CIVIL 369: "Structural Stability"

School of Architecture, Civil & Environmental Engineering
Civil Engineering Institute
Resilient Steel Structures Laboratory (RESSLab)

Euler Method & Stability of Characteristic Systems

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GC B3 485 (bâtiment GC)

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EPFL Objectives of Today's Lecture

To introduce:

- ✧ Equilibrium paths
- ✧ Method of Equilibrium (or Euler Method)
- ✧ Categorization of failure modes in structural stability
- ✧ Modeling recommendations for nonlinear geometric analysis
- ✧ Solution algorithms for stable paths, snap-back and snap-through problems.
    - ✧ Newton's method
    - ✧ Arc-Length method

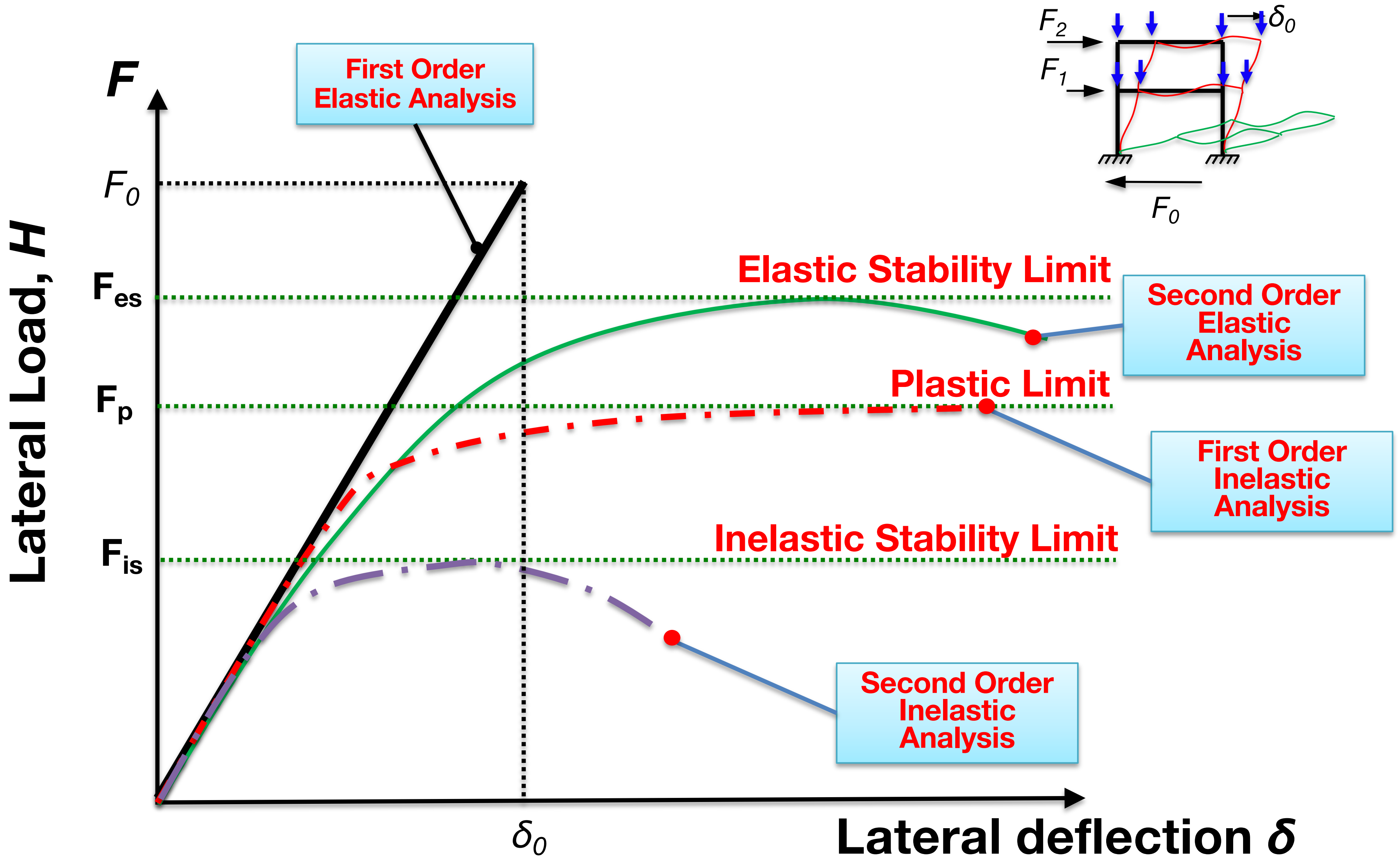
Geometric Nonlinearity – Basic Principles

✧ Large deviations of the deformation geometry of the structure (component) from the un-deformed one. This could happen when we have imperfections.

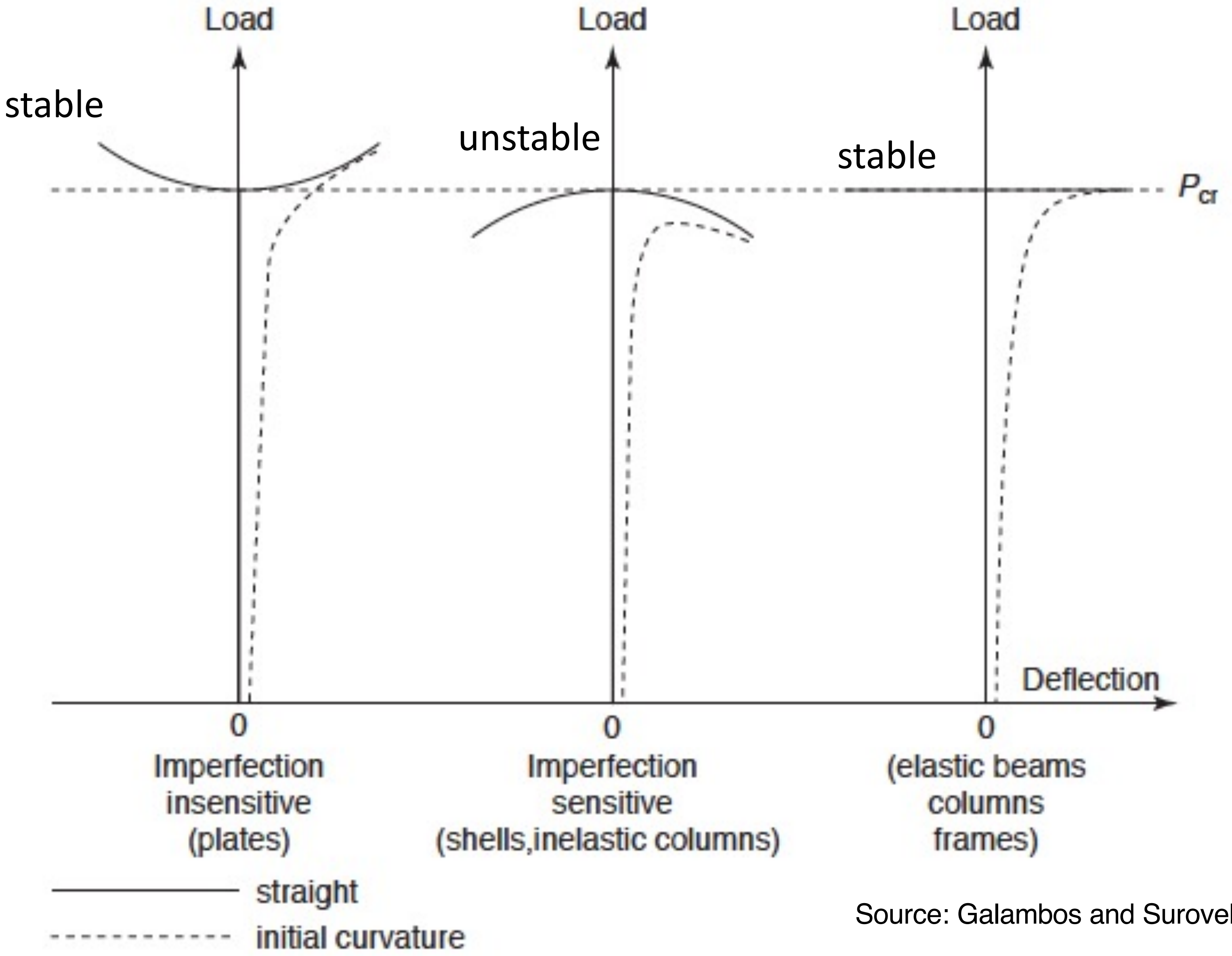
✧ Equations of equilibrium should be expressed in the deformed configuration rather than the un-deformed (second order theory or nonlinear analysis).

✧ Instability is likely to occur: compressive member (or structure) loses the ability to resist increasing loads and exhibits instead a decrease in load carrying-capacity.

Elastic Versus Inelastic Analysis

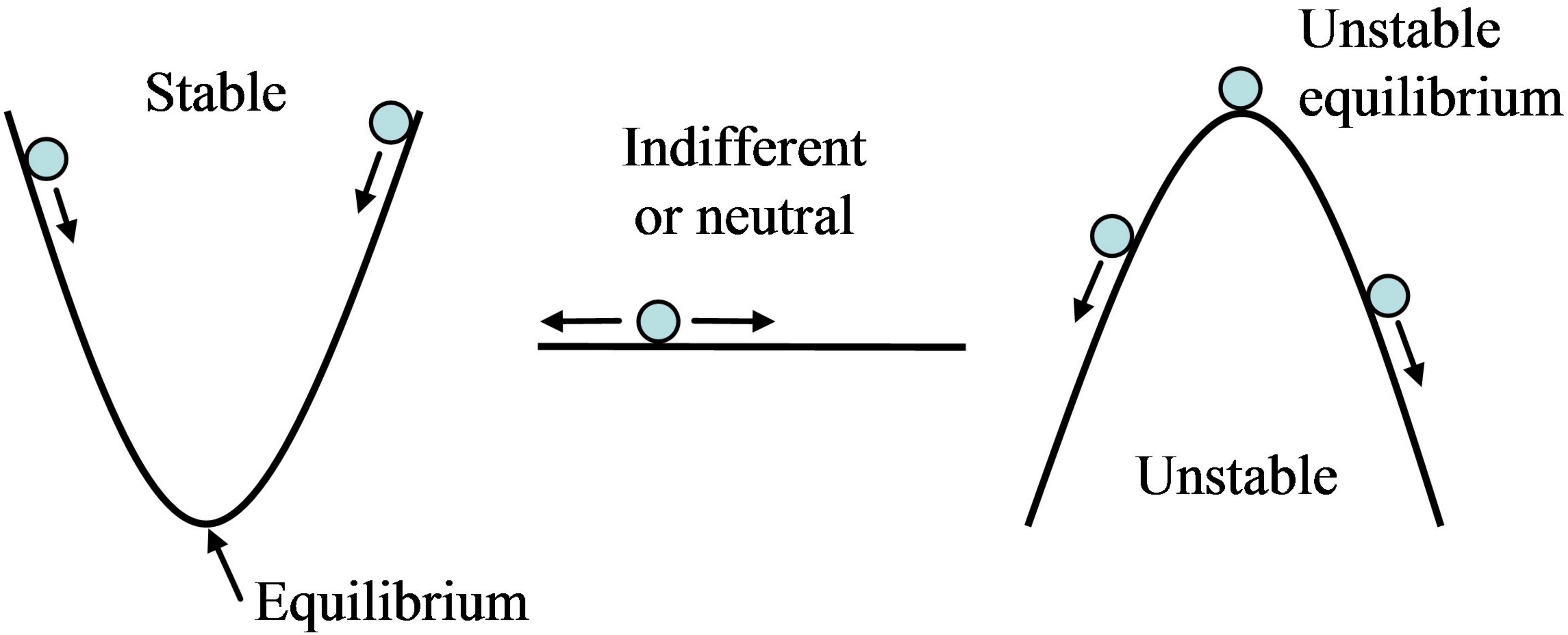


Load Equilibrium Paths and Stability



Source: Galambos and Surovek 2008

Equilibrium Paths and Stability



Source: Galambos and Surovek 2008

Major Differences Between Analysis Types

1. **First Order Elastic Analysis**: The Equations of Equilibrium are always written in the undeformed configuration and material nonlinearity is not considered.

Normally they require special software

2. **Second Order Elastic Analysis**: The Equations of Equilibrium are always written in the deformed configuration and material nonlinear is not considered.
3. **First Order inelastic Analysis**: The Equations of Equilibrium are always written in the undeformed configuration and material nonlinearity is considered.
4. **Second Order Inelastic Analysis**: The Equations of Equilibrium are always written in the deformed configuration and material nonlinear is considered.

Example on Stable / Unstable Structures – Pushover Analysis

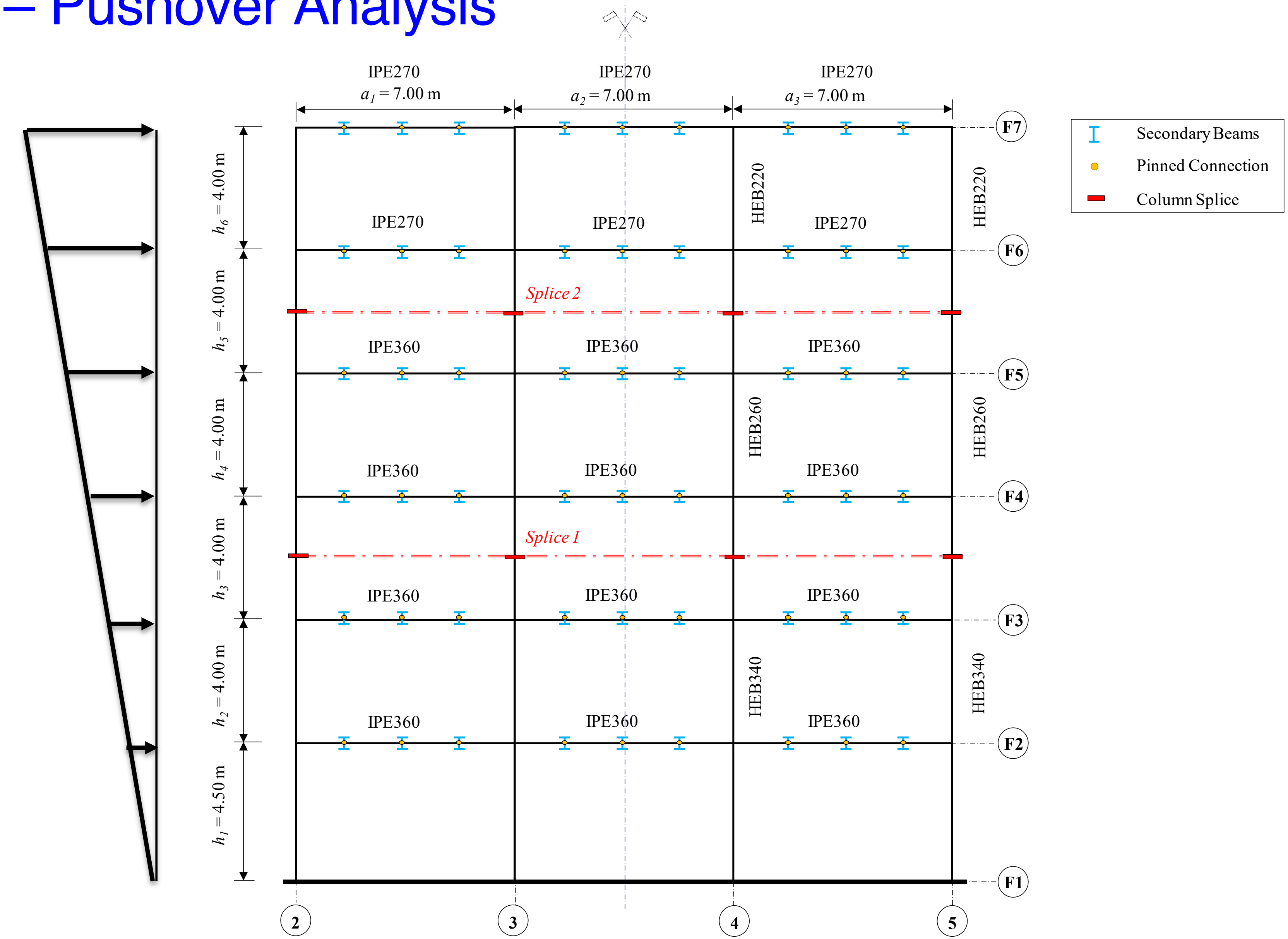


Image source: El Jisr and Lignos (2020)

Example on Stable / Unstable Structures – Pushover Analysis & Equilibrium Paths

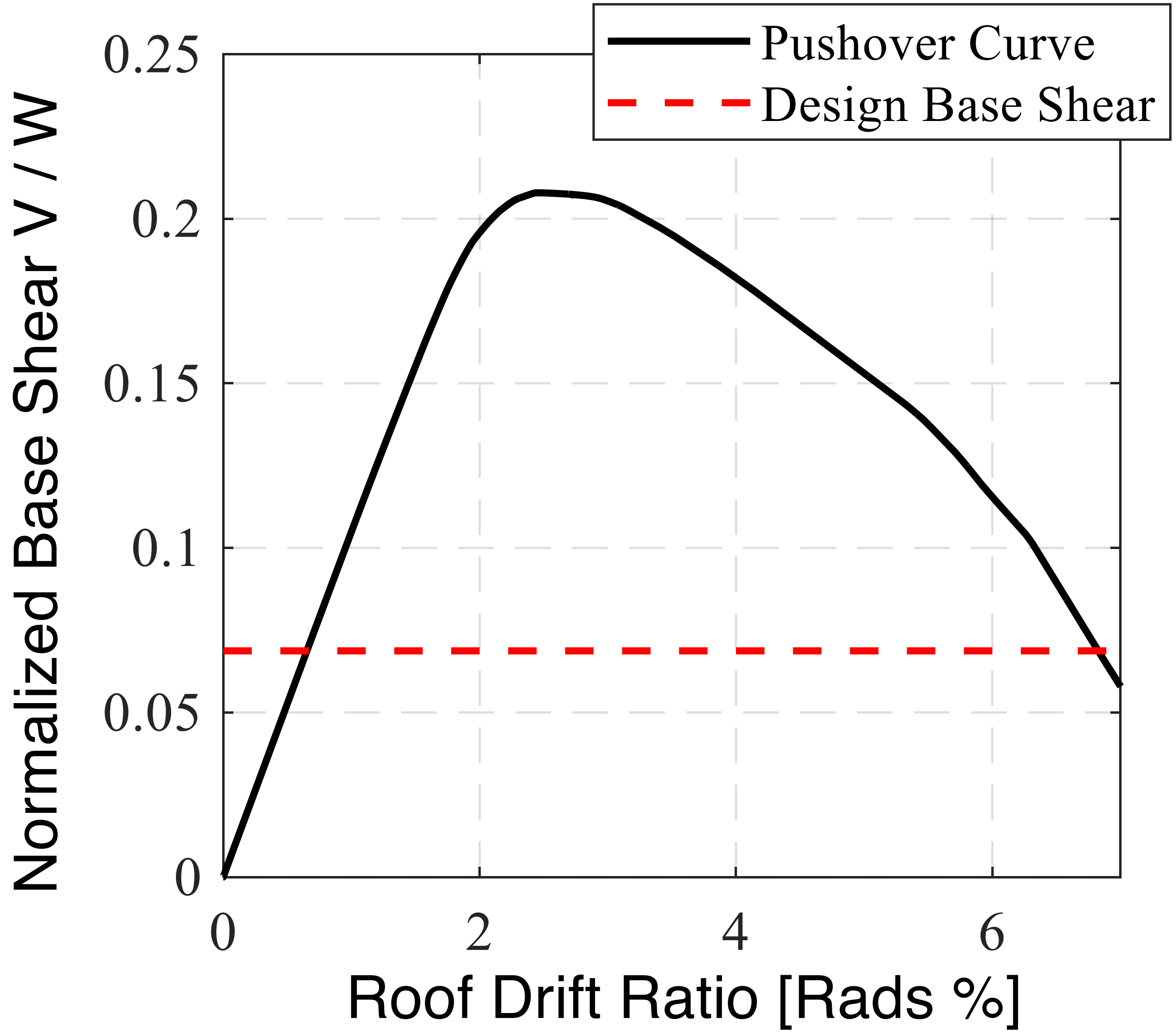


Image source: El Jisr and Lignos (2020)

EPFL Linear and Nonlinear Buckling Theory

- With linear buckling theory we can only identify the main equilibrium path and the critical buckling load.

- With nonlinear buckling theory we can identify the secondary equilibrium path.

- The main equilibrium path as well as the critical buckling load computation may not be reliable estimates depending on the magnitude of the deformations of the structure/component prior to buckling.

The Euler Method – General Principles

✧ Consider the deformed configuration of a static system.

✧ Express the equations of equilibrium and deformation compatibility at the deformed configuration of the static system.

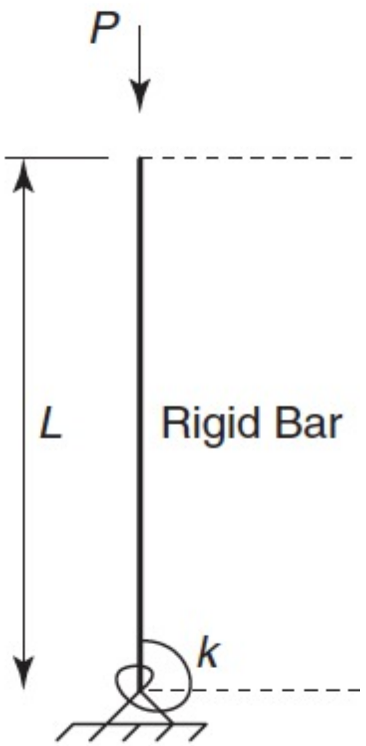
✧ Mathematically process the above equations such that we can identify if stability can be guaranteed.

✧ Assess the importance of imperfections on system performance.

Some Relevant Examples

Source: https://www.yourobserver.com/article/keep-siesta-key-bridge-traffic-moving

EPFL Moving Bridge and its Mathematical Model



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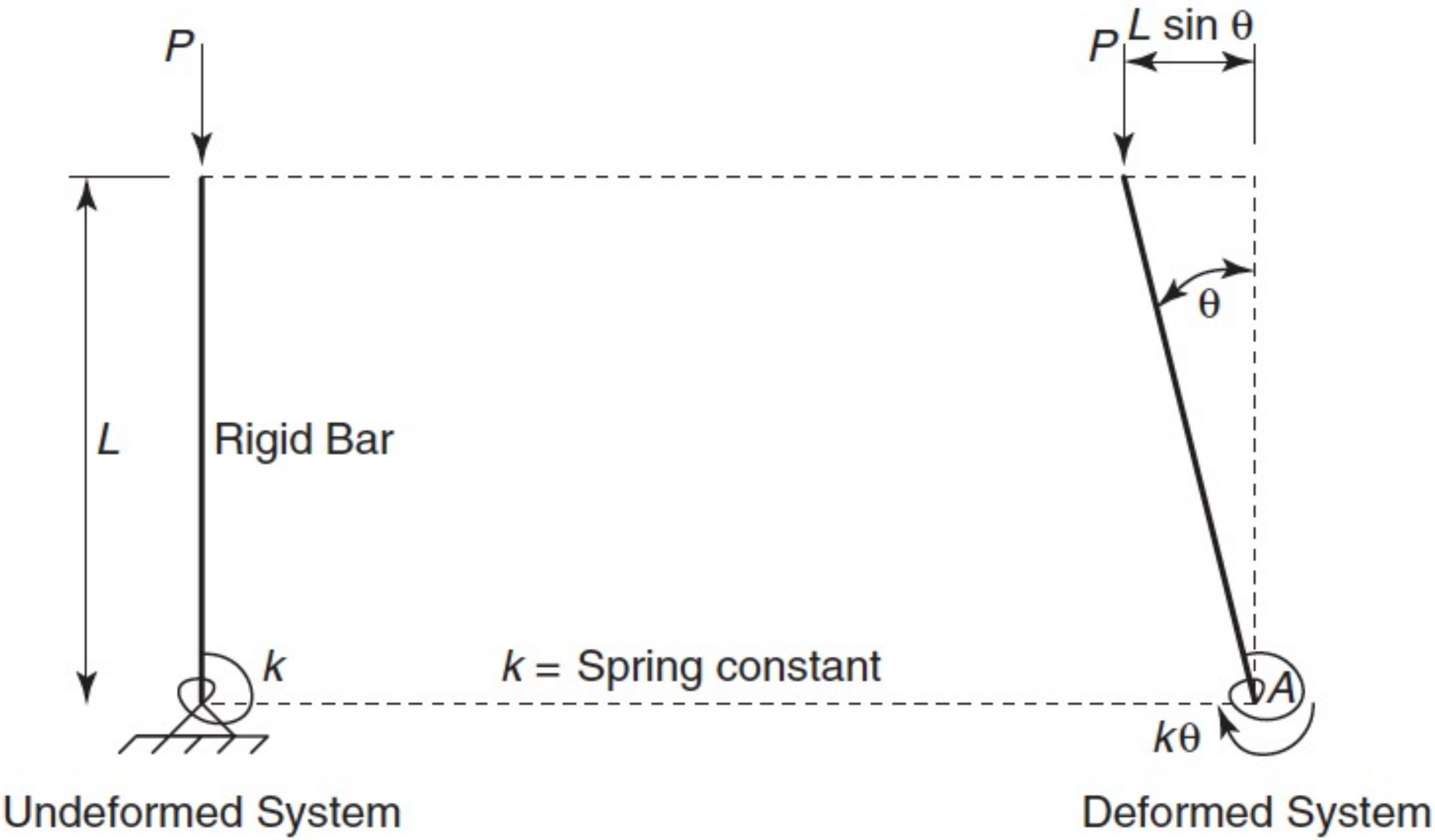
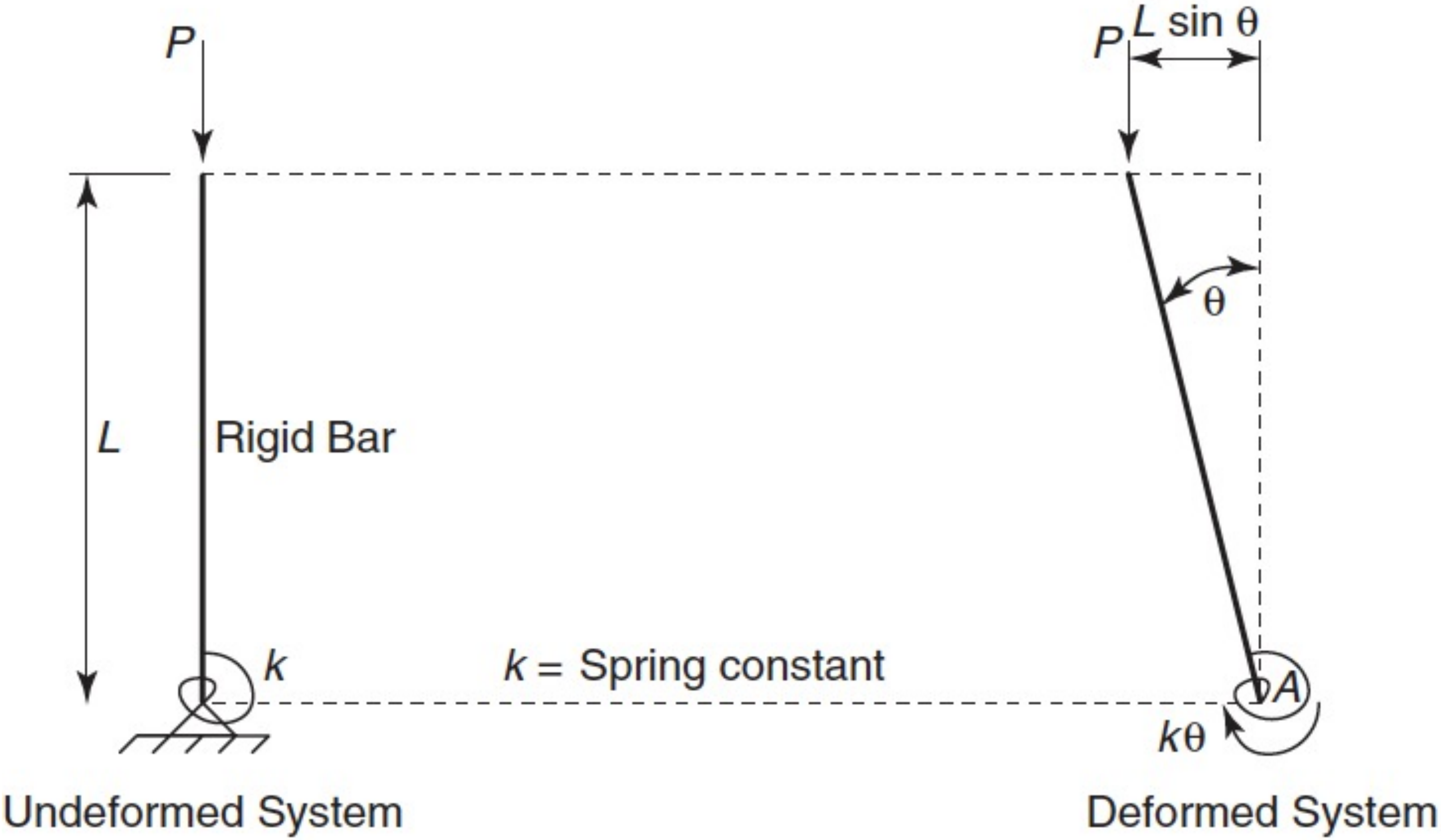
Moving Bridge and its Mathematical Model



