between lateral bracing, a modified value of χ_D may be used:

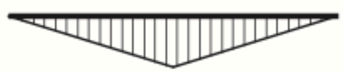
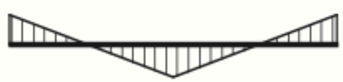
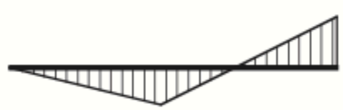
$$\chi_{D,mod} = \frac{\chi_D}{f}$$

$$f = 1 - 0.5(1 - k_c)\left[1 - 2\left(\bar{\lambda}_D - 0.8\right)^2\right] \leq 1$$

k_c depends on the moment diagram (see next page)

Lateral Torsional Buckling – According to SIA 263

-Moment Shape Effect on Reduction Factor

Distribution des moments	k_c
 $\psi = 1$	1.0
 $-1 \leq \psi \leq 1$	$\dfrac{1}{1.33 - 0.33\psi}$
	0.94
	0.90
	0.91
	0.86
	0.77
	0.82

Lateral Torsional Buckling – According to SIA 263