$$\Sigma M_A = 0 = PL sin\theta - k\theta$$

Image Source: Galambos and Surovek 2008

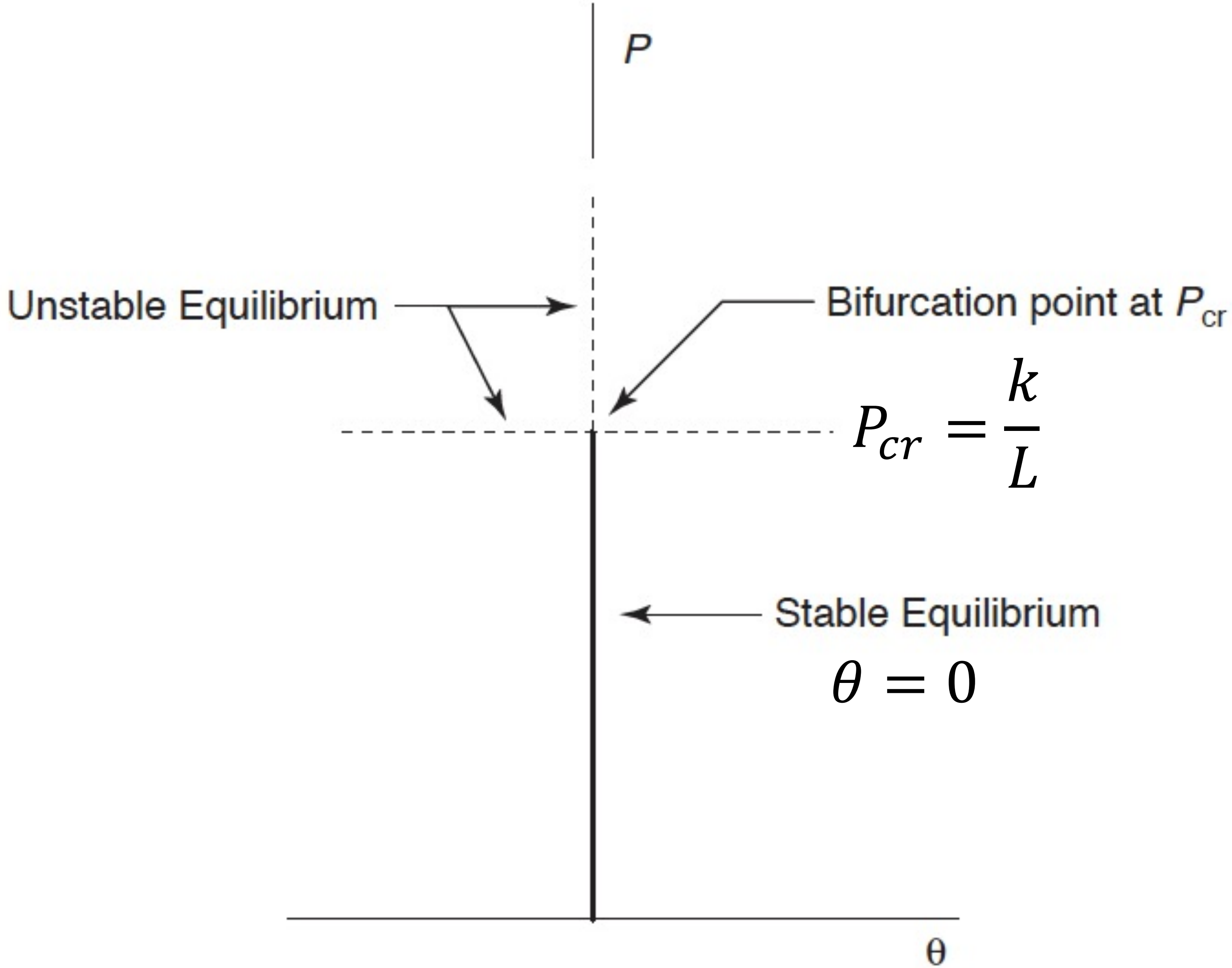
Example 1: - Linear Buckling Theory



$$PL\sin\theta - k\theta \cong PL\theta - k\theta = 0 \quad \begin{array}{l} \Longrightarrow \theta = 0 \\ \Longrightarrow P = P_{cr} = \dfrac{k}{L} \end{array}$$

Image Source: Galambos and Surovek 2008

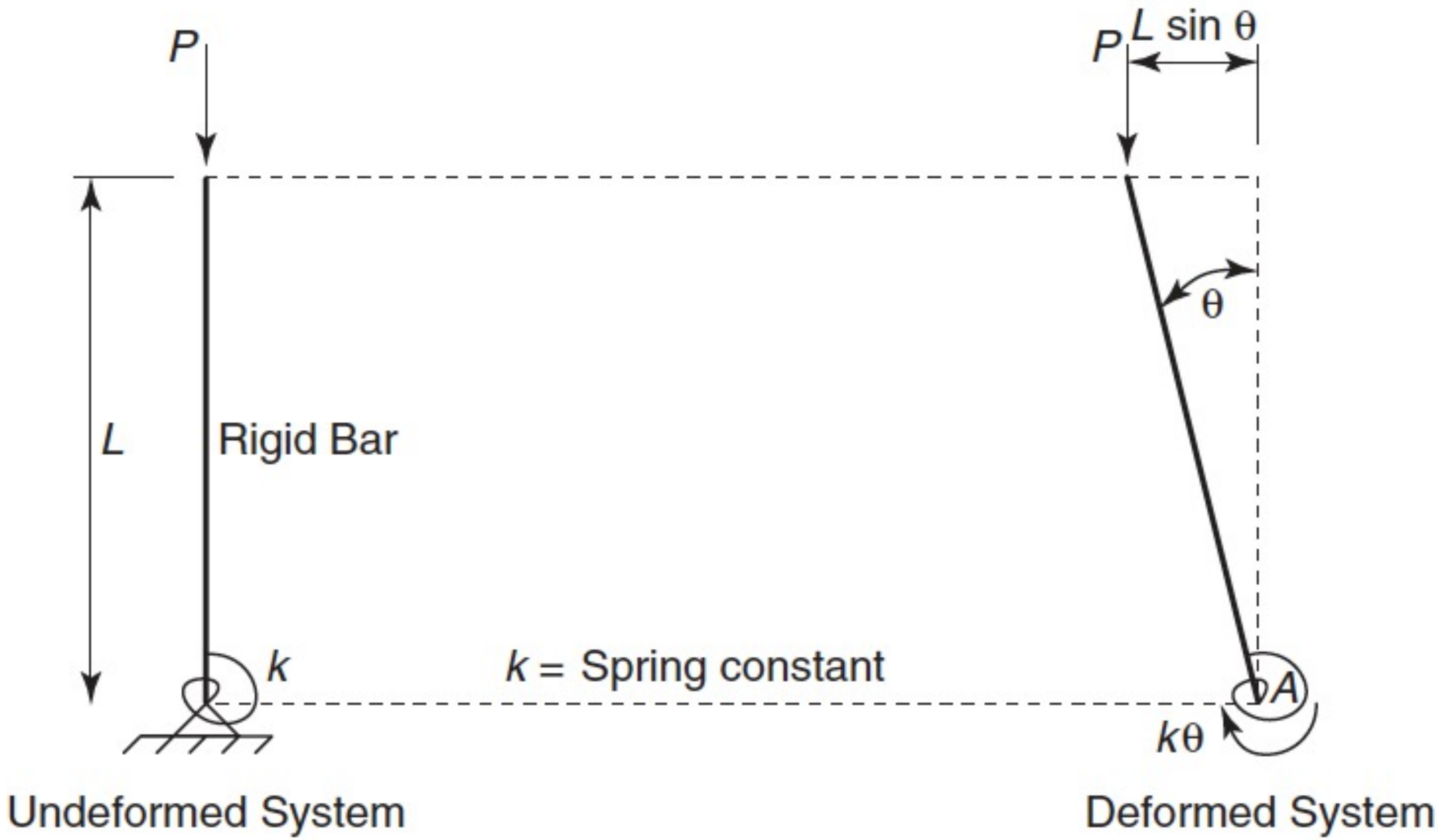
Example 1: - Linear Buckling Theory



$$P_{cr} = \frac{k}{L}$$

$$\theta = 0$$

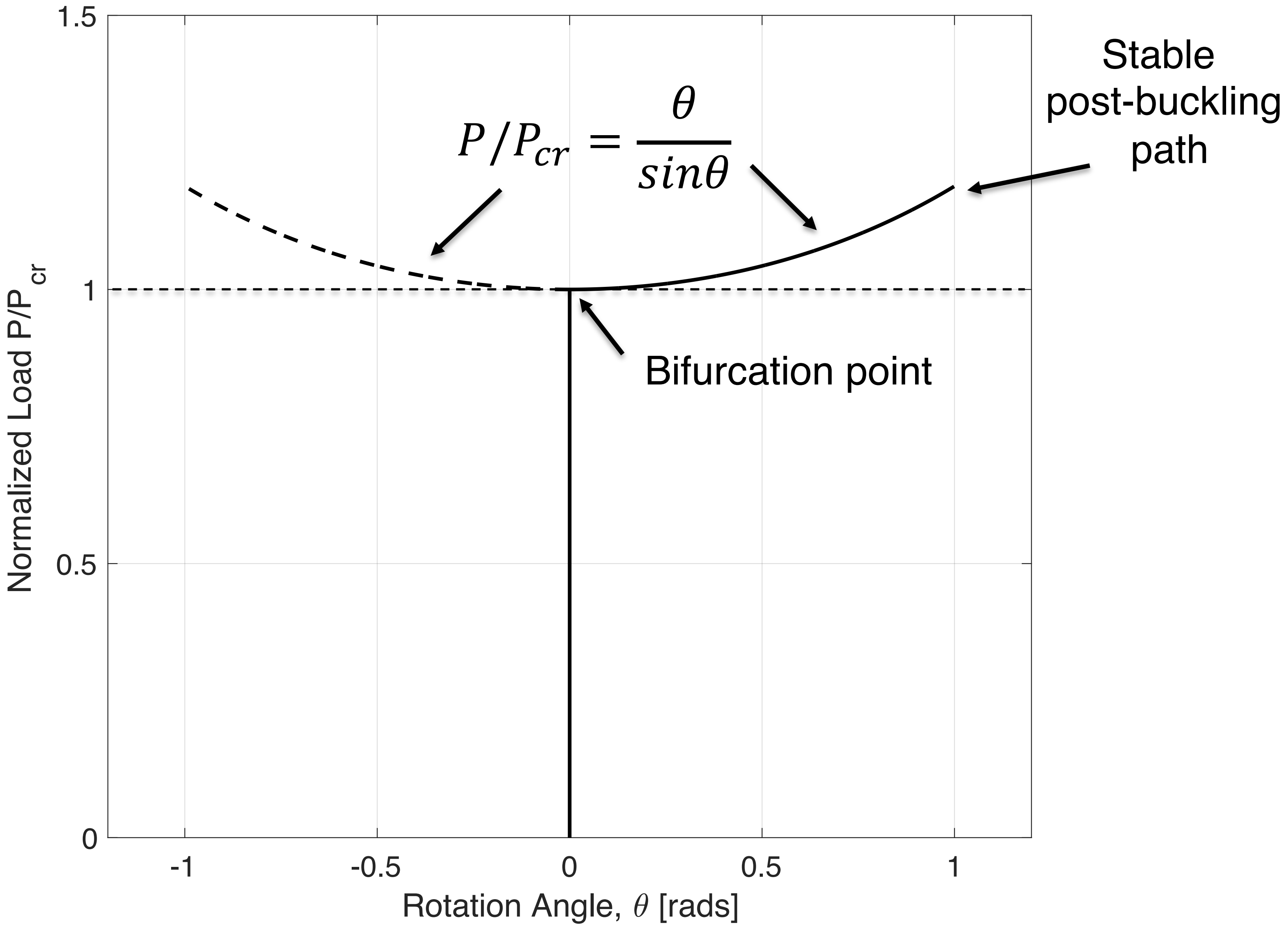
Example 1: - Nonlinear Buckling Theory



$$PLsin\theta - k\theta \qquad \Longrightarrow P = \frac{k}{L}\frac{\theta}{sin\theta}$$

Image Source: Galambos and Surovek 2008

Example 1: - Nonlinear Theory



Another Example Very Relevant to Bridge Design

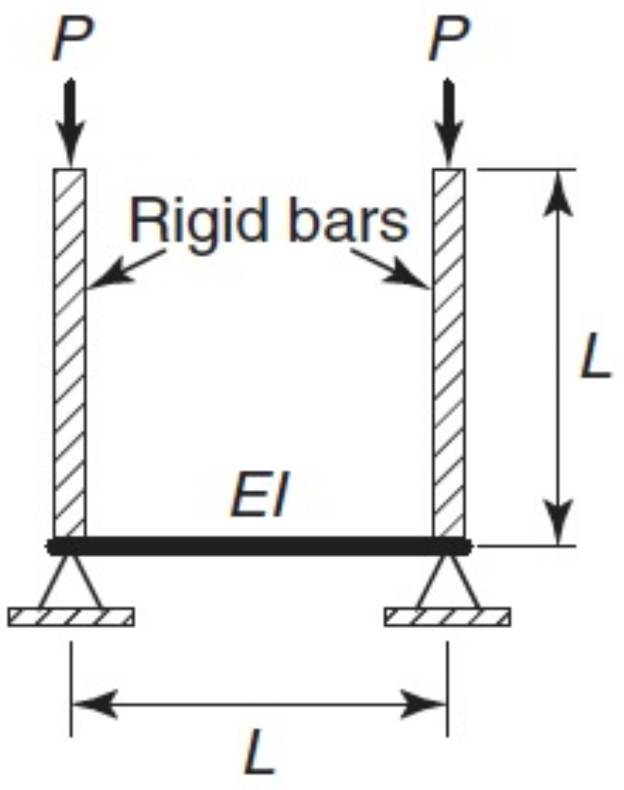
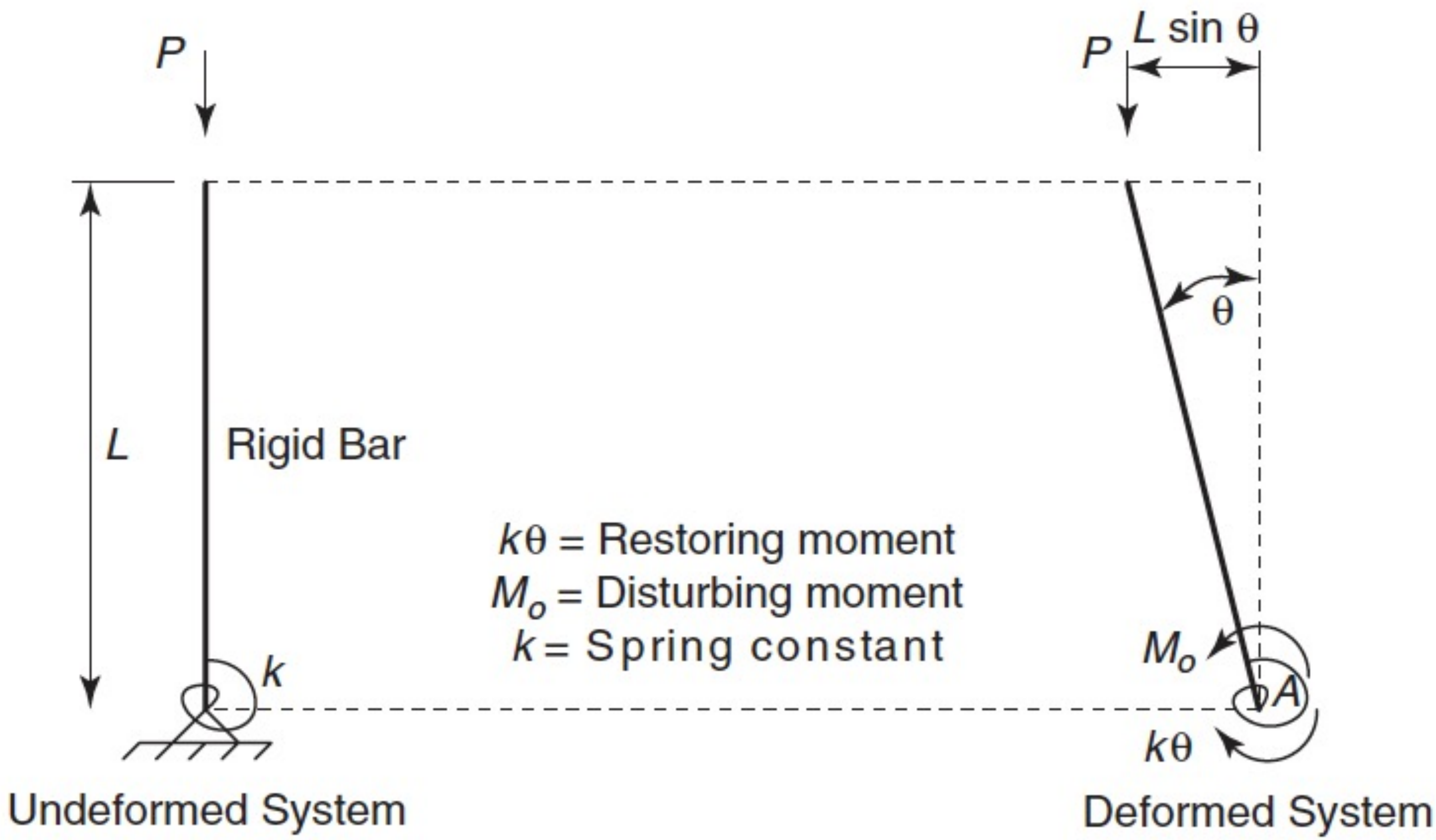


Image Source: Galambos and Surovek 2008

Example 1: System with Imperfections



$$\sum M_A = 0 = PLsin\theta + M_o - k\theta = PLsin\theta + k\theta_o - k\theta$$

Image Source: Galambos and Surovek 2008

EPFL Example 1: System with Imperfections

$$PLsin\theta + k\theta_o - k\theta = 0$$

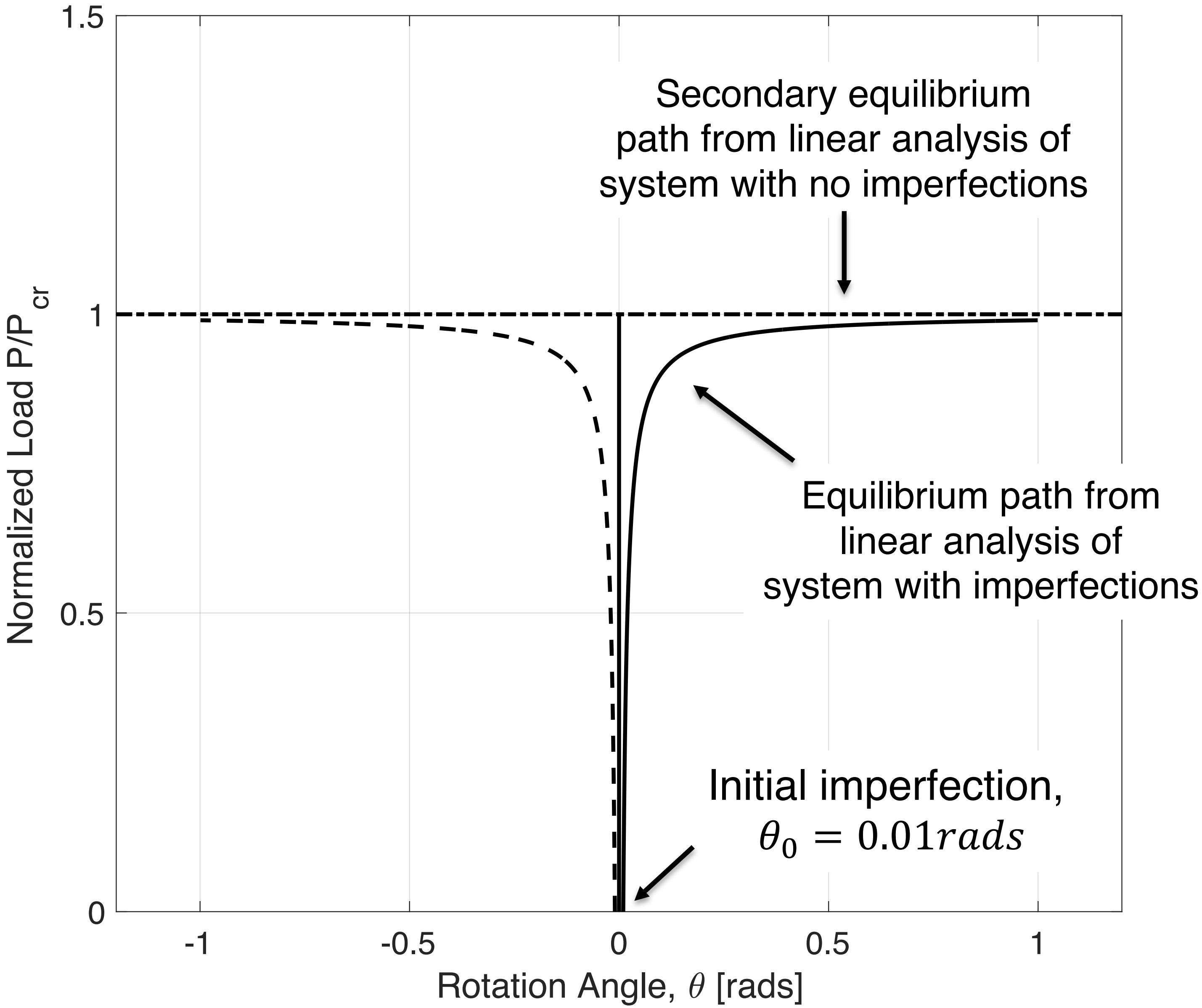
$$\frac{PL}{k} = \frac{\theta - \theta_o}{sin\theta}$$

Linear Buckling Theory

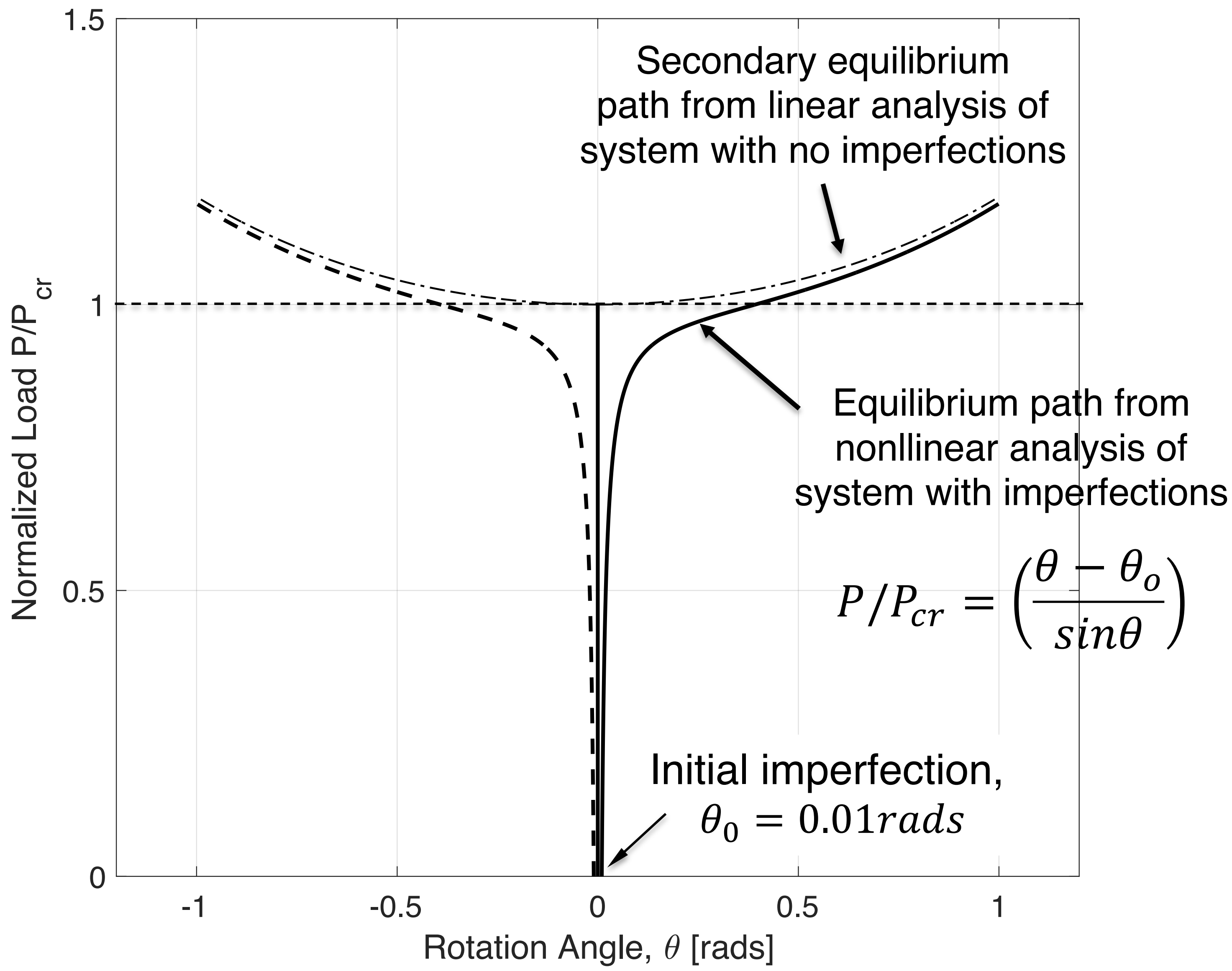
$$\frac{PL}{k} = \frac{\theta - \theta_o}{sin\theta} \cong \frac{\theta - \theta_o}{\theta} = 1 - \frac{\theta_o}{\theta}$$

$$P = \frac{k}{L}\left(1 - \frac{\theta_o}{\theta}\right)$$

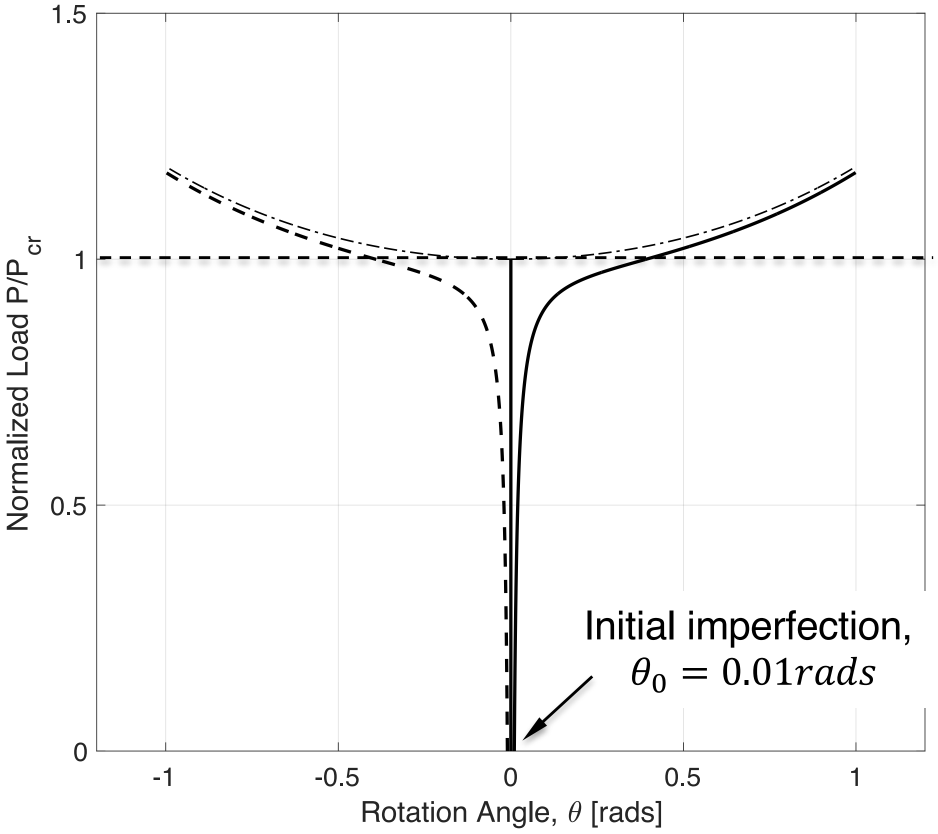
Example 1: System with Imperfections – Linear Theory

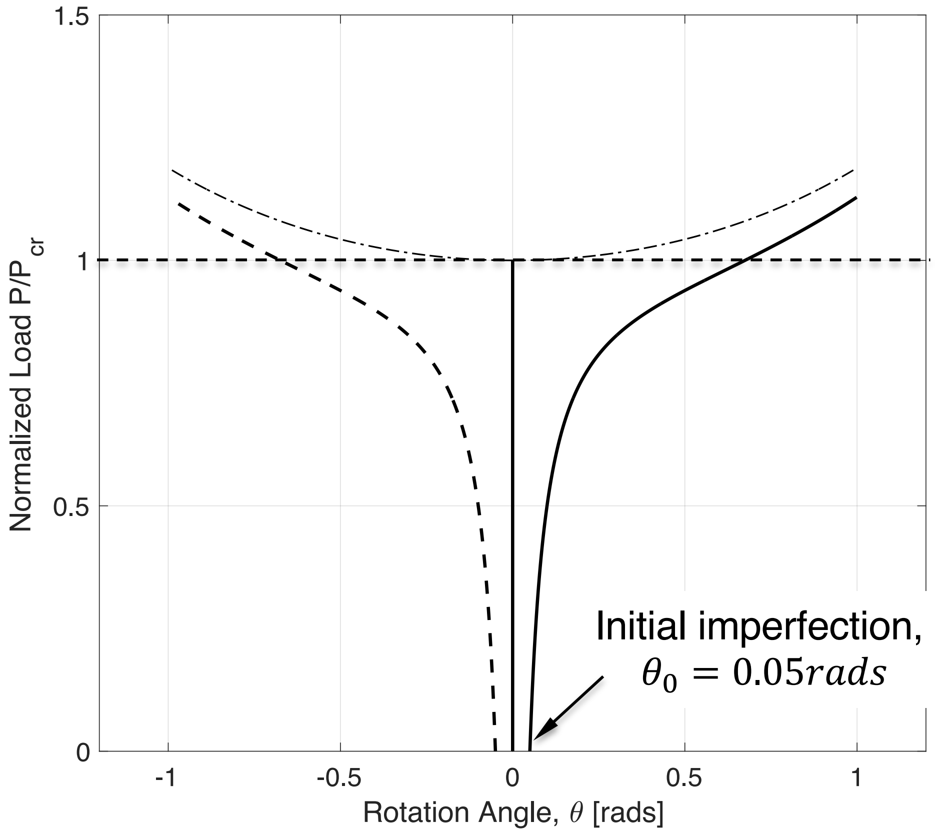


Example 1: System with Imperfections – Nonlinear Theory



Example 1: System with Imperfections – Nonlinear Theory
-Influence of Initial Imperfections





Another Relevant Example

Compute the required brace area needed to provide minimum stiffness so that "N" number of columns can support their Euler load

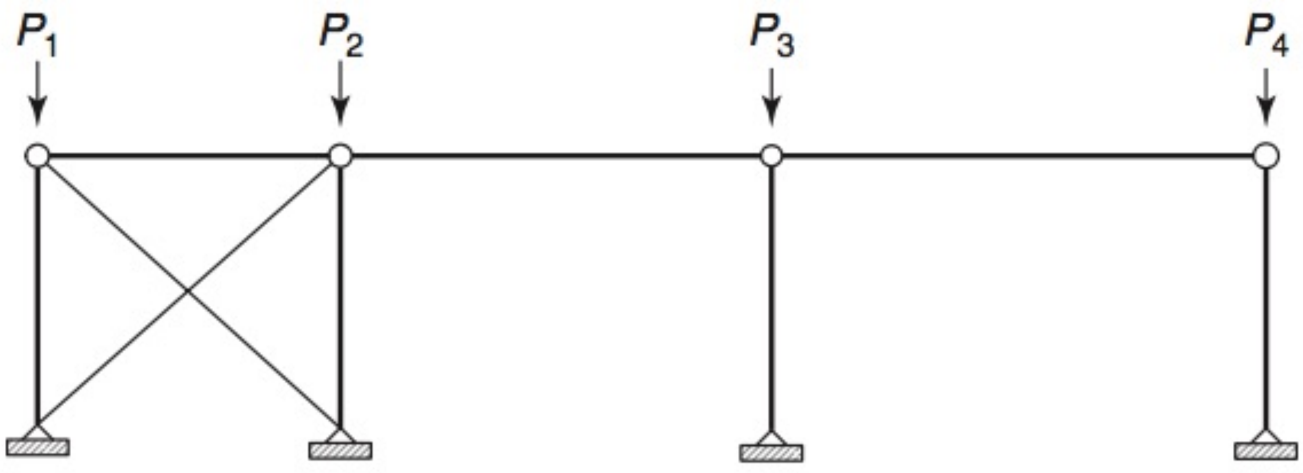
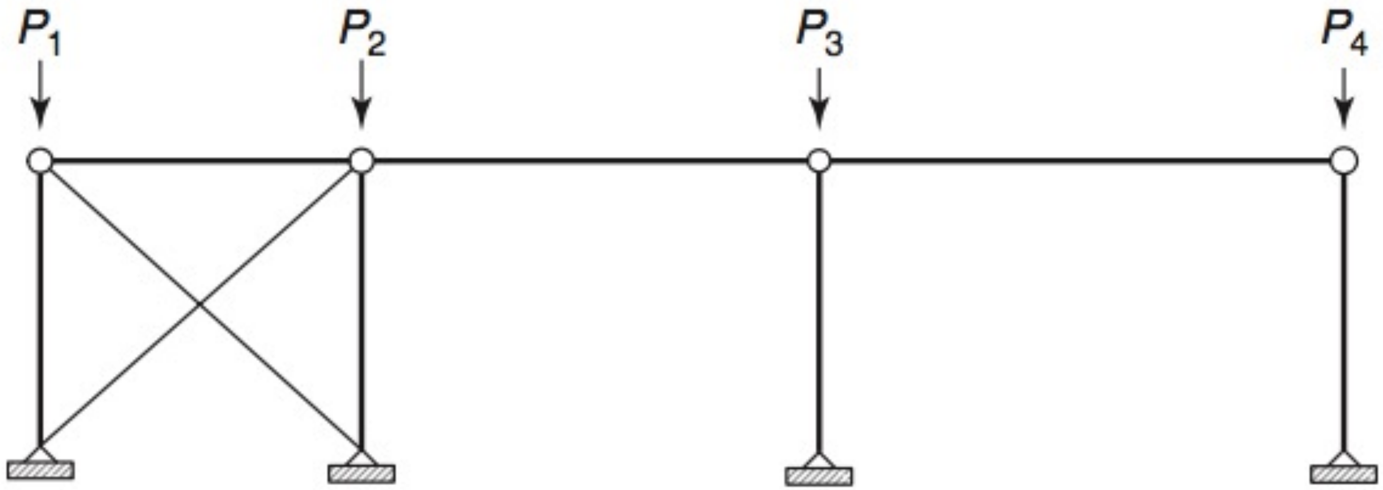


Image Source: Galambos and Surovek 2008

Steel Braced Frame and Mathematical Model



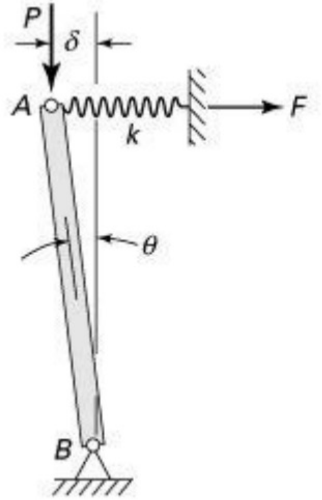
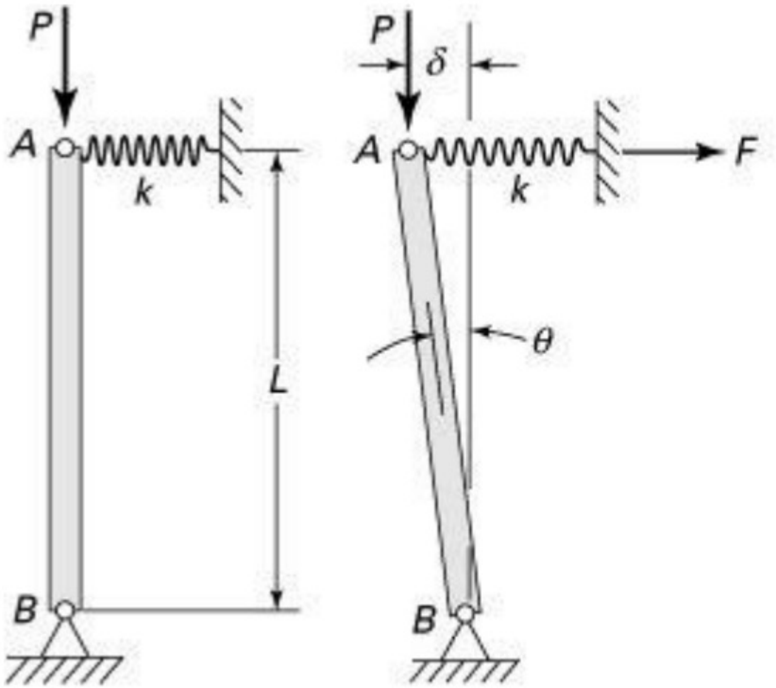


Image Source: Galambos and Surovek 2008

Example 2: Rigid Link with Elastic Support



$$PLsin\theta = k(Lsin\theta)(Lcos\theta)$$

$$P = kLcos\theta$$

Linear buckling theory

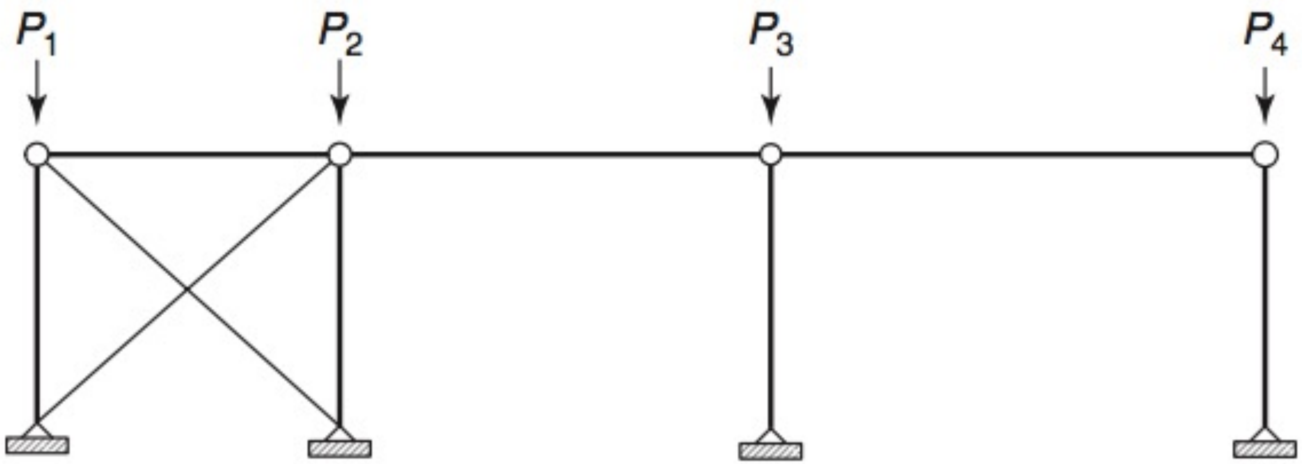
$$P = P_{cr} = kL$$

Nonlinear buckling theory

$$P = kLcos\theta$$

Stability Bracing for Non-sway Frame with Bracings

Compute the required brace area needed to provide minimum stiffness so that *N* columns can support their Euler load



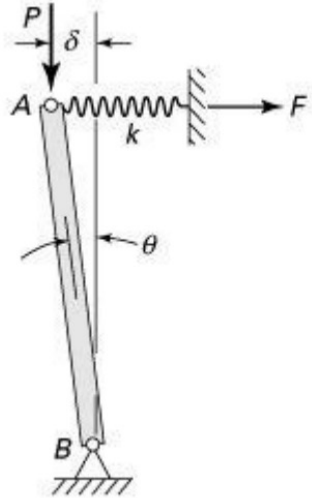


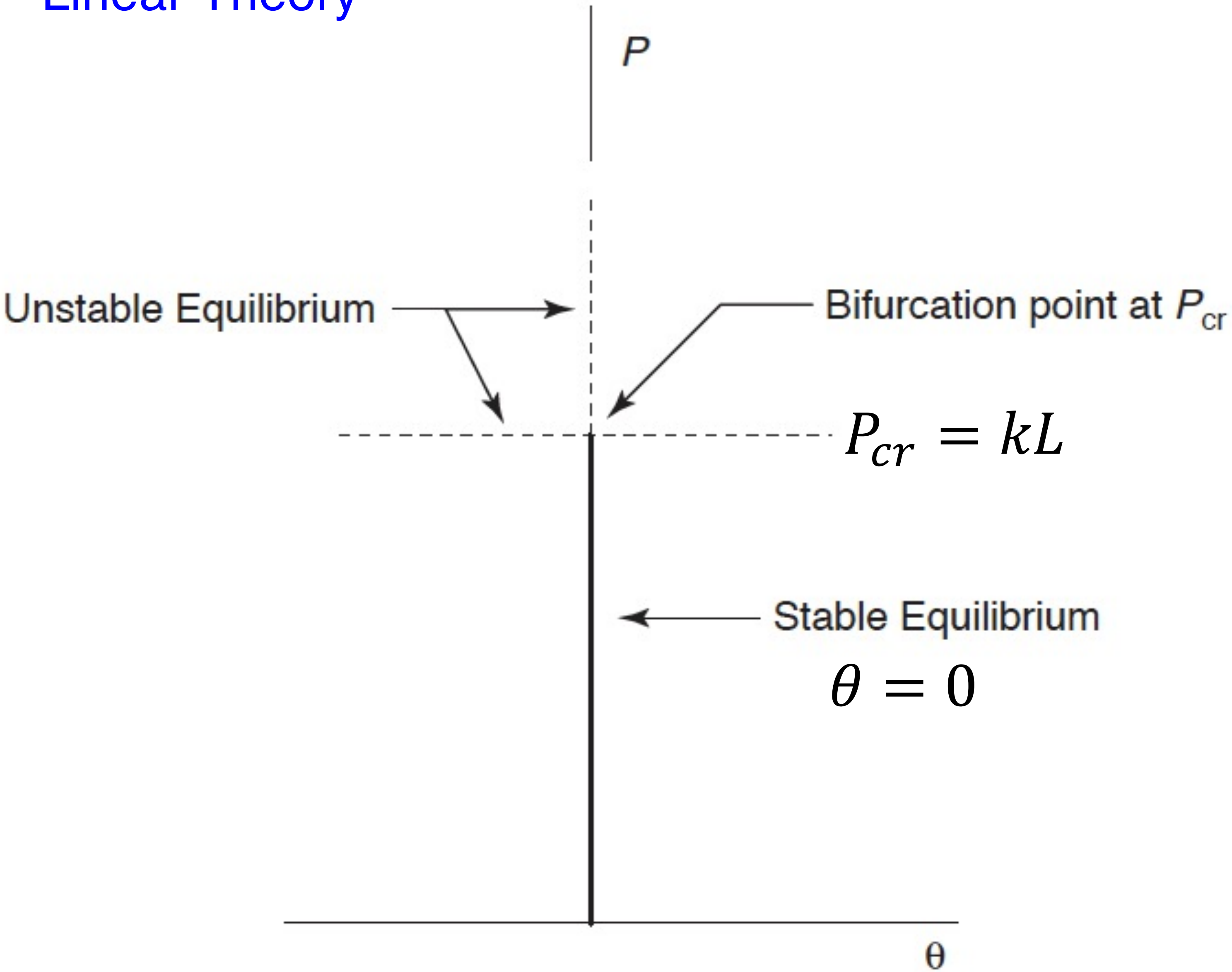
Image Source: Galambos and Surovek 2008

$$P_{total} = \sum_{i=1}^{N} P_E^i = \sum_{i=1}^{N} \frac{\pi^2 E I_i}{L_C^2}$$

$$P_{cr} = kL_c = \frac{E A_{BR} L_B^2}{(L_B^2 + L_C^2)^{3/2}} \cdot L_c$$

$$P_{cr} \geq P_{total} \rightarrow A_{BR} \geq \sum_{i=1}^{N} \frac{\pi^2 E I_i}{L_C^2} \cdot \frac{(L_B^2 + L_C^2)^{\frac{3}{2}}}{E L_B^2 L_C}$$

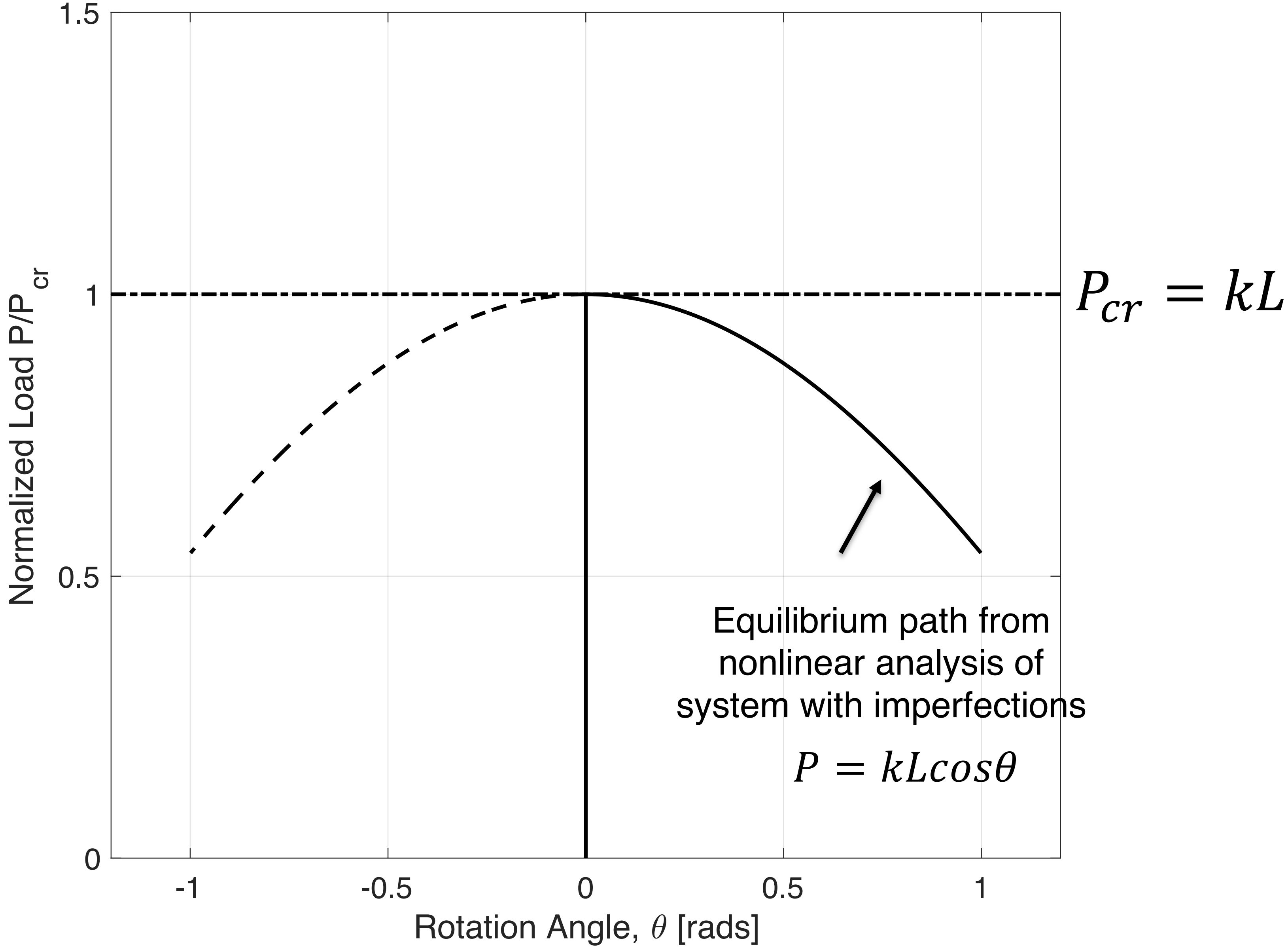
Example 2: Rigid Link with Elastic Support – Linear Theory



$$P_{cr} = kL$$

$$\theta = 0$$

Example 2: Rigid Link with Elastic Support
–Nonlinear Theory



Example 2: Rigid Link with Elastic Support – Imperfections

$$PLsin\theta = k(Lsin\theta - Lsin\theta_0)(Lcos\theta)$$

$$Psin\theta = kL(sin\theta - sin\theta_0)cos\theta$$

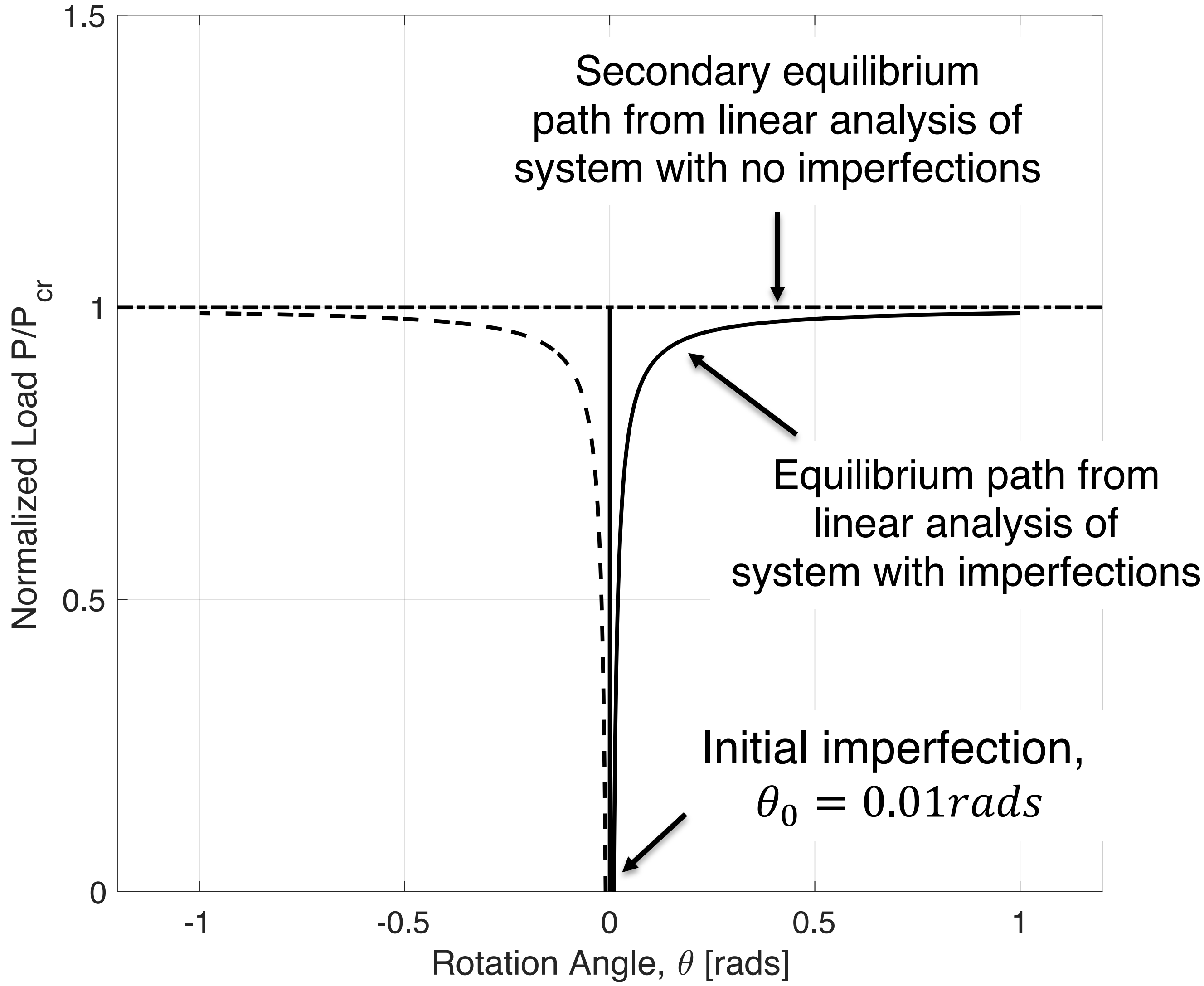
Linear buckling theory

$$P\theta \cong kL(\theta - \theta_0)$$

Nonlinear buckling theory

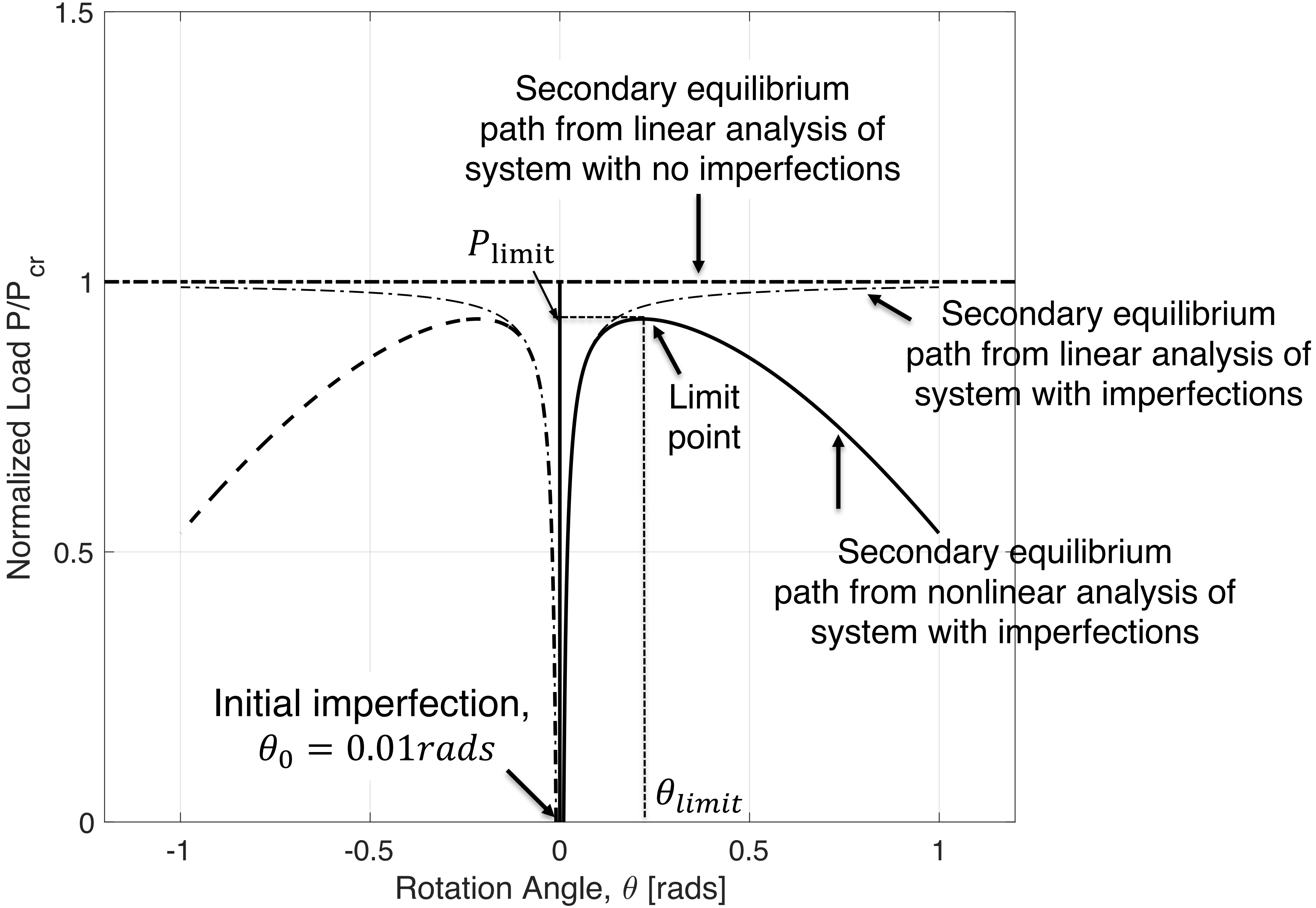
$$P = kL\left(1 - \frac{sin\theta_0}{sin\theta}\right)cos\theta$$

Example 2: Rigid Link with Elastic Support – Imperfections, Linear Theory

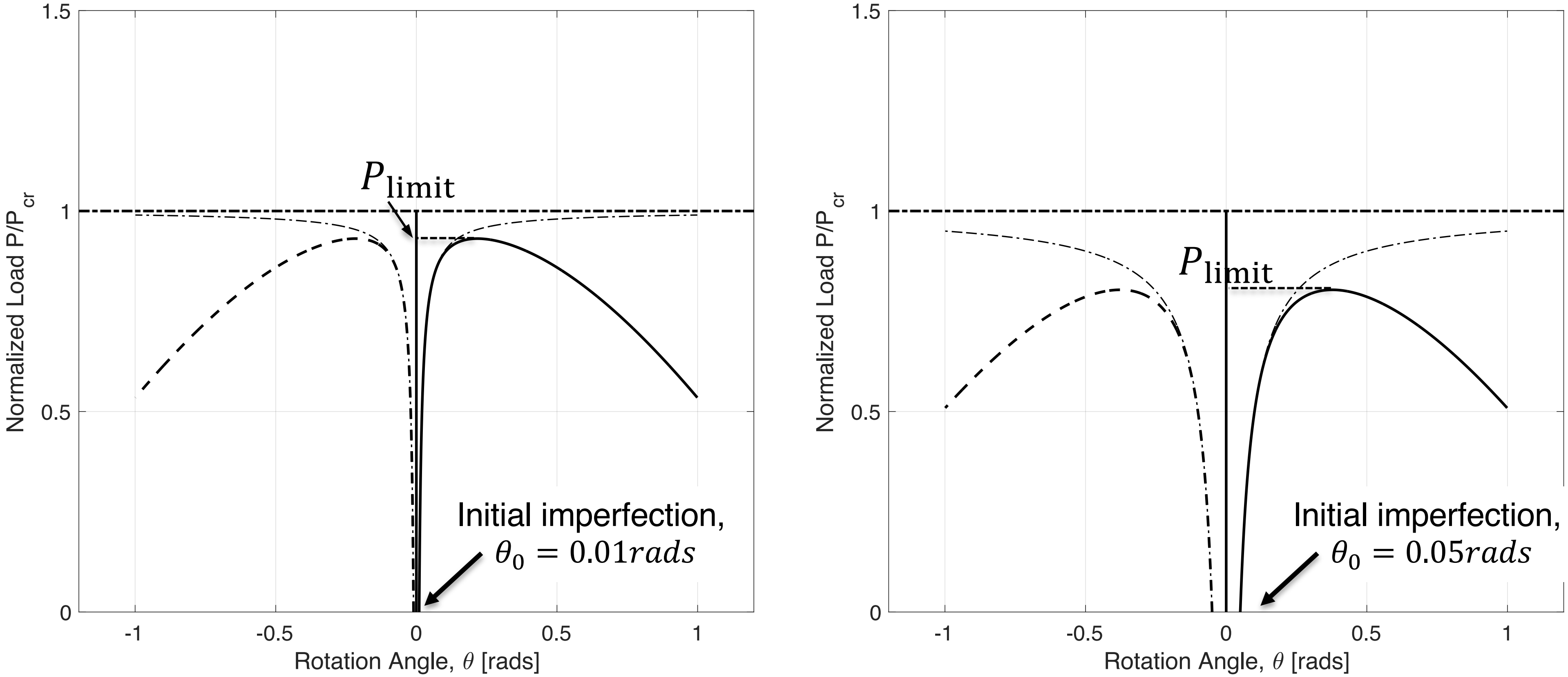


Example 2: Rigid Link with Elastic Support

– Imperfections, Nonlinear Theory



Example 2: Rigid Link with Elastic Support
– Imperfections, Nonlinear Theory



EPFL

Example 2: Rigid Link with Elastic Support
– Effect of Imperfections on Limit Point

$$\frac{\vartheta P}{\vartheta \theta} = kL\left[-sin\theta - sin\theta_o\left(-\frac{1}{\sin^2\theta}\right)\right]$$

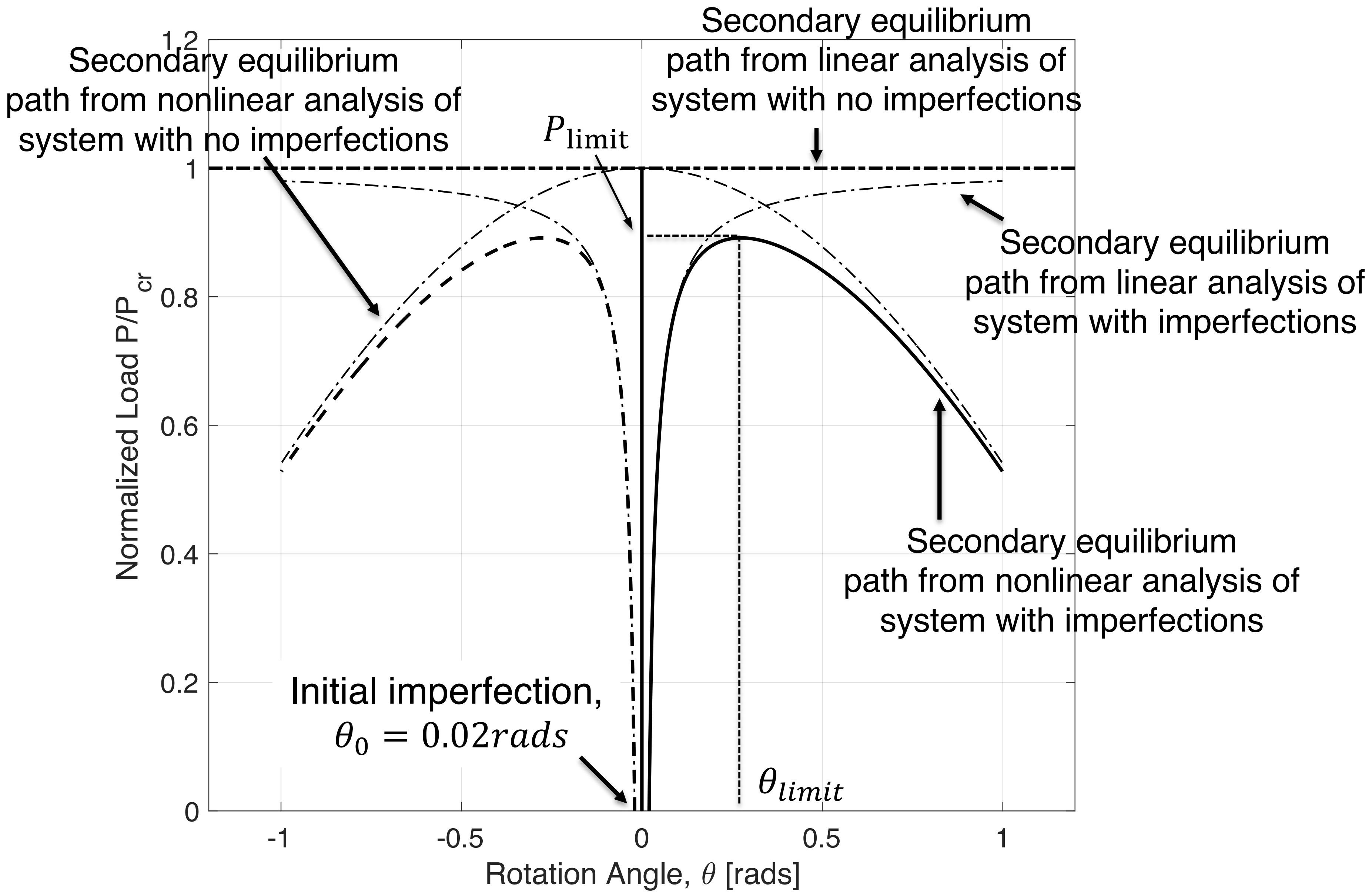
$$\Rightarrow \frac{sin\theta_o}{\sin^2\theta_{limit}} - sin\theta_{limit} = 0$$

$$\Rightarrow \sin^3\theta_{limit} = sin\theta_o \Rightarrow sin\theta_{limit} = (sin\theta_o)^{1/3}$$

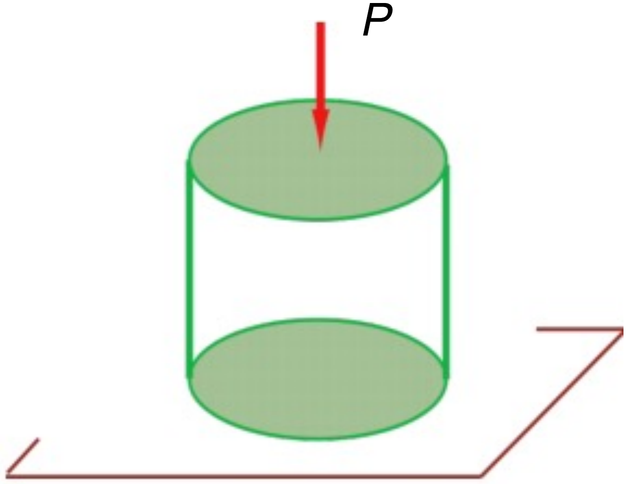
$$P_{limit} = P(\theta_{limit}) = kL\left[1 - \frac{sin\theta_o}{(sin\theta_o)^{\frac{1}{3}}}\right]\sqrt{1-(sin\theta_o)^{2/3}}$$

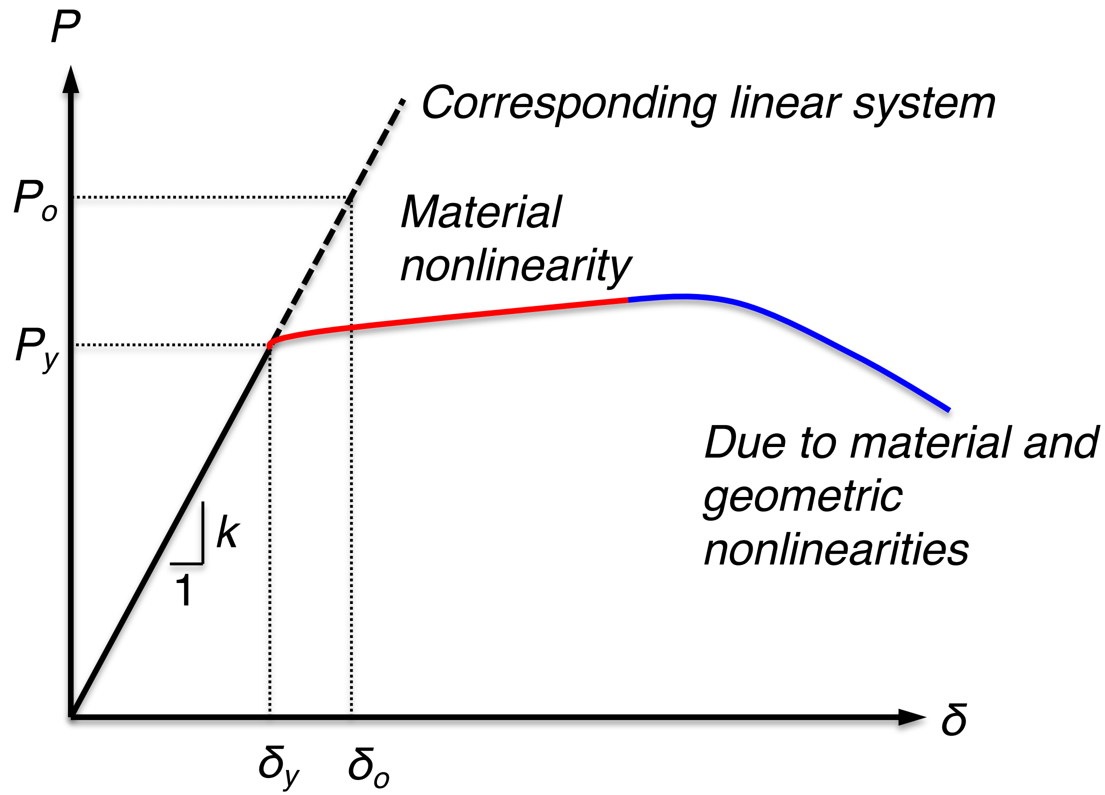
$$P_{limit} = kL\left[1-(sin\theta_o)^{\frac{2}{3}}\right]^{3/2}$$

Example 2: Rigid Link with Elastic Support
– Effect of Imperfections on Limit Point



Members with Small Slenderness

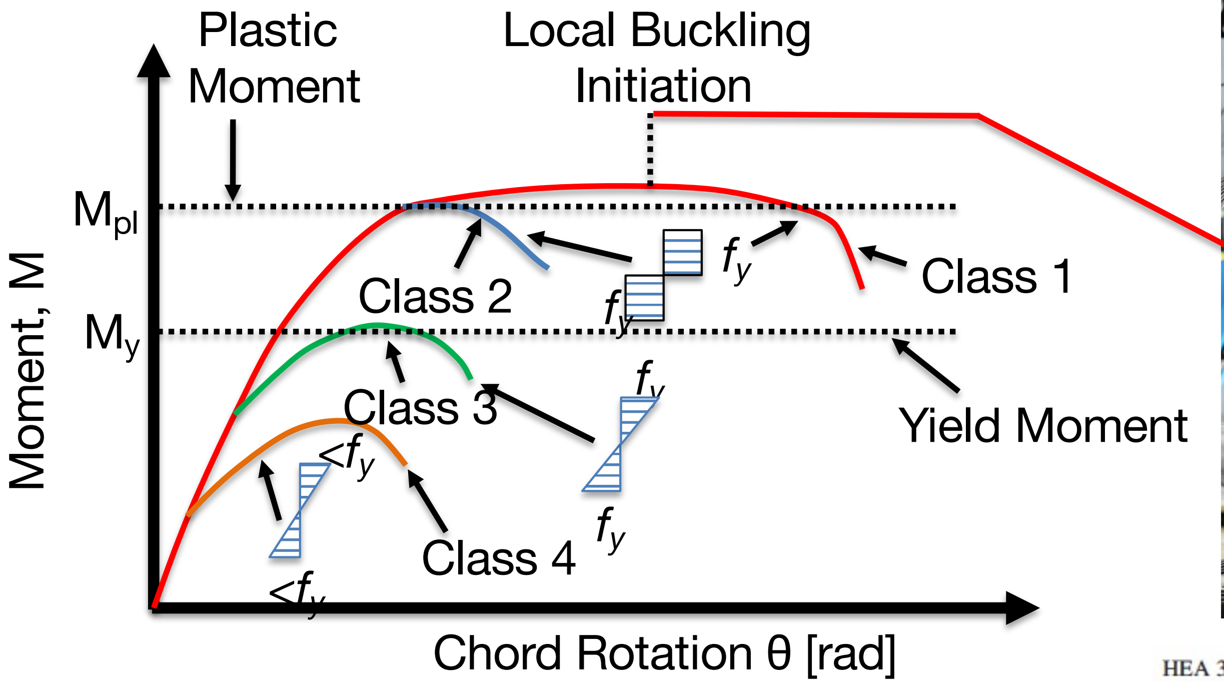


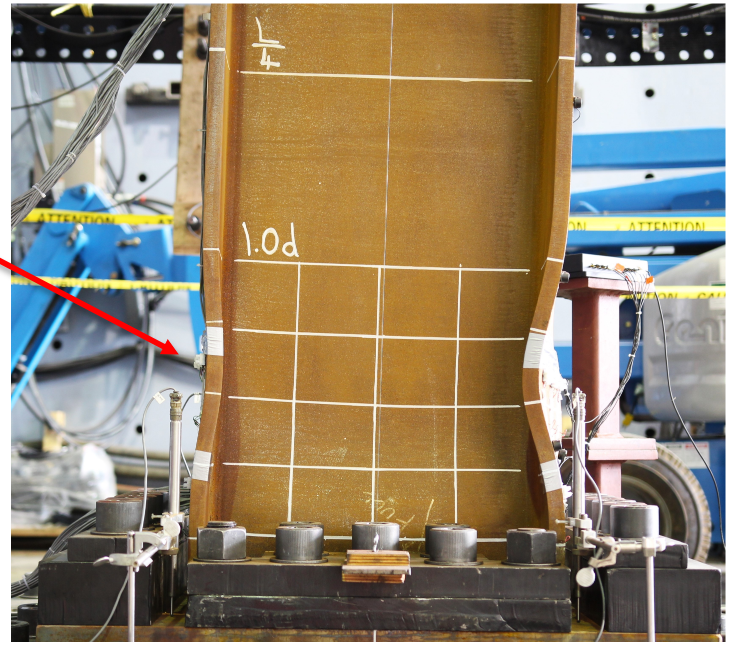


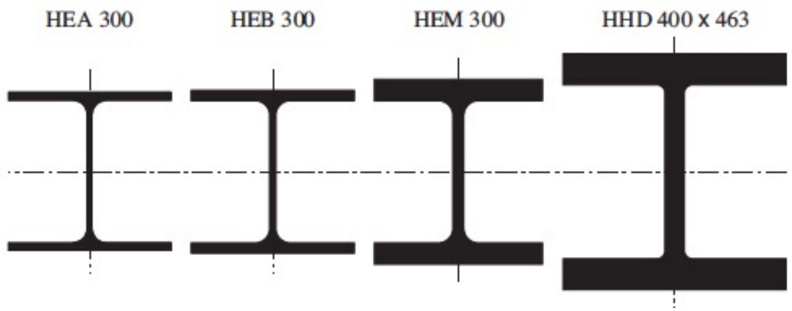
Plastic Design Against Local Buckling

-Cross-Section Classification (Controlling “Plate” Buckling)

We control the equilibrium path based on cross-section classification

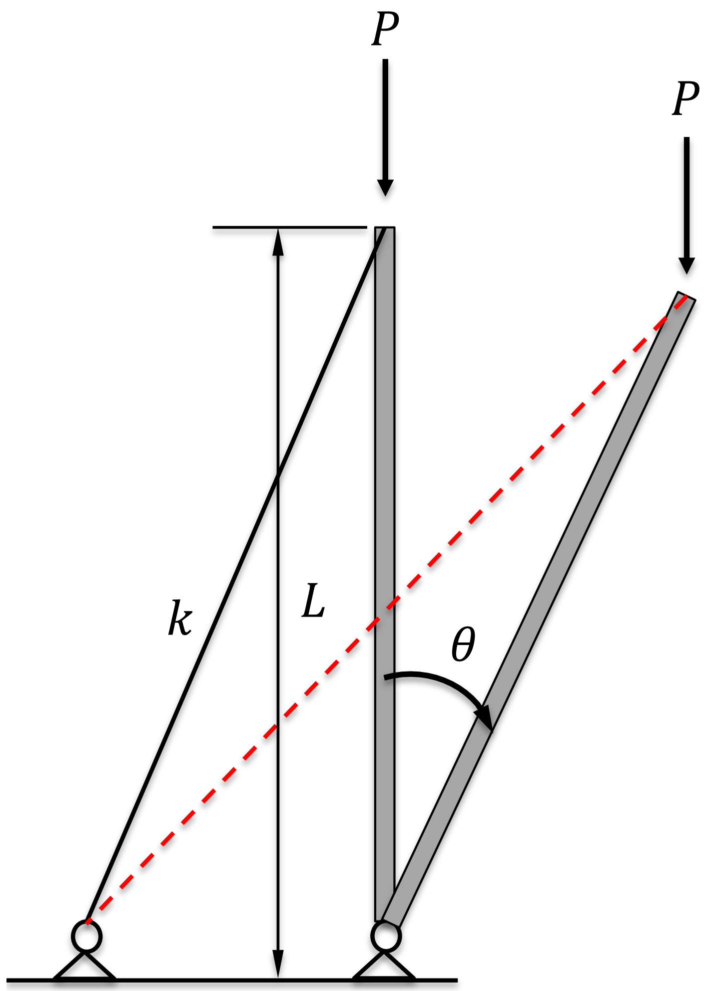






Example 3: Steel Column under Construction

Example 3: Rigid Link with Elastic Support in an Angle



Nonlinear buckling theory

$$PL\sin\theta = k\left[\sqrt{(L+L\sin\theta)^2+(L\cos\theta)^2}-L\sqrt{2}\right]\sqrt{L^2-\left(\frac{L}{2}\sqrt{2(1+\sin\theta)}\right)^2}$$

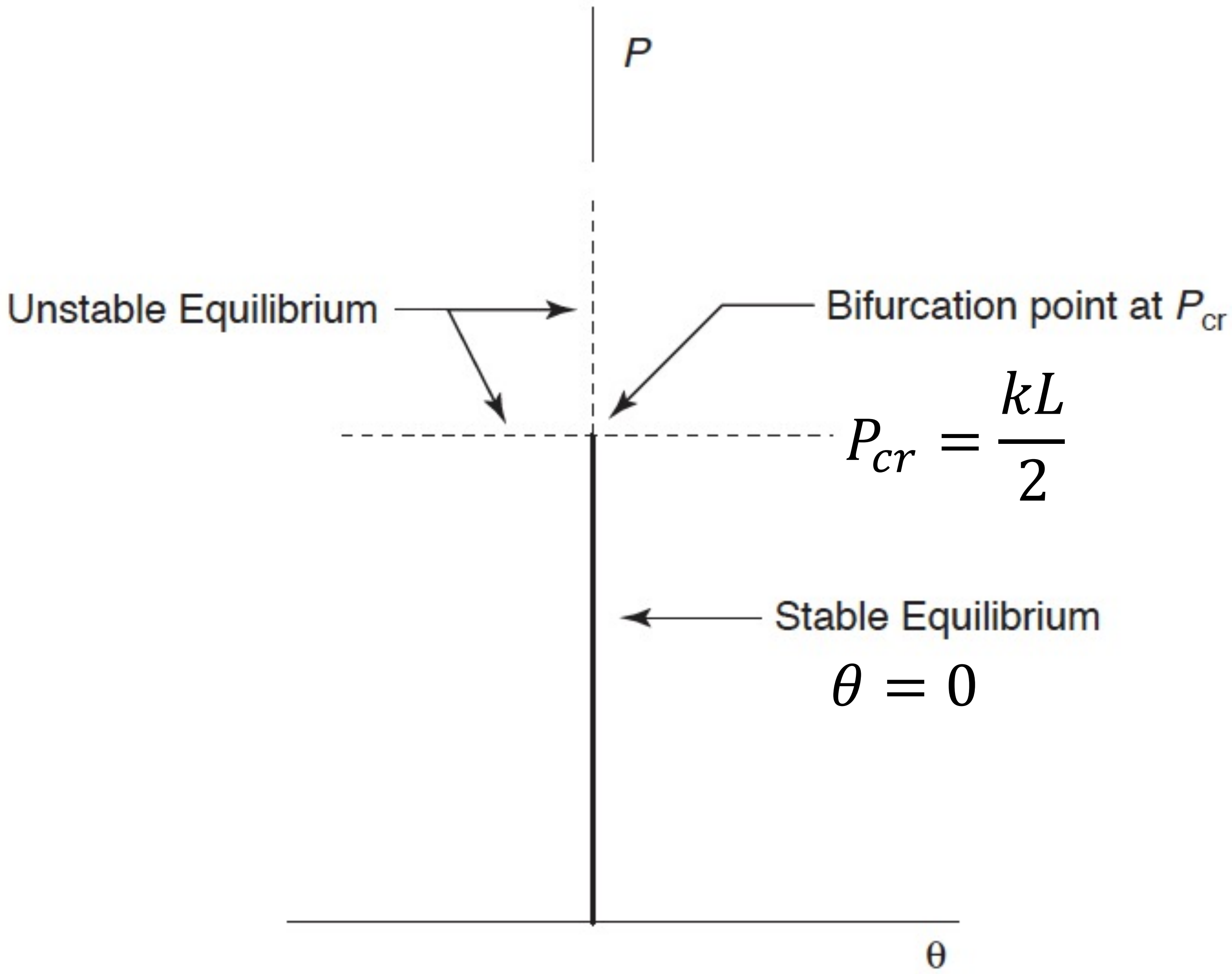
$$P = kL\frac{\left(\cos\theta-\sqrt{1-\sin\theta}\right)}{\sin\theta}$$

Linear buckling theory

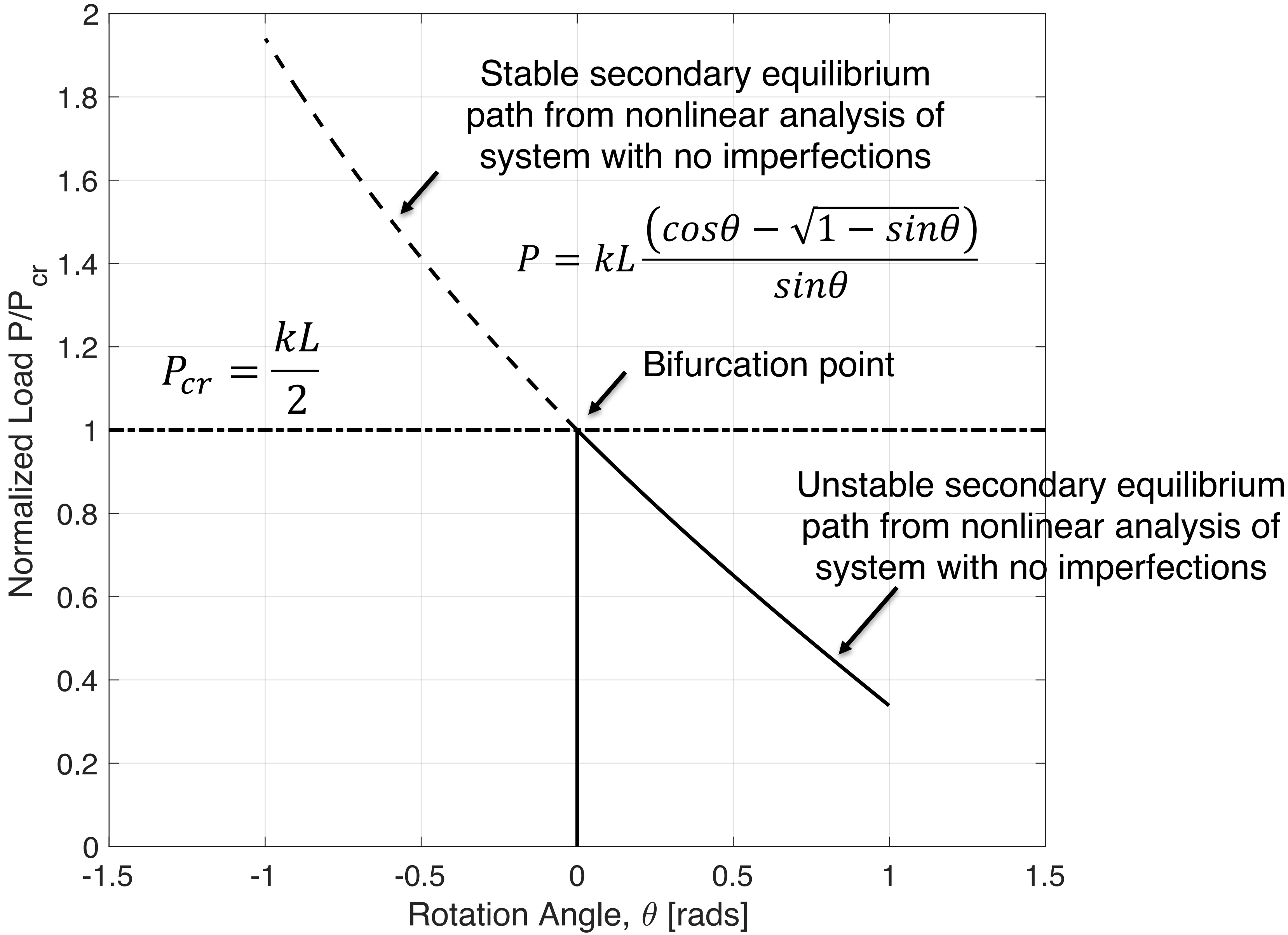
$$\left(\sqrt{1-\theta}\sim 1-\frac{\theta}{2}\right)$$

$$P = P_{cr} = \frac{kL}{2}$$

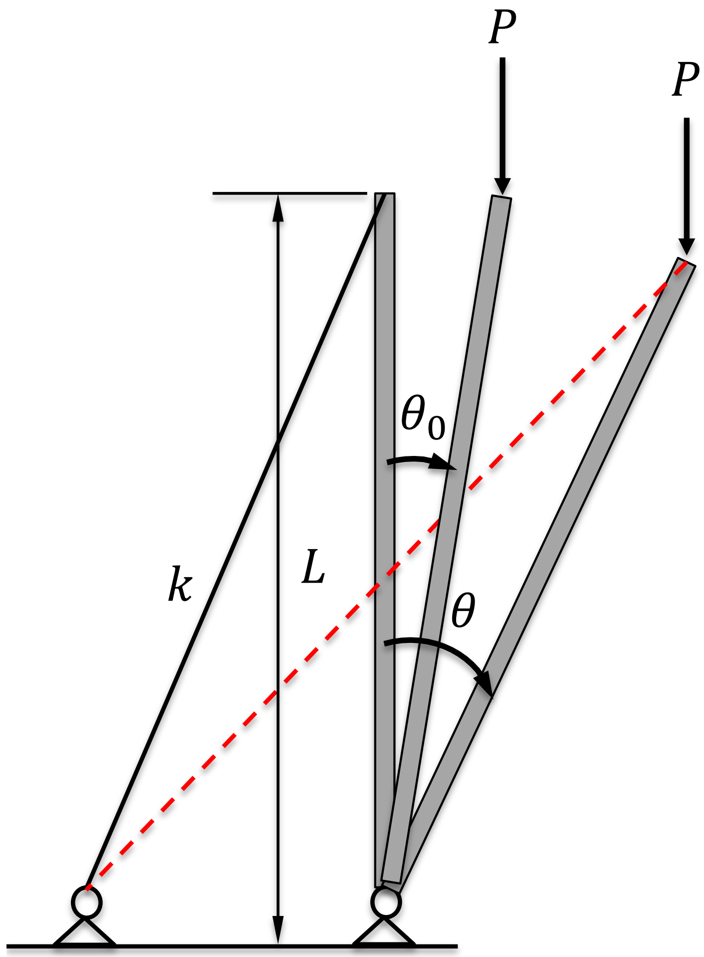
Example 3: Rigid Link with Elastic Support-Linear Theory



Example 3: Rigid Link with Elastic Support-Nonlinear Theory



EPFL Example 3: Rigid Link with Elastic Support-Imperfections



Nonlinear theory

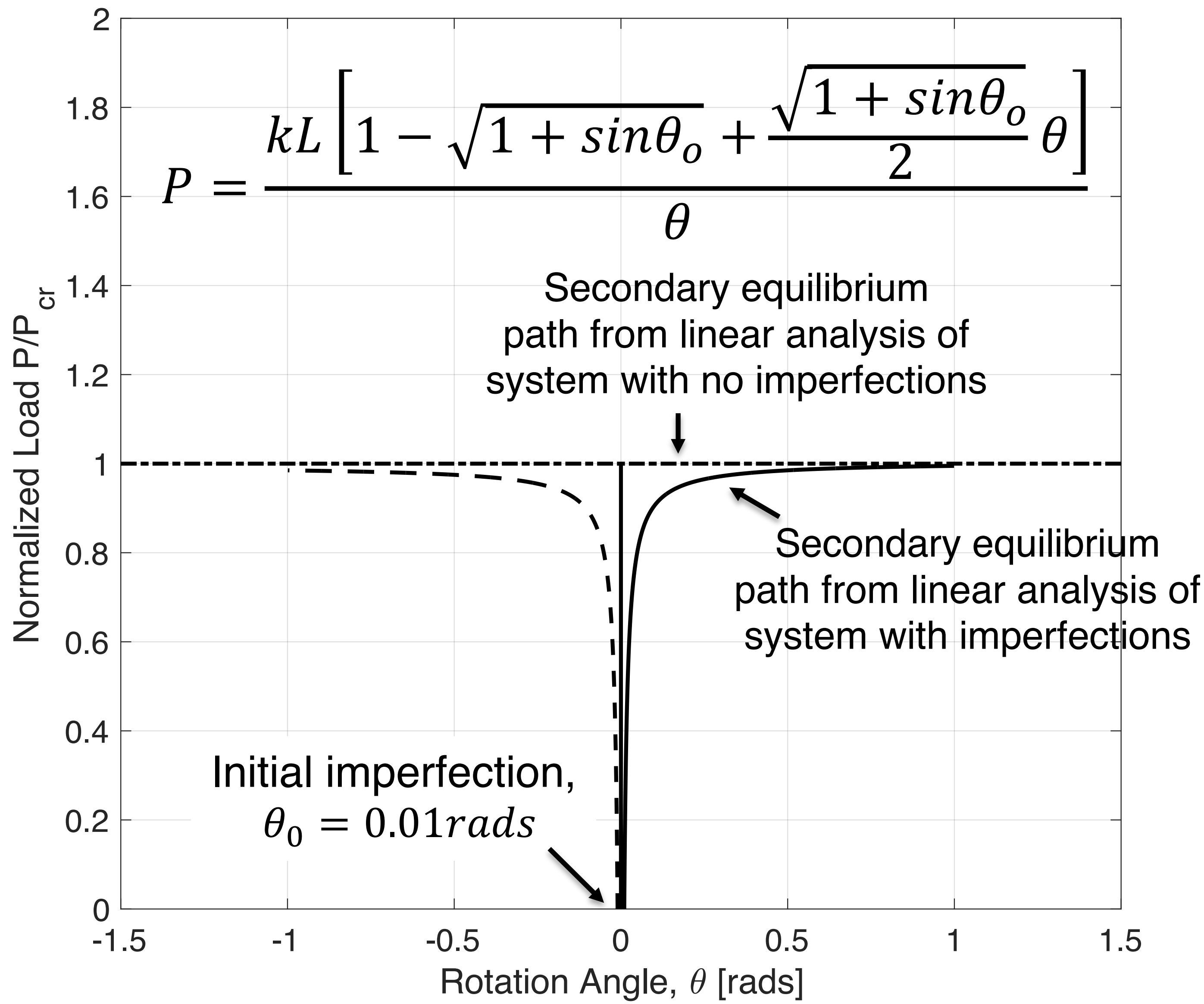
$$P = \frac{kL\left(\sqrt{1+sin\theta} - \sqrt{1+sin\theta_o}\right)\sqrt{1-sin\theta}}{sin\theta}$$

Linear theory (applies for small angles)

$$P = \frac{kL\left[1 - \sqrt{1+sin\theta_o} + \frac{\sqrt{1+sin\theta_o}}{2}\theta\right]}{\theta}$$

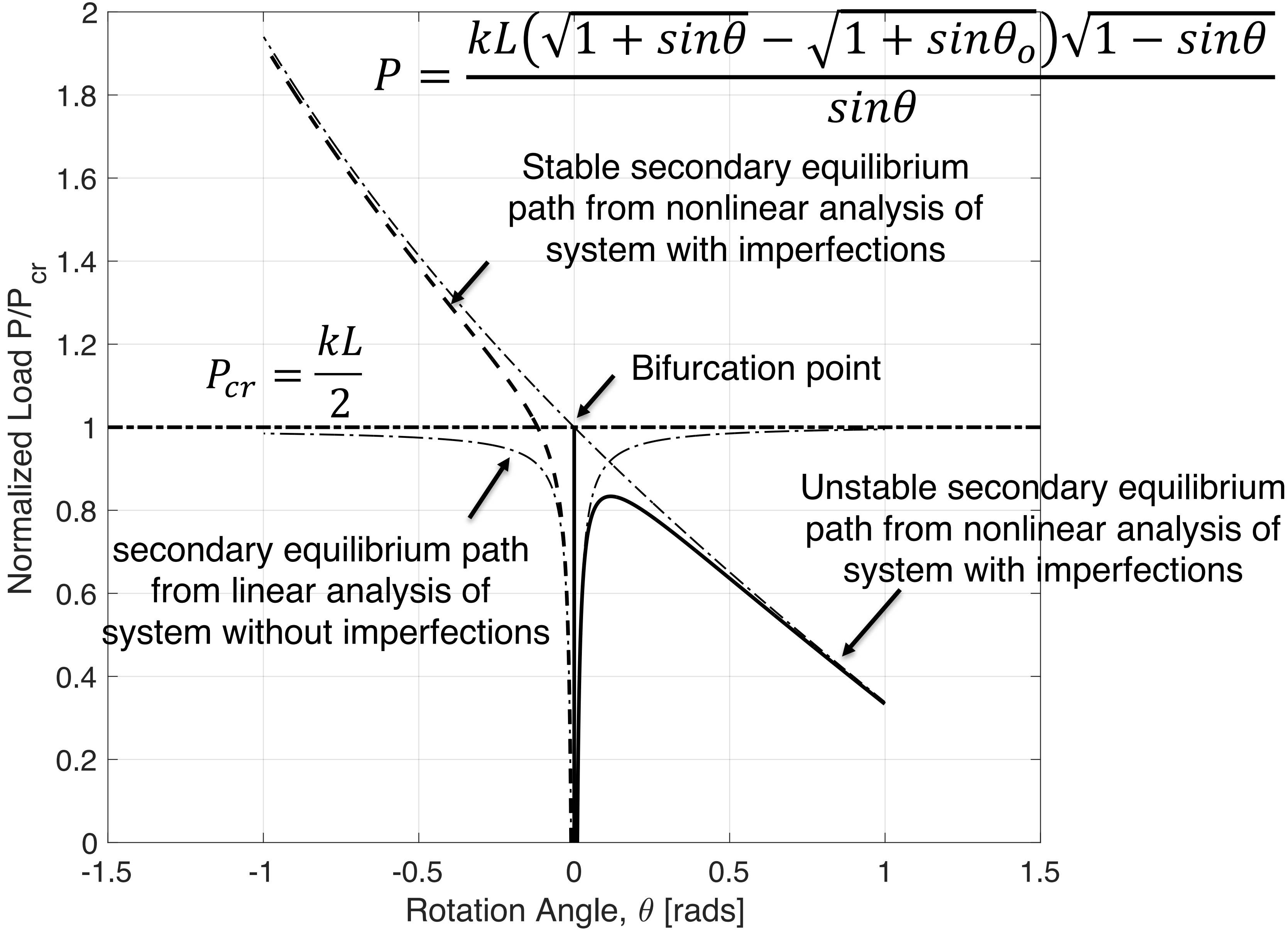
Example 3: Rigid Link with Elastic Support-Imperfections

-Linear Theory



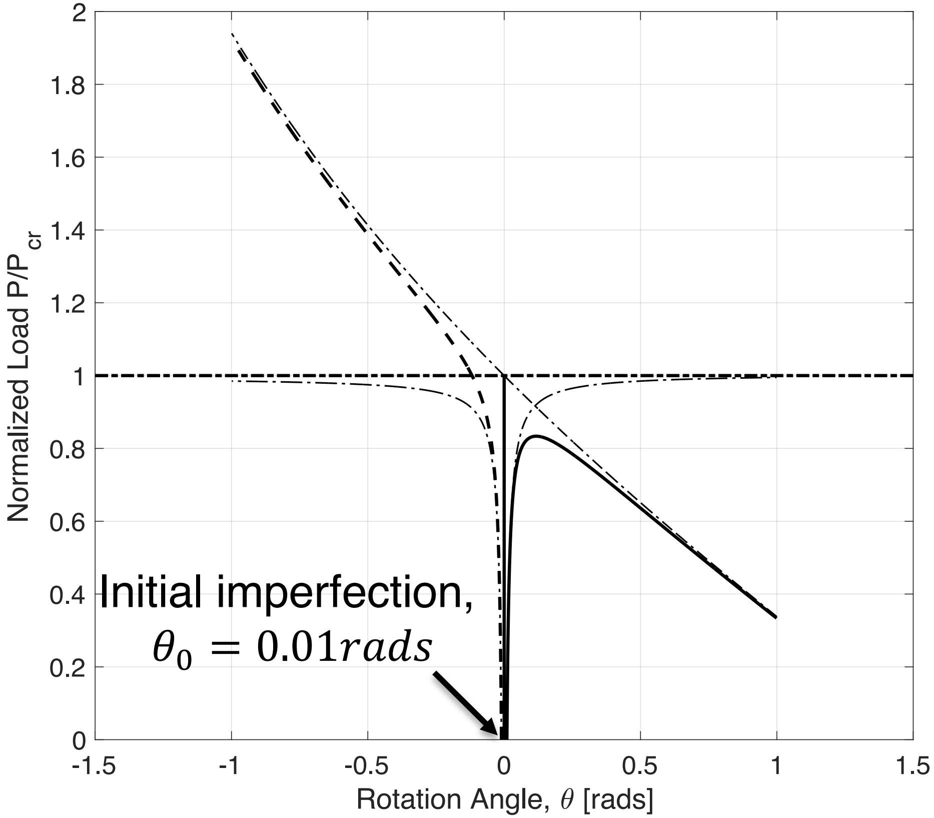
Example 3: Rigid Link with Elastic Support-Imperfections

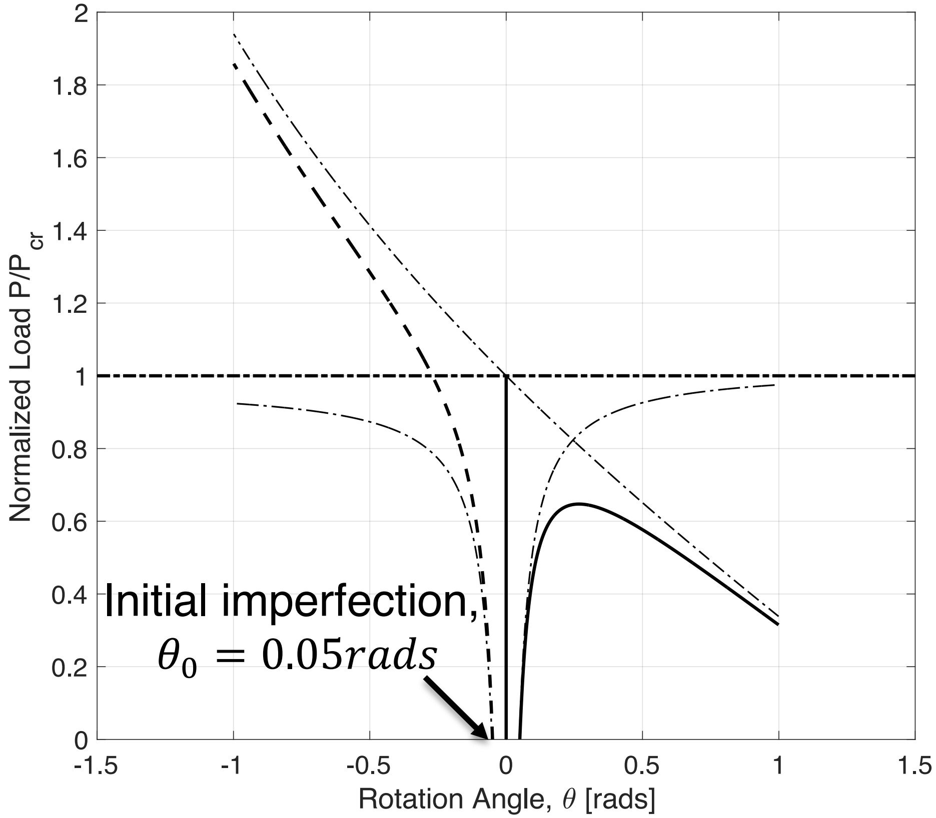
-Nonlinear Theory



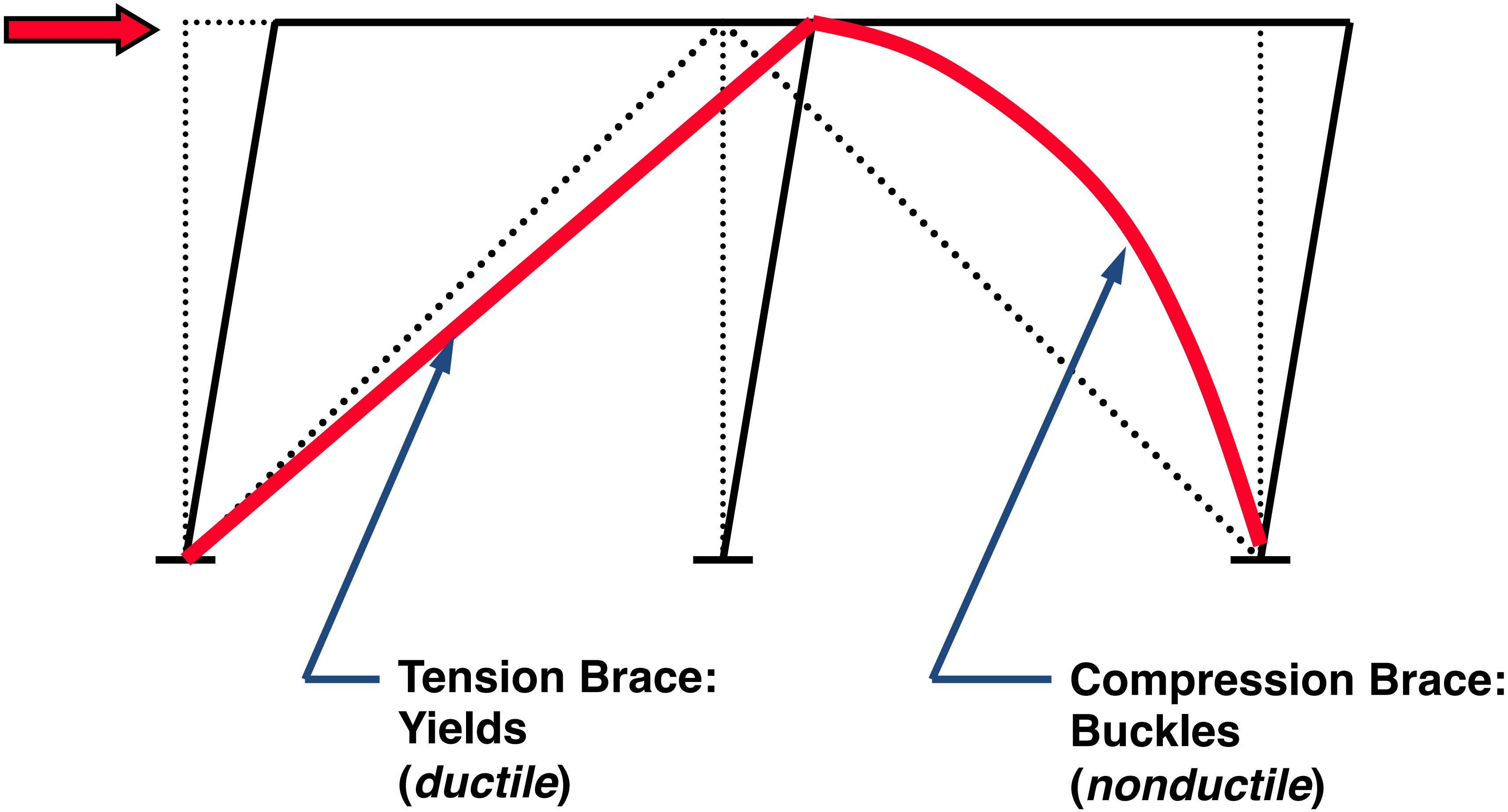
Example 3: Rigid Link with Elastic Support

-Sensitivity to Imperfections



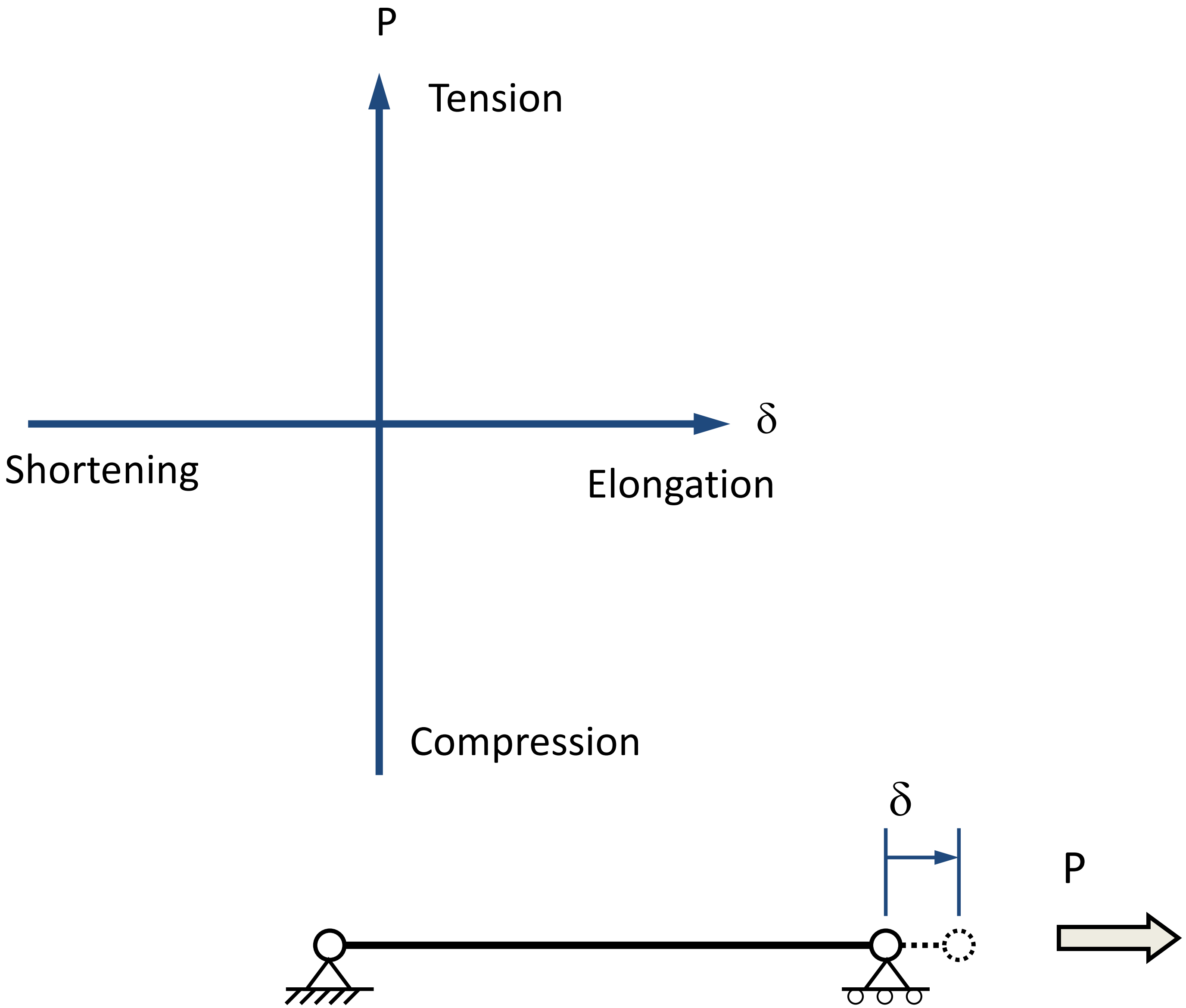


EPFL Inelastic Response of Bracing Members

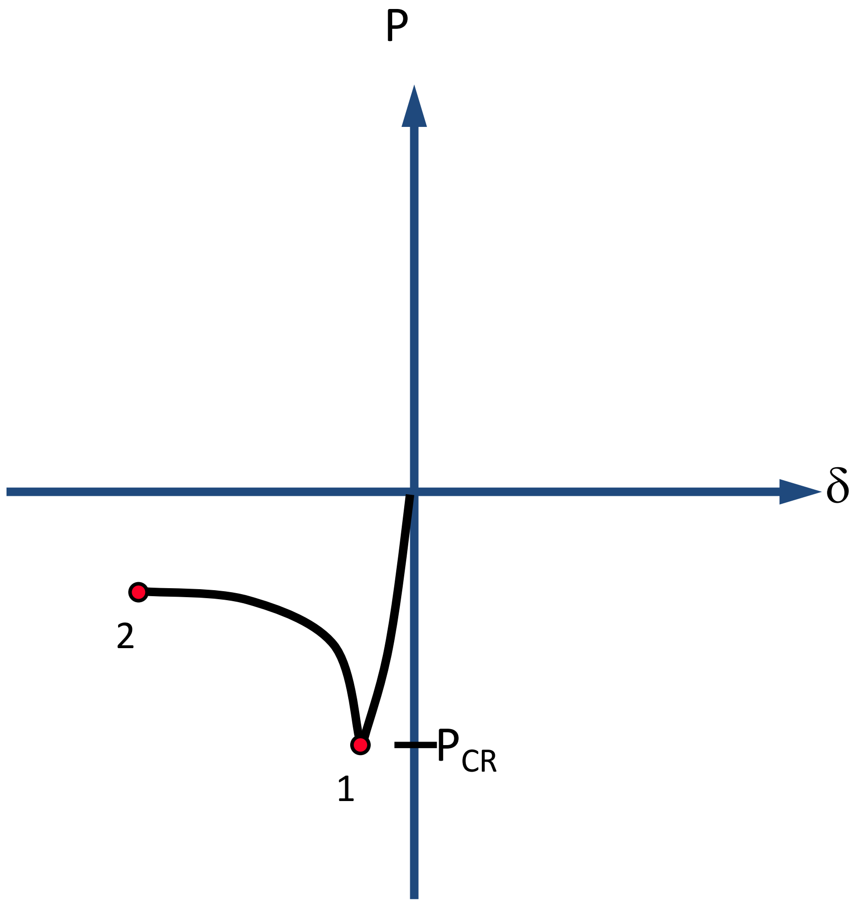


Columns and beams: remain essentially elastic

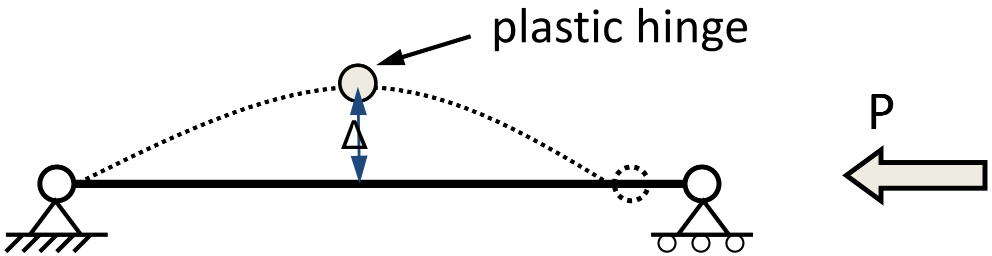
Stability of Steel Brace Under Cyclic Loading



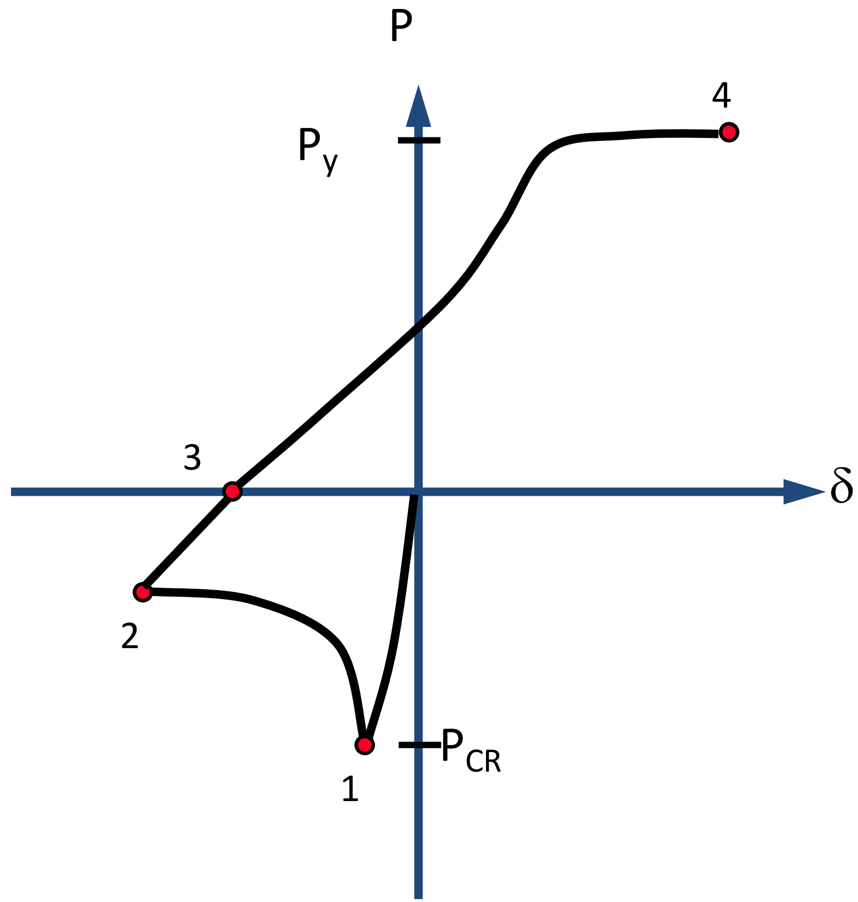
Stability of Steel Brace Under Cyclic Loading



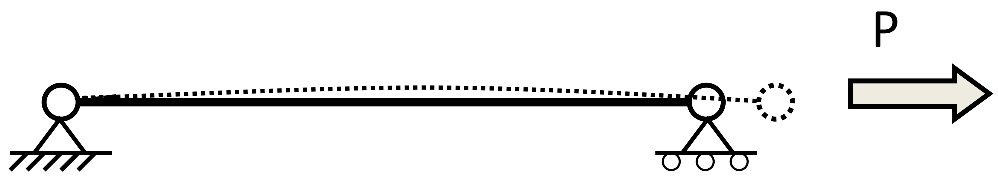
1. Brace loaded in compression to peak compression capacity (buckling).
2. Continue loading in compression. Compressive resistance drops rapidly. Flexural plastic hinge forms at mid-length (due to P-Δ moment in member).



Stability of Steel Brace Under Cyclic Loading



4. Brace loaded in tension to yield.



Example 4: Retractable Arch

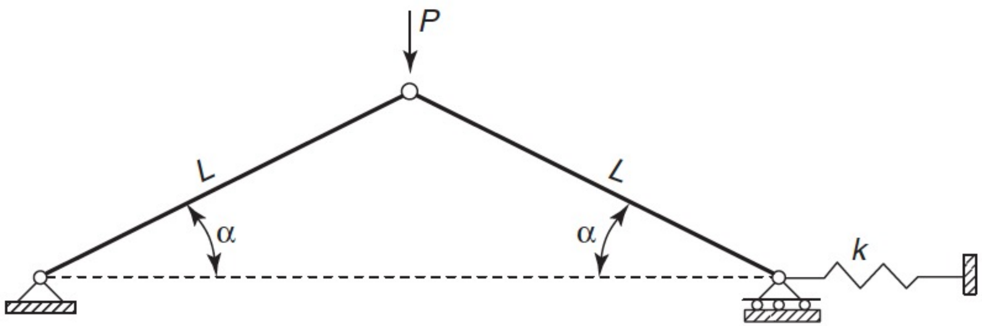
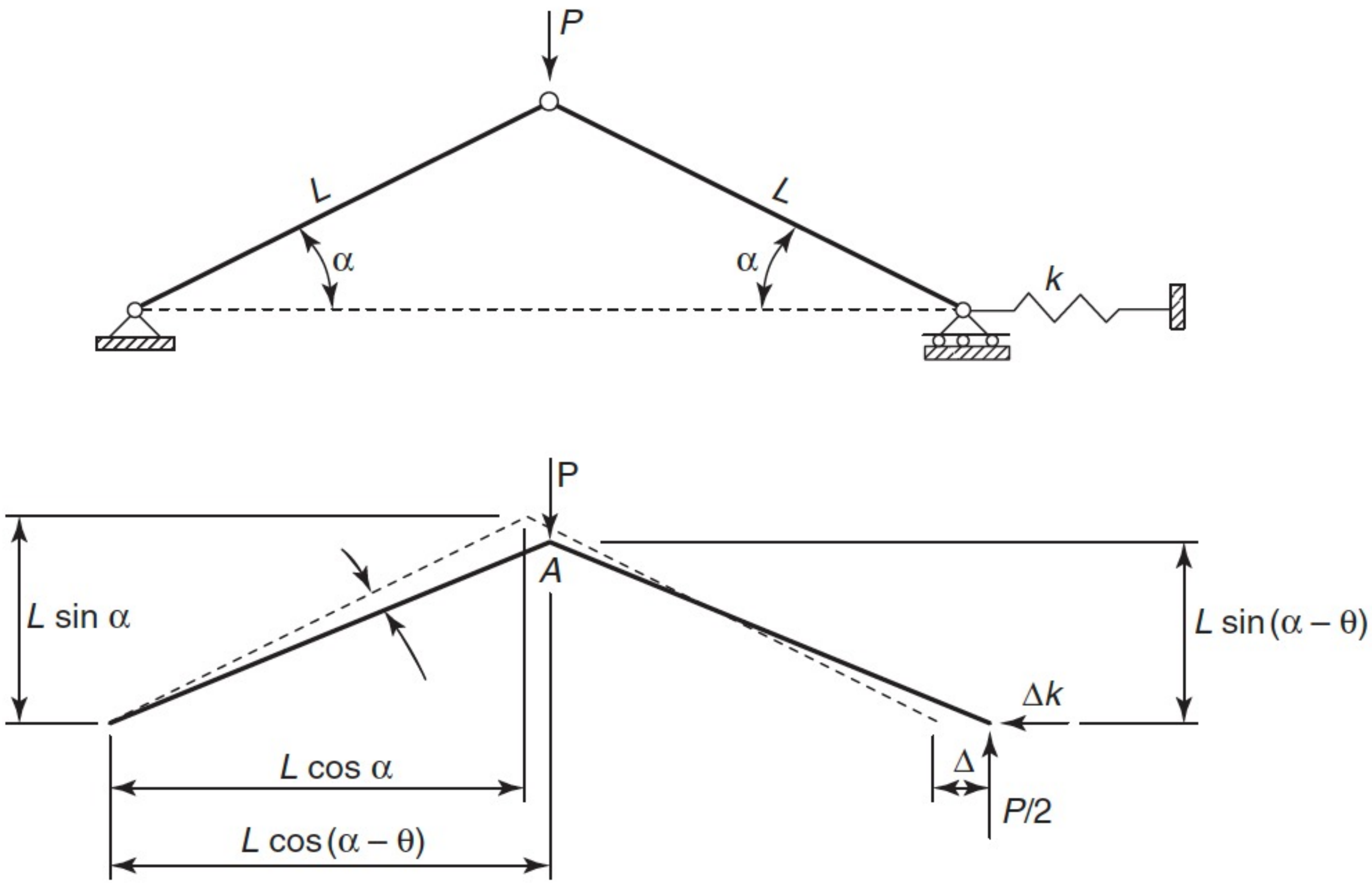


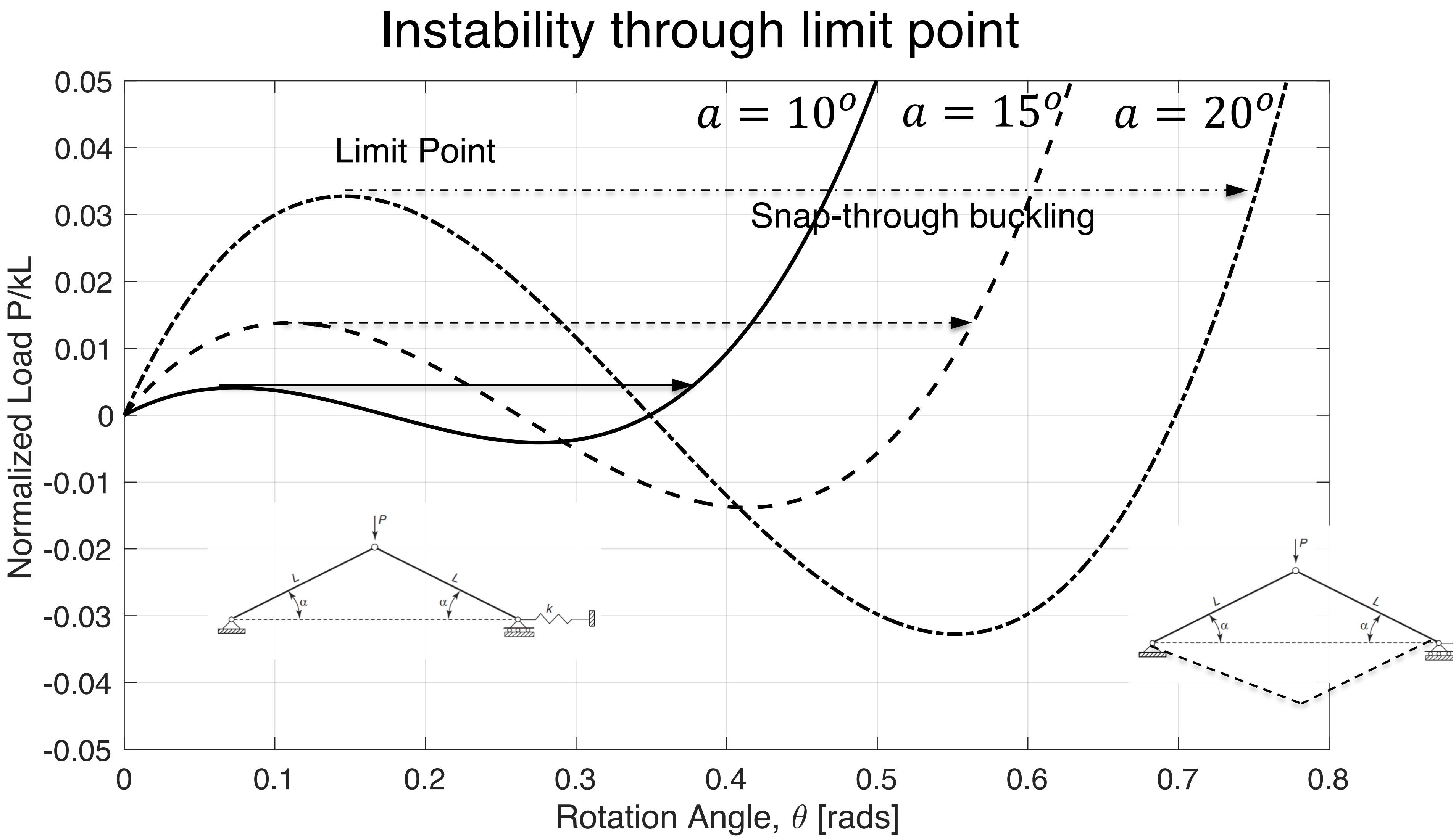
Image via DP Architects

Example 4: Arch with Horizontal Spring



$$\frac{P}{2} Lcos(a-\theta) = \Delta k sin(a-\theta) \Longrightarrow P = 4kL[cos(a-\theta) - \cos a] \tan(a-\theta)$$

Example 4: Arch with Horizontal Spring



Example 4: Arch with Horizontal Spring

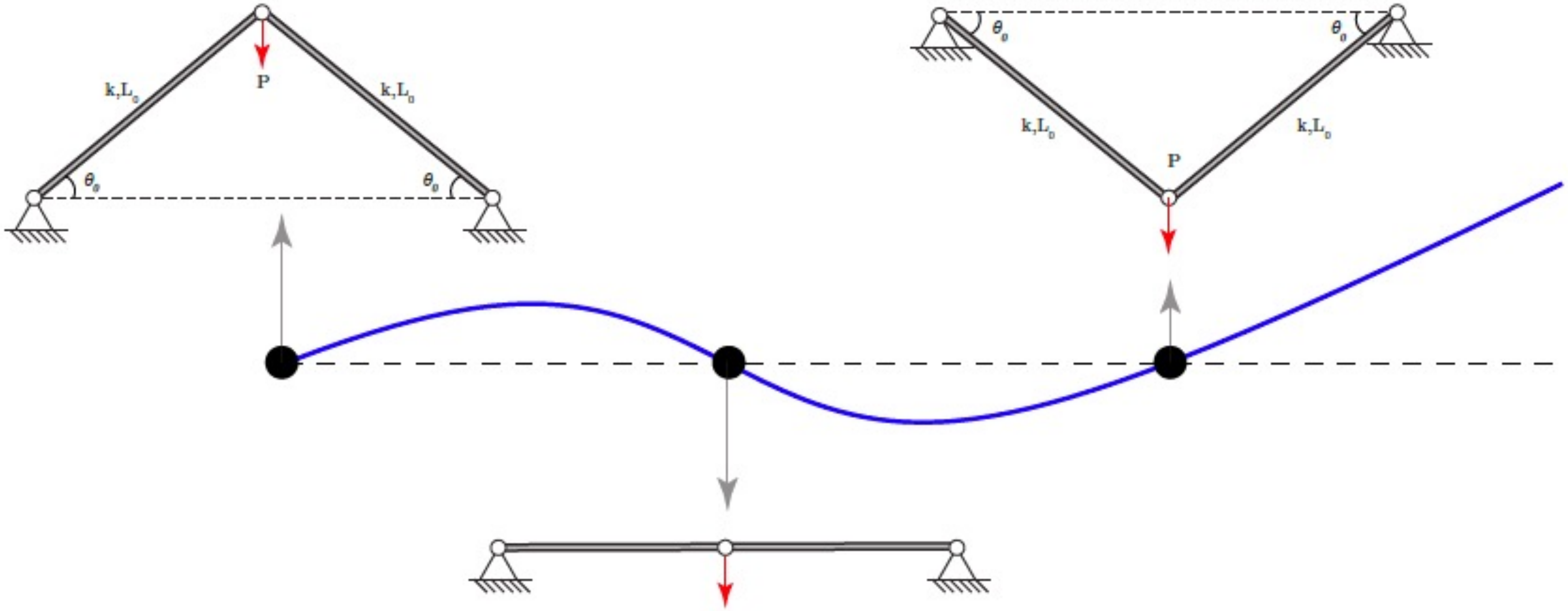
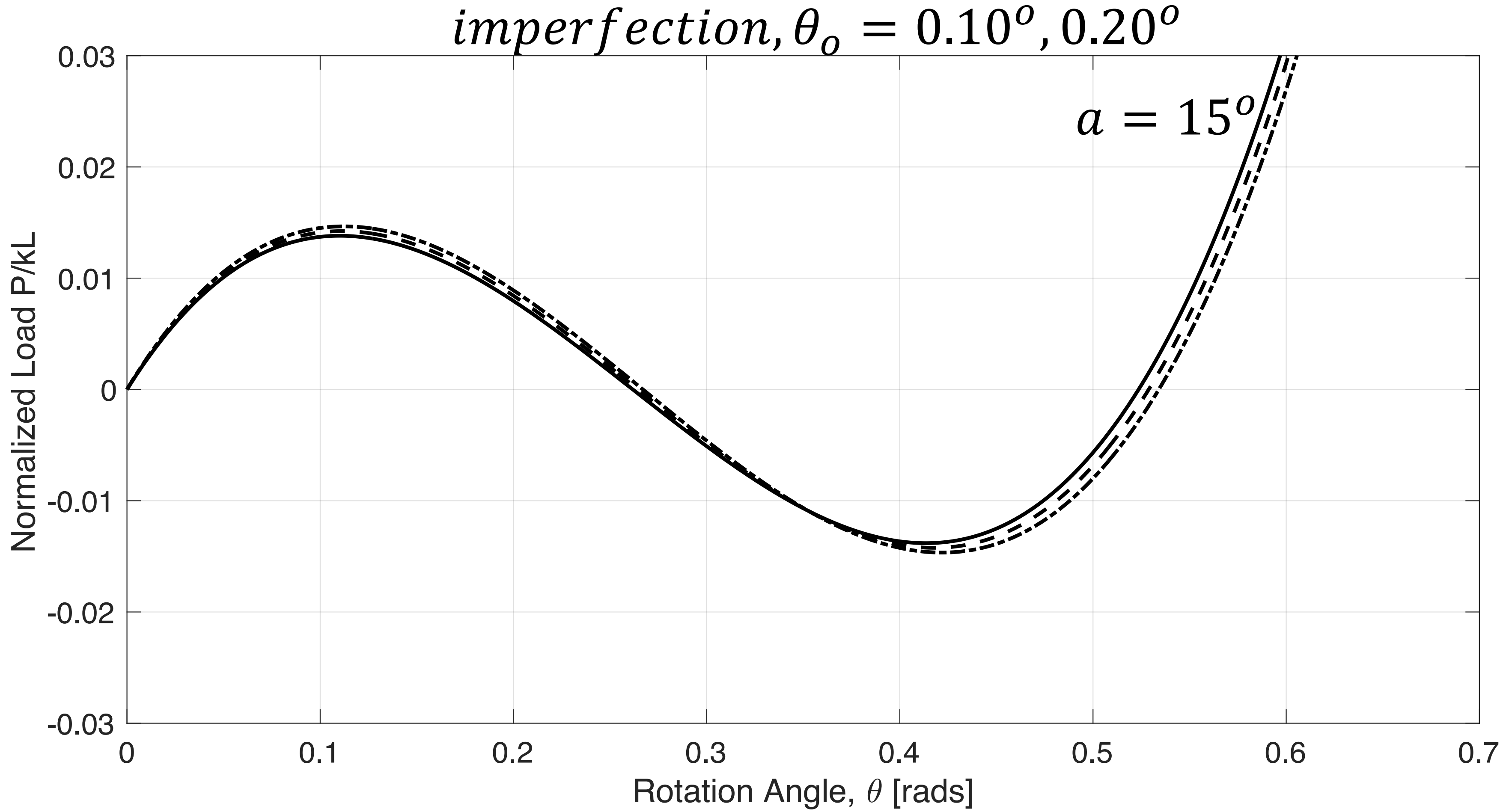
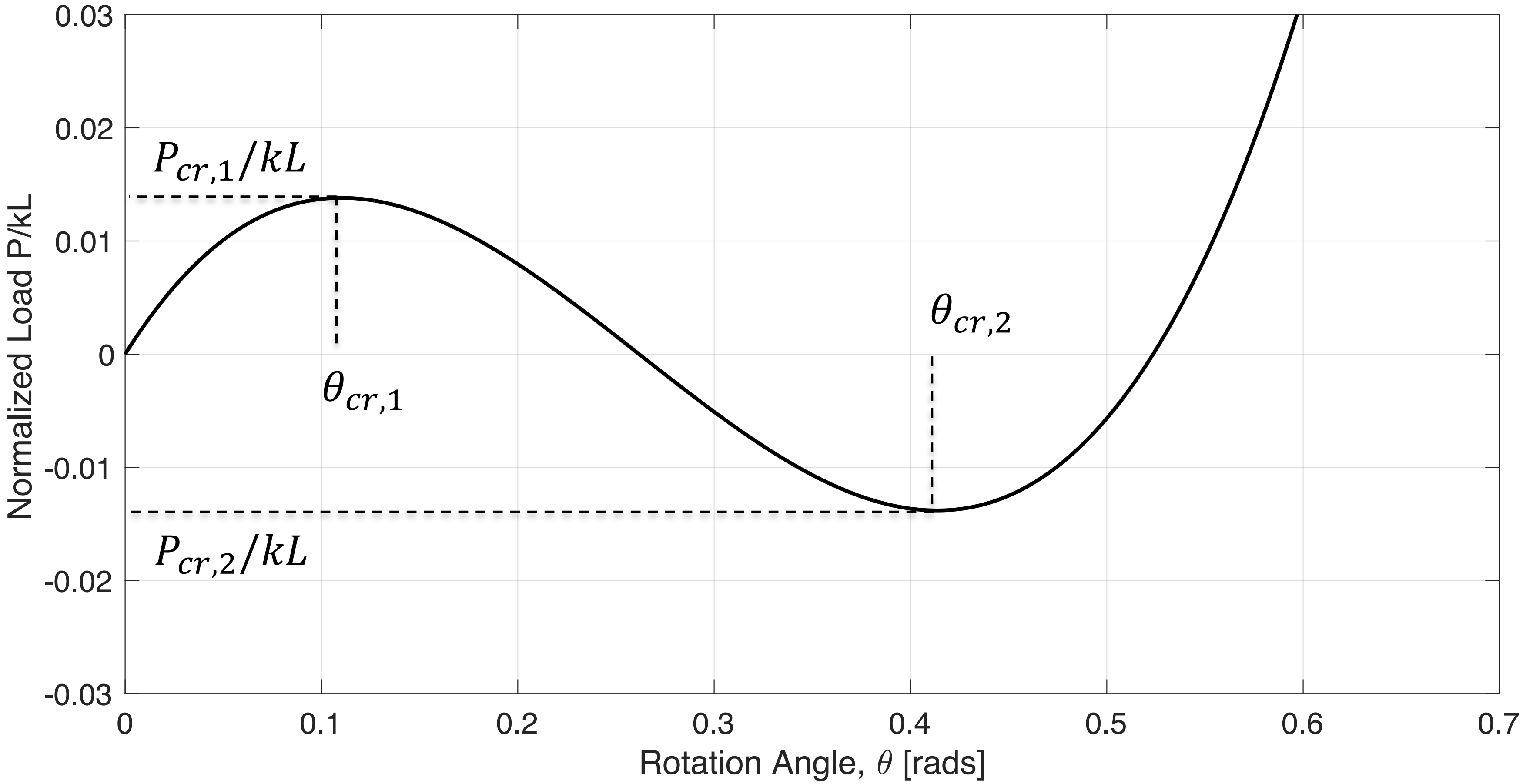


Image source: Vasios (2015)

Example 4: Arch with Horizontal Spring – Imperfections



Example 4: Arch with Horizontal Spring
– Computation of Limit Points

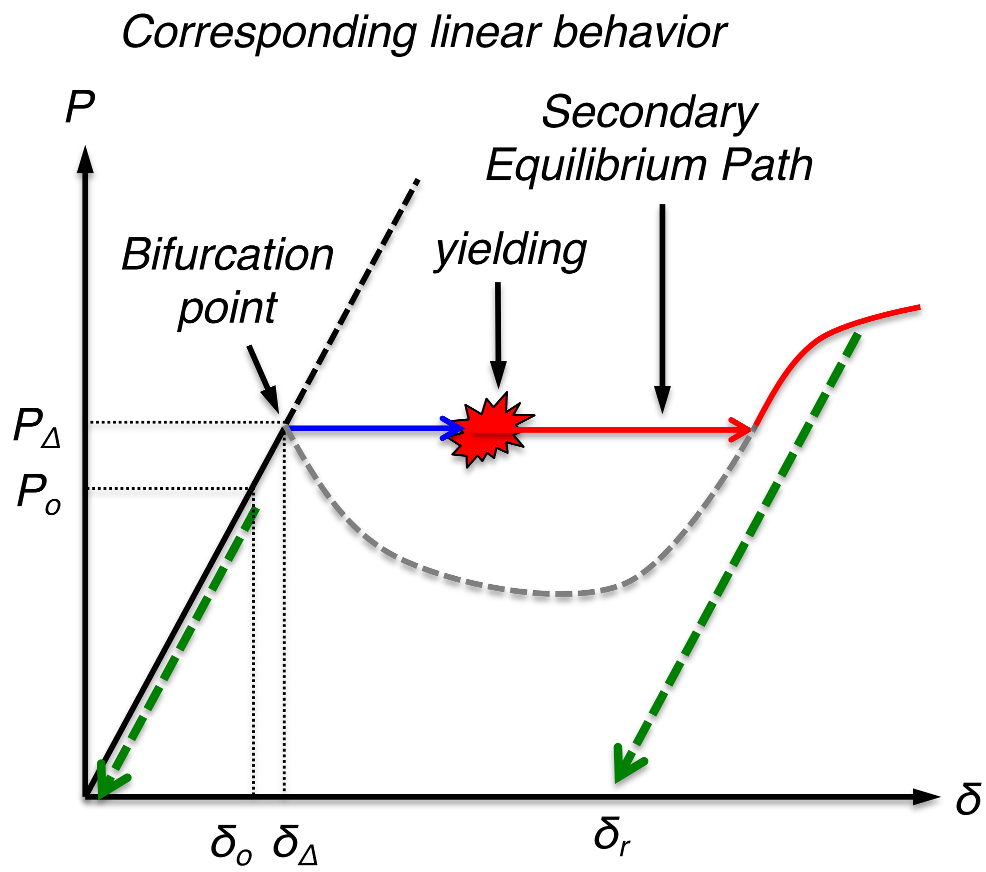


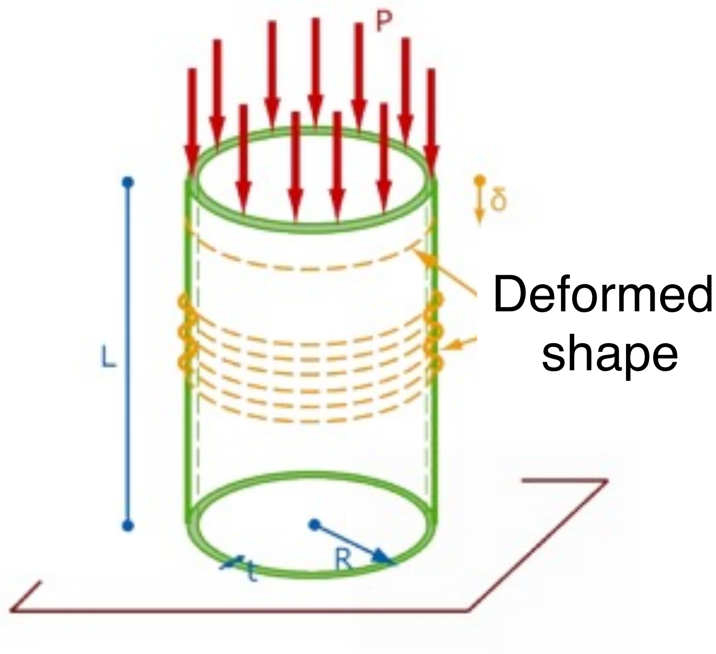
$$\frac{dP}{d\theta} = 4kL[\sin(a-\theta)\tan(a-\theta) + (\cos(a-\theta) - cosa)(1+\tan^2(a-\theta))(-1)]$$

$$\theta_{cr,1,2} = a \pm \cos^{-1}(\sqrt[3]{cosa}) \quad P_{cr,1,2} = \pm 4kL[\sqrt[3]{cosa} - cosa]\tan(\cos^{-1}(\sqrt[3]{cosa}))$$

Visualization of Snap-Through Buckling

Slender Cylinder with Small Member Slenderness
-By Controlling Applying Force (Measure the Displacement)





Steel Column Behaviour under Cyclic Loading

-By Controlling Displacement (measuring force)

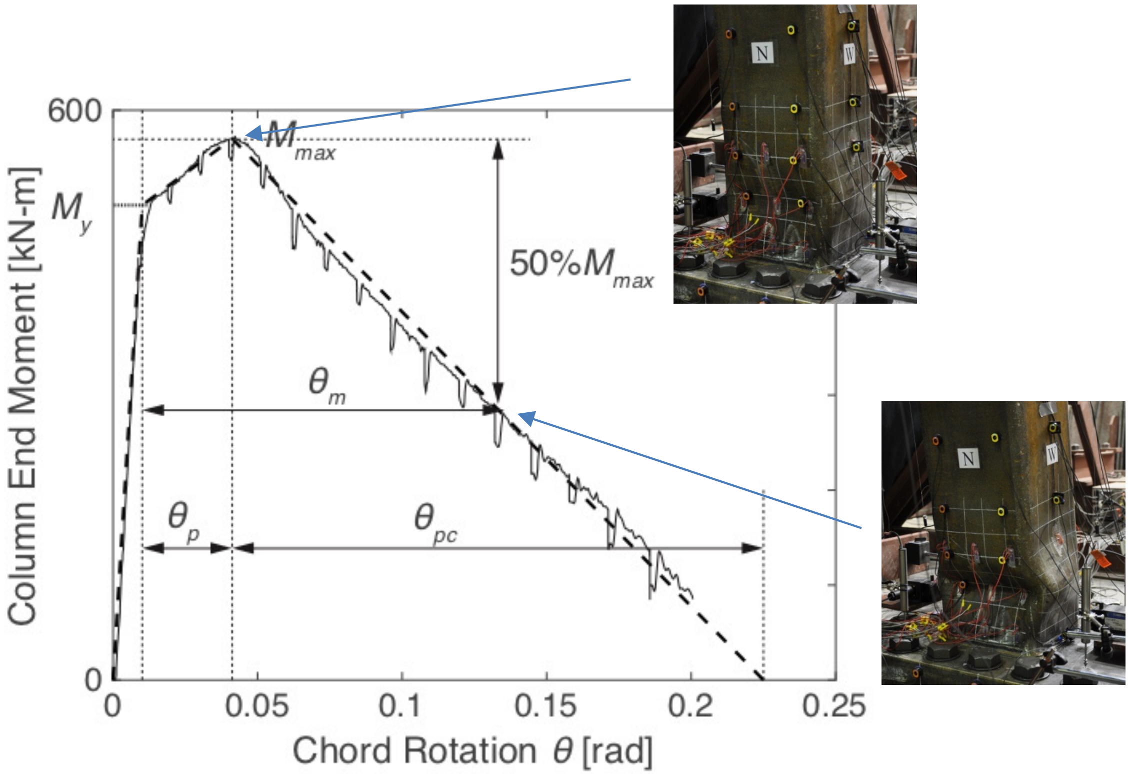


Image source: Suzuki and Lignos (2015)

Steel Column Behaviour under Cyclic Loading

-By Controlling Displacement (measuring force)

Video source: Suzuki and Lignos (2015)

Possible Failure Modes

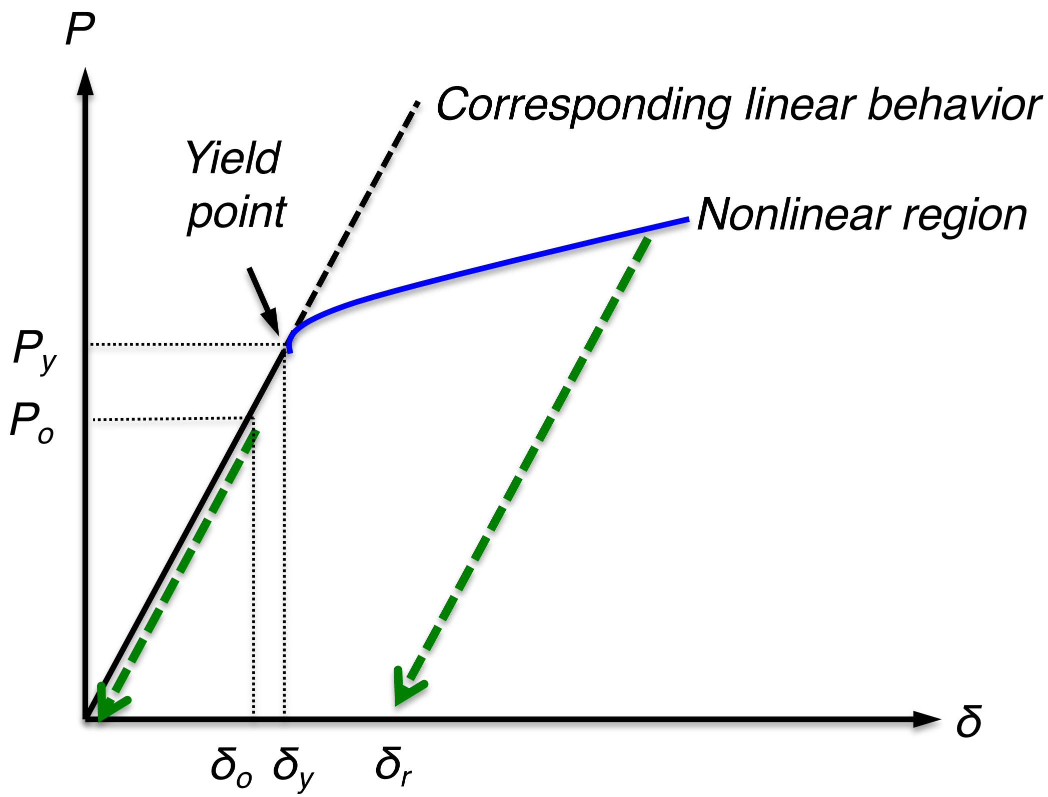
✧ Failure due to material nonlinearity

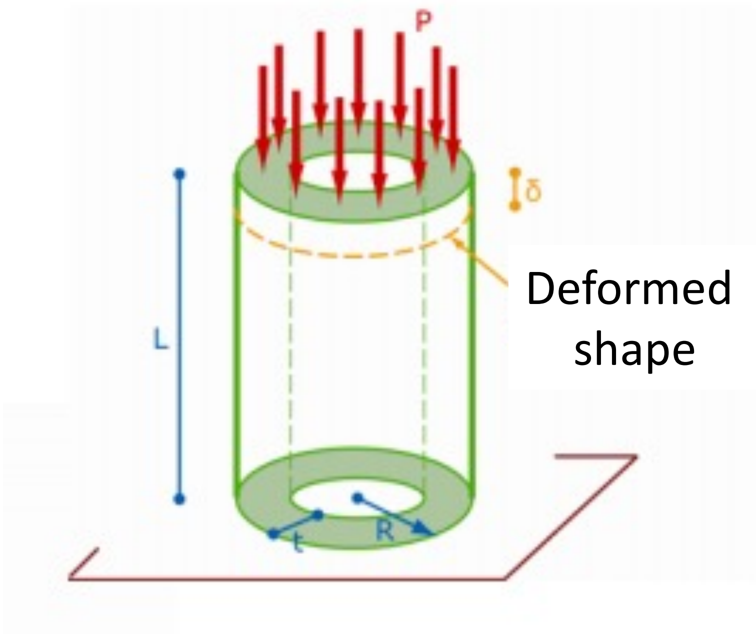
✧ Failure through stable path (bifurcation point)

✧ Failure through unstable path (bifurcation point)

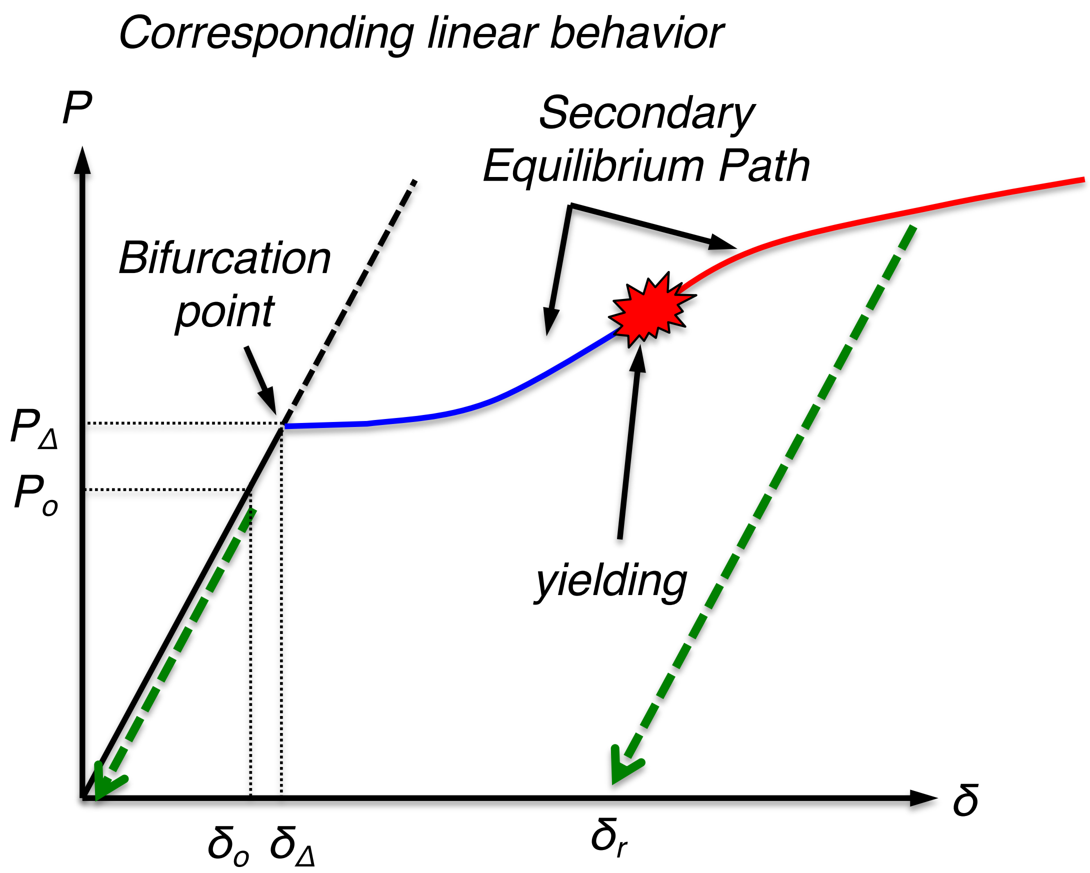
✧ Failure through limit point

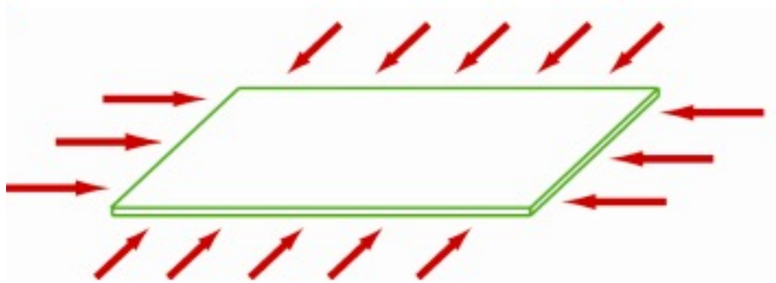
Failure due to Material Nonlinearity



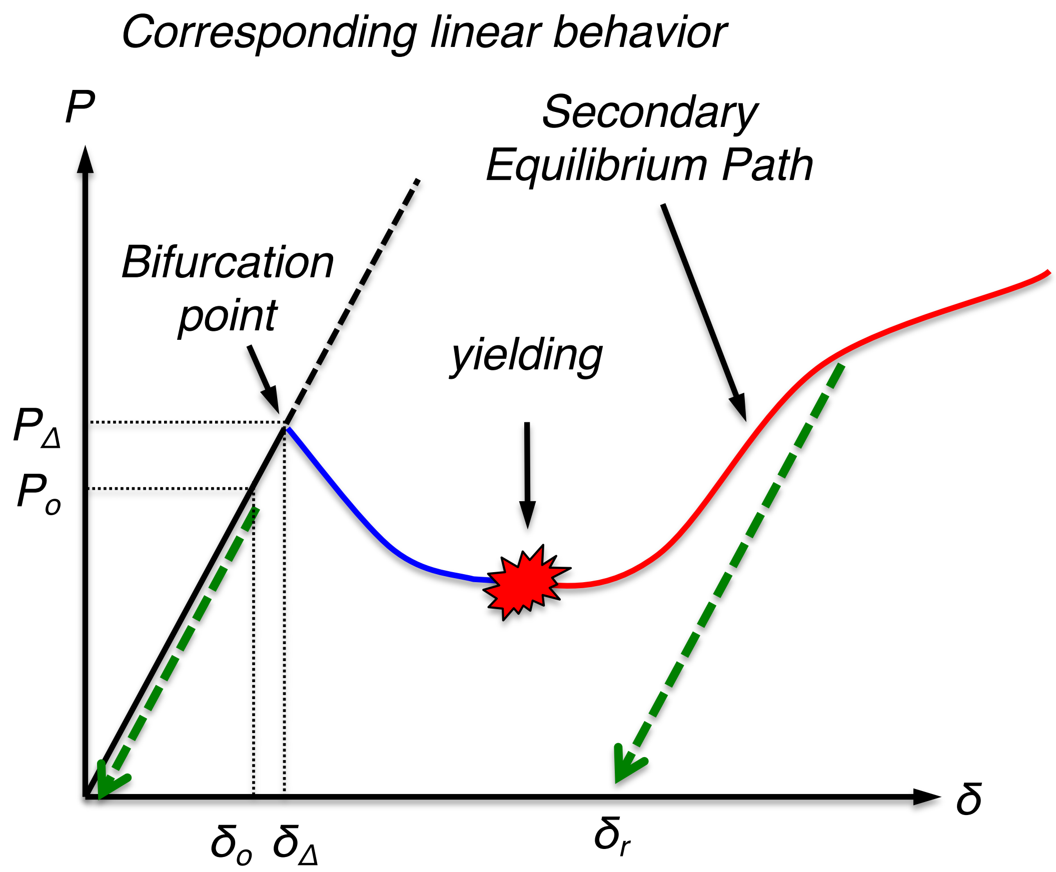


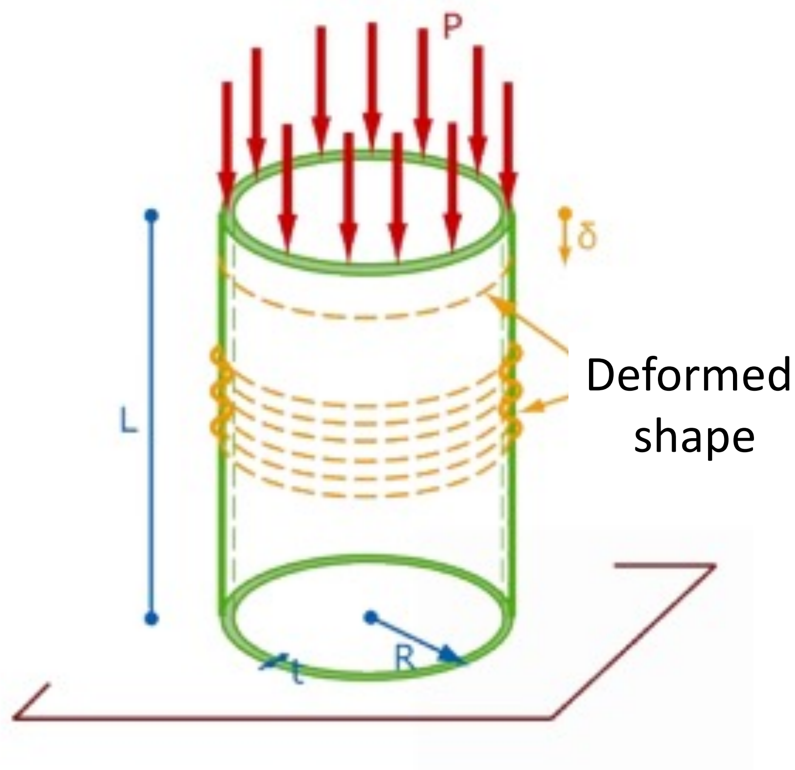
Failure Through Stable Path



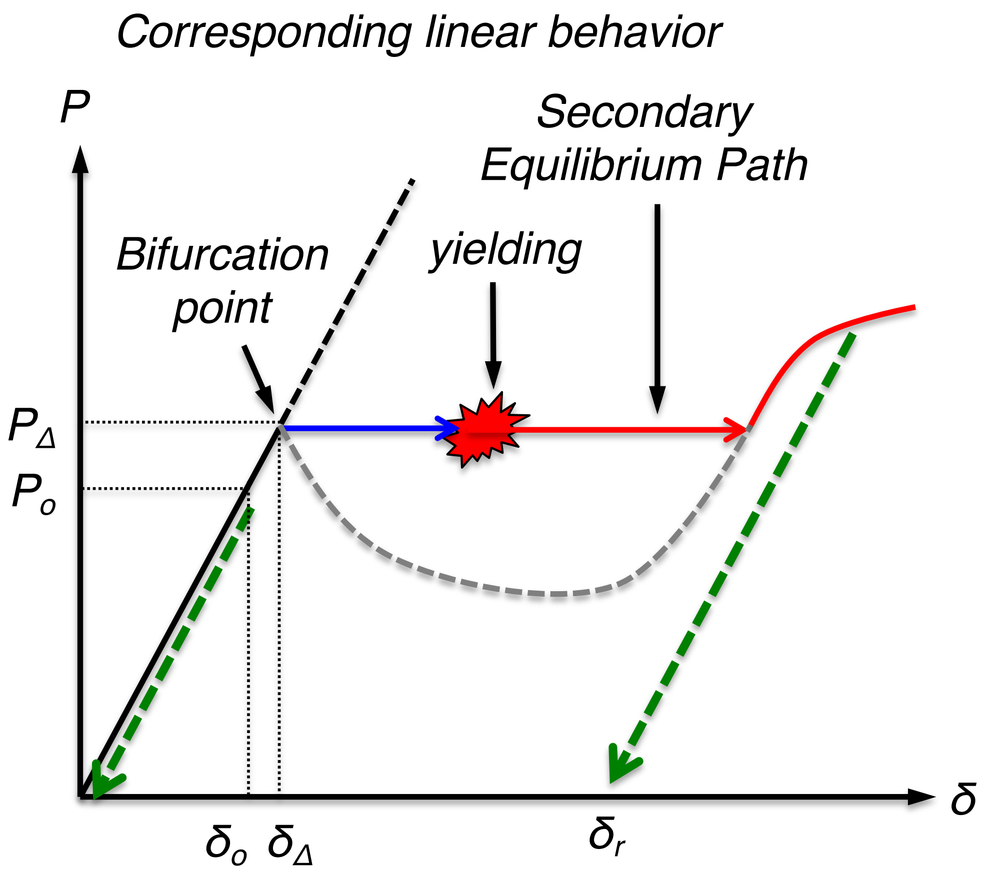


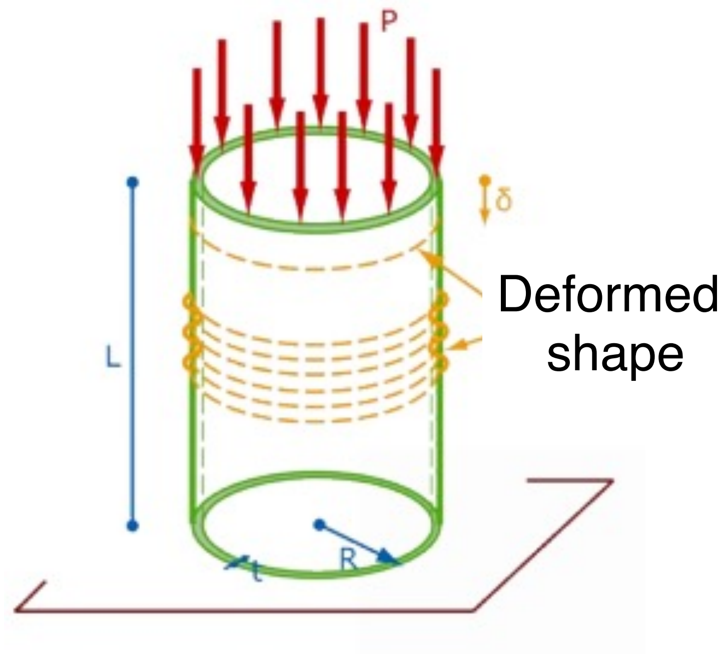
Failure Through Unstable Path





EPFL Failure Through a Limit Point (Snap—Through)





Summary of Modeling Recommendations

Type of Buckling	Importance of Imperfection	Suggested Analysis
Symmetric stable bifurcation point	Small	Linear with or without imperfections
Symmetric unstable bifurcation point	Large	Nonlinear with imperfections
Asymmetric bifurcation point	Large	Nonlinear with imperfections
Limit Point Snap-through	Small	Nonlinear with or without imperfections

Solution Techniques

When an analytical solution is not possible to be retrieved, a numerical solution should be retrieved. This is the case in most structural engineering problems associated with stability (or instability).

✧ Failure through stable paths: Newton's Method (Newton-Raphson iteration)

✧ Failure through unstable paths: Arc-Length

Newton's Method

- In Newton's method, the incremental loading is expressed as follows:
- The external load vector $\mathbf{F}^{ext}$ is gradually increased from 0 in order to reach a desired value $\mathbf{F}^{*}$.
- Assuming, $\mathbf{F}^{*}$ itself remains constant during the analysis in terms of its direction and only its magnitude is changing, we can write, $\mathbf{F}^{ext} = \mathbf{q} = known$ to simplify our expression for the system of equations.

Newton's Method

✧ Then we control how the external load vector increases or decreases by introducing a scalar quantity λ and express the system as follows,

$$\mathbf{R}(\mathbf{u}) = \mathbf{F}^{int}(\mathbf{u}) - \lambda\mathbf{F}^{ext} = 0 \qquad (1)$$

✧ Thus by increasing of decreasing λ we can control our **load incrementation**.

EPFL Newton's Method

- ✧ We introduced the term $\mathbf{R}(\mathbf{u})$ because we are interested in the general case were the system of equations is not in equilibrium and the difference expresses the residual vector. We will then use this to find corrections to our solution.
- ✧ In this system of equations we are interested in $\mathbf{u}$ and λ. As such, at every increment, we change slightly the value of λ and try to determine u so that the system of equations below is satisfied.

$$\mathbf{R}(\mathbf{u}) = \mathbf{F}^{int}(\mathbf{u}) - \lambda\mathbf{F}^{ext} = 0 \quad (2)$$

EPFL Newton's Method

✧ The load increment is initiated by postulating:

$$\lambda' = \lambda_0 + \Delta\lambda \qquad (3)$$

✧ Where $\Delta\lambda$ is a known predefined incrementation parameter. This variation $\Delta\lambda$ violates the equation in the previous slide and thus we need to update the displacements $\mathbf{u}_0$ by,

$$\mathbf{u}' = \mathbf{u}_0 + \Delta\mathbf{u} \qquad (4)$$

$$\mathbf{R}(\mathbf{u}') = \mathbf{R}(\mathbf{u}_0 + \Delta\mathbf{u}) = \mathbf{0} \Longrightarrow \mathbf{F}^{int}(\mathbf{u}_0 + \Delta\mathbf{u}) - (\lambda_0 + \Delta\lambda)\mathbf{q} = 0 \qquad (5)$$

✧ But, $\mathbf{F}^{int}(\mathbf{u}_0 + \Delta\mathbf{u})$ can be expressed in terms of $\mathbf{F}^{int}(\mathbf{u}_0)$ by a Taylor series expansion.

Newton's Method

✧ As such,

$$\mathbf{F}^{int}(\mathbf{u}_0 + \Delta\mathbf{u}) = \mathbf{F}^{int}(\mathbf{u}_0) + \left[\frac{\vartheta \mathbf{F}(\mathbf{u})}{\vartheta \mathbf{u}}\right]_{\mathbf{u}_0} \cdot \Delta\mathbf{u} = \mathbf{F}^{int}(\mathbf{u}_o) + [K_T]_{\mathbf{u}_o} \cdot \Delta\mathbf{u} \qquad (6)$$

✧ With $[K_T] = \left[\frac{\vartheta \mathbf{F}(\mathbf{u})}{\vartheta \mathbf{u}}\right]$ is the "Jacobian" matrix of the system of equations and is commonly referred to as the *Stiffness Matrix* of a structure. Now combining the equations above,

$$\mathbf{F}^{int}(\mathbf{u}_o) + [K_T]_{\mathbf{u}_o} \cdot \Delta\mathbf{u} - (\lambda_0 + \Delta\lambda)\mathbf{q} = 0 \qquad (7)$$

$$\Longrightarrow \underbrace{\mathbf{F}^{int}(\mathbf{u}_o) - \lambda_0 \mathbf{q}}_{0} + [K_T]_{\mathbf{u}_o} \cdot \Delta\mathbf{u} - \Delta\lambda\mathbf{q} = 0 \qquad (8)$$

✧ Hence,

$$\Delta\mathbf{u} = [K_T]_{\mathbf{u}_o}^{-1} \cdot (\Delta\lambda\mathbf{q}) \qquad (9)$$

Newton's Method

- From the previous equation, we can calculate the displacement correction, $\Delta\mathbf{u}$. However, even though we postulated that $\Delta\mathbf{u}$ would be such that Equation (5) is satisfied, the linear approximation in Taylor expansion prevents the immediate achievement (linear-response) of equilibrium.
- Thus, if we evaluate the system of Equations (5) at the new point $(\mathbf{u}', \lambda')$ we will obtain a non-zero residual vector, $\widehat{\mathbf{R}}(\mathbf{u}')$. Using this residual vector, we can calculate a new displacement correction, $\delta\mathbf{u}$.

EPFL Newton's Method

✧ This can be done as follows,

$$\mathbf{R}(\mathbf{u}_0 + \Delta\mathbf{u} + \boldsymbol{\delta}\mathbf{u}) = \mathbf{0} \Longrightarrow \mathbf{F}^{int}(\mathbf{u}_0 + \Delta\mathbf{u}) + [K_T]_{\mathbf{u}'} \cdot \delta\mathbf{u} - (\lambda + \Delta\lambda)\mathbf{q} = 0$$

$$\Longrightarrow [K_T]_{\mathbf{u}'} \cdot \delta\mathbf{u} = -\left(\mathbf{F}^{int}(\mathbf{u}_0 + \Delta\mathbf{u}) - (\lambda + \Delta\lambda)\mathbf{q}\right) \Longrightarrow [K_T]_{\mathbf{u}'} \cdot \delta\mathbf{u} = -\hat{\mathbf{R}}(\mathbf{u}')$$

$$\Longrightarrow \delta\mathbf{u} = -[K_T]_{\mathbf{u}'}^{-1} \cdot \hat{\mathbf{R}}(\mathbf{u}') \qquad (10)$$

✧ Hence, a new displacement correction is determined and evaluating the system of Equations (5) at the new points $(\mathbf{u}' + \delta\mathbf{u}, \lambda')$ would in general result to a new and smaller residual vector $\widehat{\mathbf{R}}(\mathbf{u}'')$.

Newton's Method

✧ We continue to provide displacement corrections until a norm (usually Euclidean) of the residual vector is less than the specified tolerance. A schematic representation of the Newton-Raphson method is shown below.

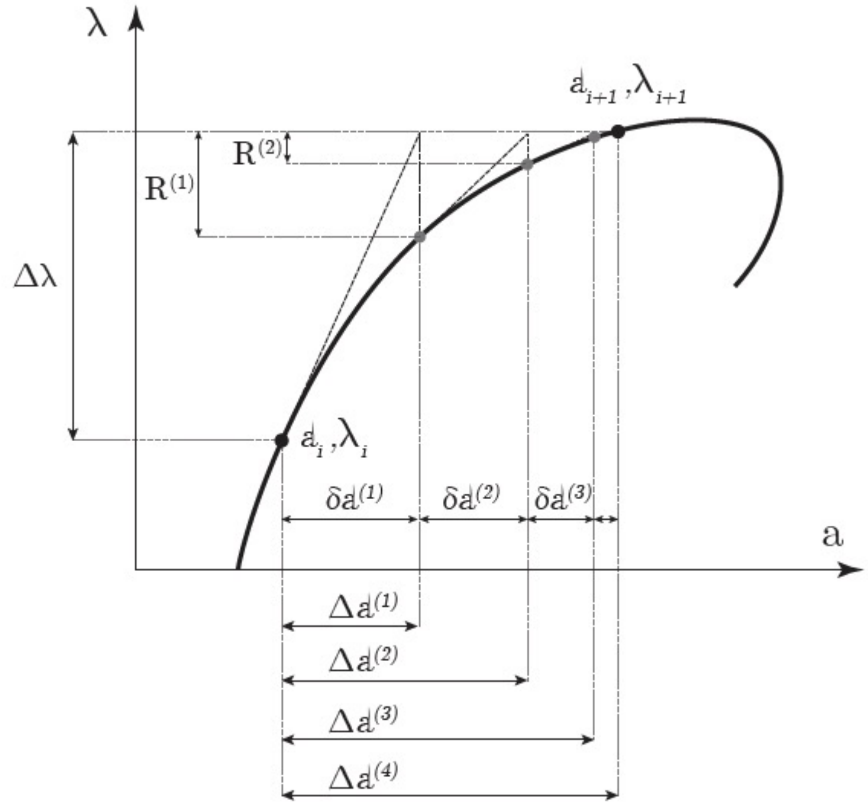


Image source: Vasios (2015)

Newton's Method

- **Major drawback:** it fails to accurately follow the equilibrium path once the tangent stiffness reaches zero. That happens due to the formulation of Newton's method, and in particular that is restricts the parameter λ to change monotonically every increment (i.e., in most structural mechanics problems, for instance, we tend to increase the external loads)
- The definition of a limit point then suggests that in order to reman on the equilibrium path you need to change your loading pattern depending on whether the limit point is a local maximum or maximum in the $\mathbf{u} - \lambda$ space.

Unstable Systems

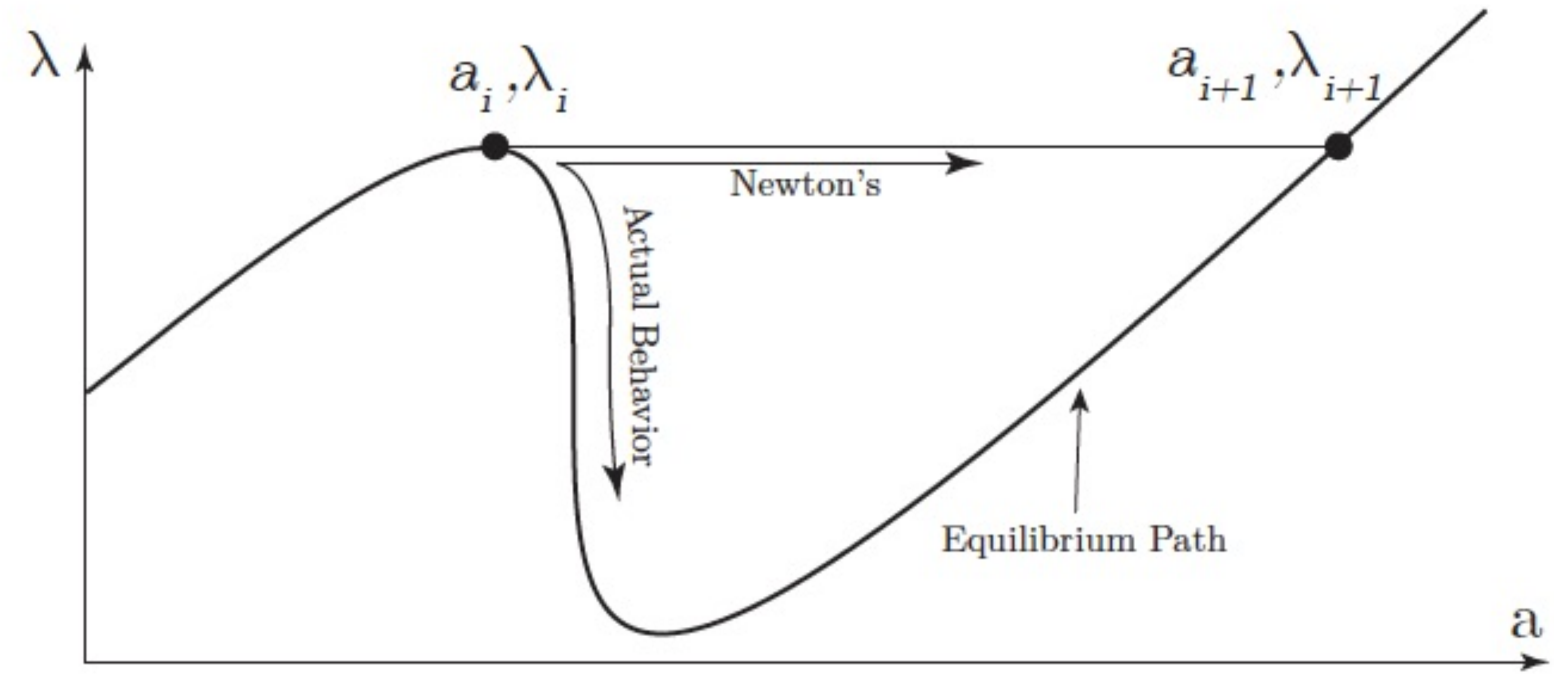


Image source: Vasios (2015)

Unstable Systems (2)

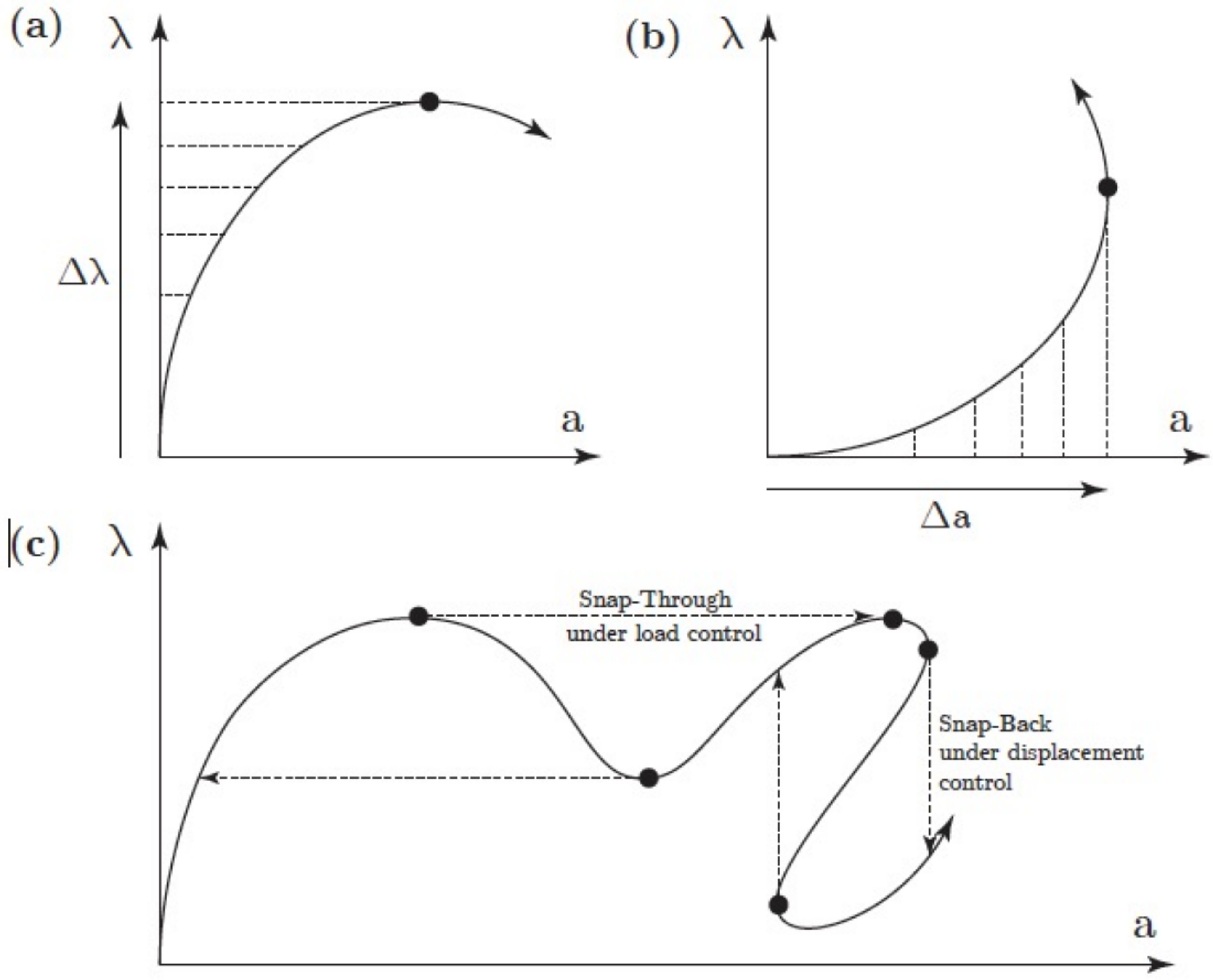


Image source: Vasios (2015)

The Arc Length Method

- The Arc-Length method (Riks 1979) is a very efficient method in solving non-linear systems of equations when the problem under consideration exhibits one or more critical points.
- In terms of a simple mechanical loading-unloading problem, a critical point could be interpreted as the point at which the loaded body cannot support an increase of the external forces and an instability occurs.

$$\mathbf{F}^{int}(\mathbf{u}) - \mathbf{F}^{ext} = 0 \Longrightarrow \mathbf{F}^{int}(\mathbf{u}) - \lambda \mathbf{q} = 0 \qquad (11)$$

The Arc Length Method

- Suppose at the point $(\mathbf{u}_0, \lambda_0)$ is such to satisfy the system of equations and thus belongs to the equilibrium path that we are trying to identify. Unlike the Newton's method, the Arc-Length method postulates a simultaneous variation in both the displacements $\Delta\mathbf{u}$ and the load vector coefficient, $\Delta\lambda$.
- The main different is that both $\Delta\mathbf{u}$ and $\Delta\lambda$ are unknowns in contrast to Newton's method where $\Delta\lambda$ was given and we had to iterate for $\Delta\mathbf{u}$.

$$\mathbf{R}(\mathbf{u}', \lambda') = \mathbf{F}^{int}(\mathbf{u}_0 + \Delta\mathbf{u}) - (\lambda_0 + \Delta\lambda)\mathbf{q} = 0 \qquad (12)$$

- If the above equation is satisfied for $(\mathbf{u}_0 + \Delta\mathbf{u}, \lambda_0 + \Delta\lambda)$ then this point also belongs to the equilibrium path.

The Arc Length Method

✧ Immediate satisfaction of Equation (12) is not achievable. As a result, we need to provide necessary corrections, $(\delta\mathbf{u}, \delta\lambda)$, aiming that the new point, $(\mathbf{u_0} + \Delta\mathbf{u} + \delta\mathbf{u}, \lambda_0 + \Delta\lambda + \delta\lambda)$, will satisfy Equation (12). Hence,

$$\mathbf{R}(\mathbf{u}'', \lambda'') = \mathbf{F}^{int}(\mathbf{u_0} + \Delta\mathbf{u} + \delta\mathbf{u}) - (\lambda_0 + \Delta\lambda + \delta\lambda)\mathbf{q} = 0 \quad (13)$$

✧ Using a Taylor's series expansion and retaining only the linear terms, the last equation becomes,

$$\mathbf{F}^{int}(\mathbf{u_0} + \Delta\mathbf{u}) + \underbrace{\left[\frac{\vartheta\mathbf{F}^{int}(\mathbf{u})}{\vartheta\mathbf{u}}\right]_{\mathbf{u}_0+\Delta\mathbf{u}}}_{[K_T]} \cdot \delta\mathbf{u} - (\lambda_0 + \Delta\lambda + \delta\lambda)\mathbf{q} = 0 \quad (14)$$

The Arc Length Method

✧ Hence,

$$[K_T]_{\mathbf{u}_0+\Delta\mathbf{u}} \cdot \delta\mathbf{u} - \delta\lambda\mathbf{q} = -\left[\mathbf{F}^{int}(\mathbf{u_0} + \Delta\mathbf{u}) - (\lambda_0 + \Delta\lambda)\mathbf{q}\right] = -\mathbf{R}(\mathbf{u}', \lambda') \quad (15)$$

✧ Recall that $\delta\mathbf{u}$ and $\delta\lambda$ are the unknowns for whom we need to solve. If the u vector has dimensions Nx1 then we have a total of N equations that we need to solve for N+1 unknowns. Equations (15) are not sufficient.

✧ The supplementary equation that completes the system is called the **Arc Length Equation** and has the following form:

$$(\Delta\mathbf{u} + \delta\mathbf{u})^T \cdot (\Delta\mathbf{u} + \delta\mathbf{u}) + \psi^2(\Delta\lambda + \delta\lambda)^2\left(\mathbf{q}^T \cdot \mathbf{q}\right) = \Delta l^2 \quad (16)$$

The Arc Length Method

✧ In Equation (16), ψ and Δl are user defined parameters. In a sense, Δl defines how far to search for the next equilibrium point and it is analogous (but not directly equivalent) to the load increment $\Delta\lambda$ we used in Newton's method.

✧ Collecting up Equations (16) and (15), we can write the system of equations we need to solve in a compact form,

$$\begin{bmatrix} [K_T] & -\mathbf{q} \\ 2\Delta\mathbf{u}^T & 2\psi^2\Delta\lambda(\mathbf{q}^T\cdot\mathbf{q}) \end{bmatrix} \cdot \begin{Bmatrix} \delta\mathbf{u} \\ \delta\lambda \end{Bmatrix} = -\begin{Bmatrix} \mathbf{R} \\ A \end{Bmatrix} \qquad (17)$$

$$\mathbf{R} = \mathbf{F}^{int}(\mathbf{u_0} + \Delta\mathbf{u}) - (\lambda_0 + \Delta\lambda)\mathbf{q}$$

$$A = -\left(\Delta\mathbf{u}^T \cdot \Delta\mathbf{u} + \psi^2\Delta\lambda^2\left(\mathbf{q}^T\cdot\mathbf{q}\right) - \Delta l^2\right)$$

The Arc Length Method

- The system of equations in (17) is solved and updates the previous corrections $\Delta\mathbf{u}, \Delta\lambda$ to be $\Delta\mathbf{u} = \Delta\mathbf{u} + \delta\mathbf{u}$ and $\Delta\lambda' = \Delta\lambda + \delta\lambda$, respectively.
- The method continues to provide such incremental corrections $\delta\mathbf{u}, \delta\lambda$ until convergence is achieved in (13).
- When $\psi = 1$, the method is called spherical Arc-Length method because Equation (16) suggests that the points $\Delta\mathbf{u}', \Delta\lambda'$ belong to a circle with radius Δl.
- In its most general form for arbitrary ψ, Equation (16) can be geometrically interpreted as a hyper-ellipse in the multidimensional displacement-load space $(\mathbf{u} - \lambda)$.

The Arc Length Method

✧ The user decides which value should be assigned to the radius and the next converged point is then obtained as the point of intersection between the equilibrium path and that sphere.

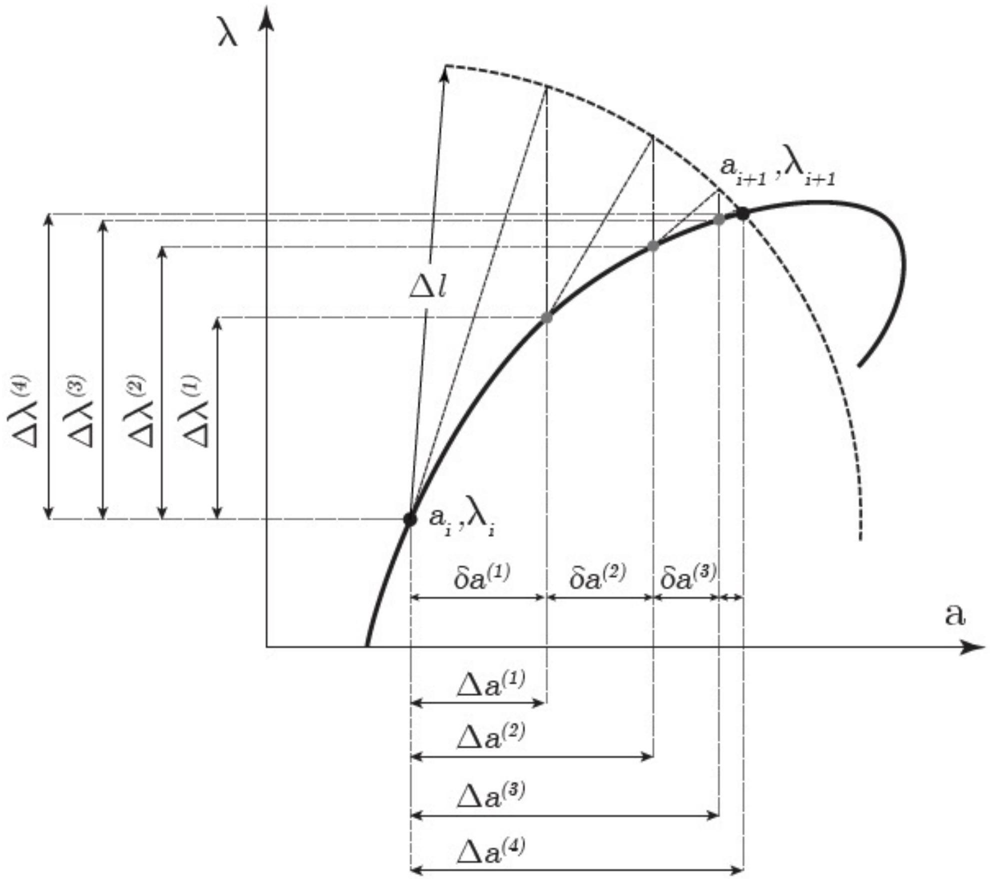
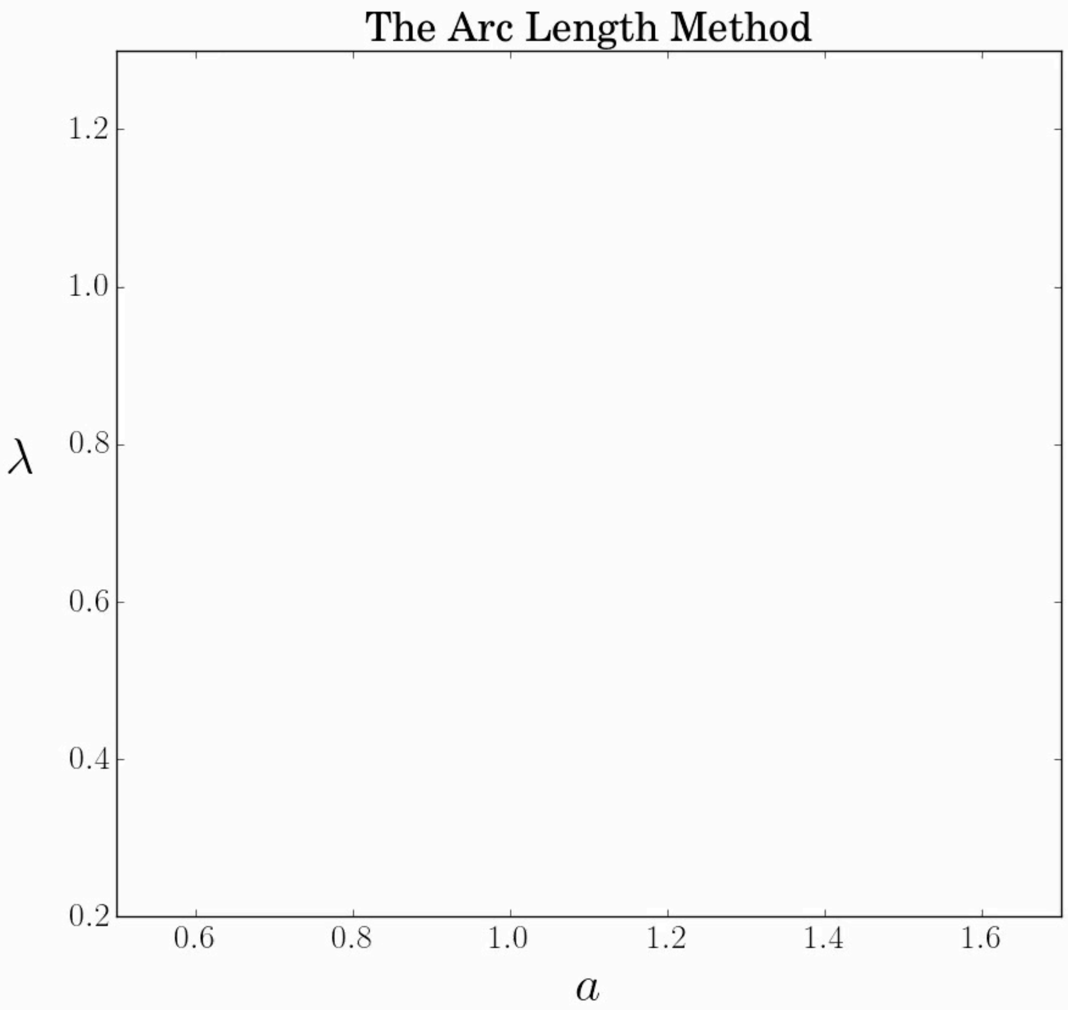


Image source: Vasios (2015)

The Arc Length Method



Video source: Vasios (2015)

Going Back to the Arch Problem Now…

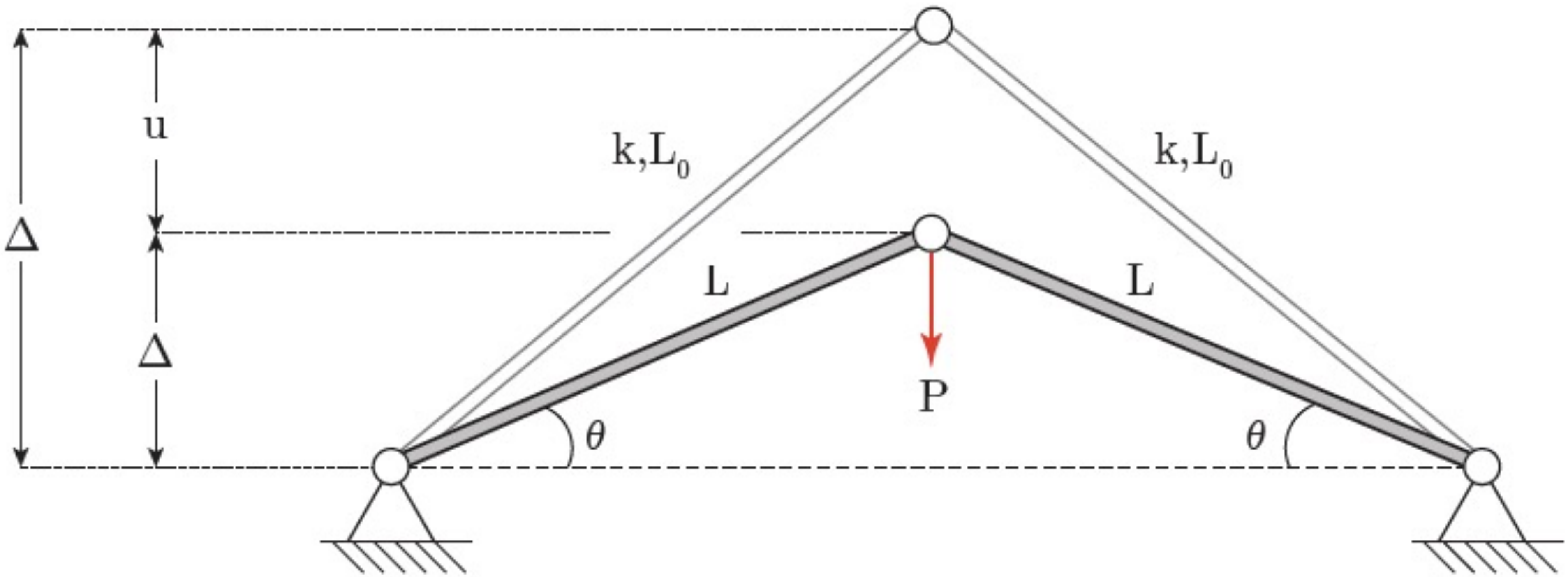


Image source: Vasios (2015)

Going Back to the Arch Problem Now…

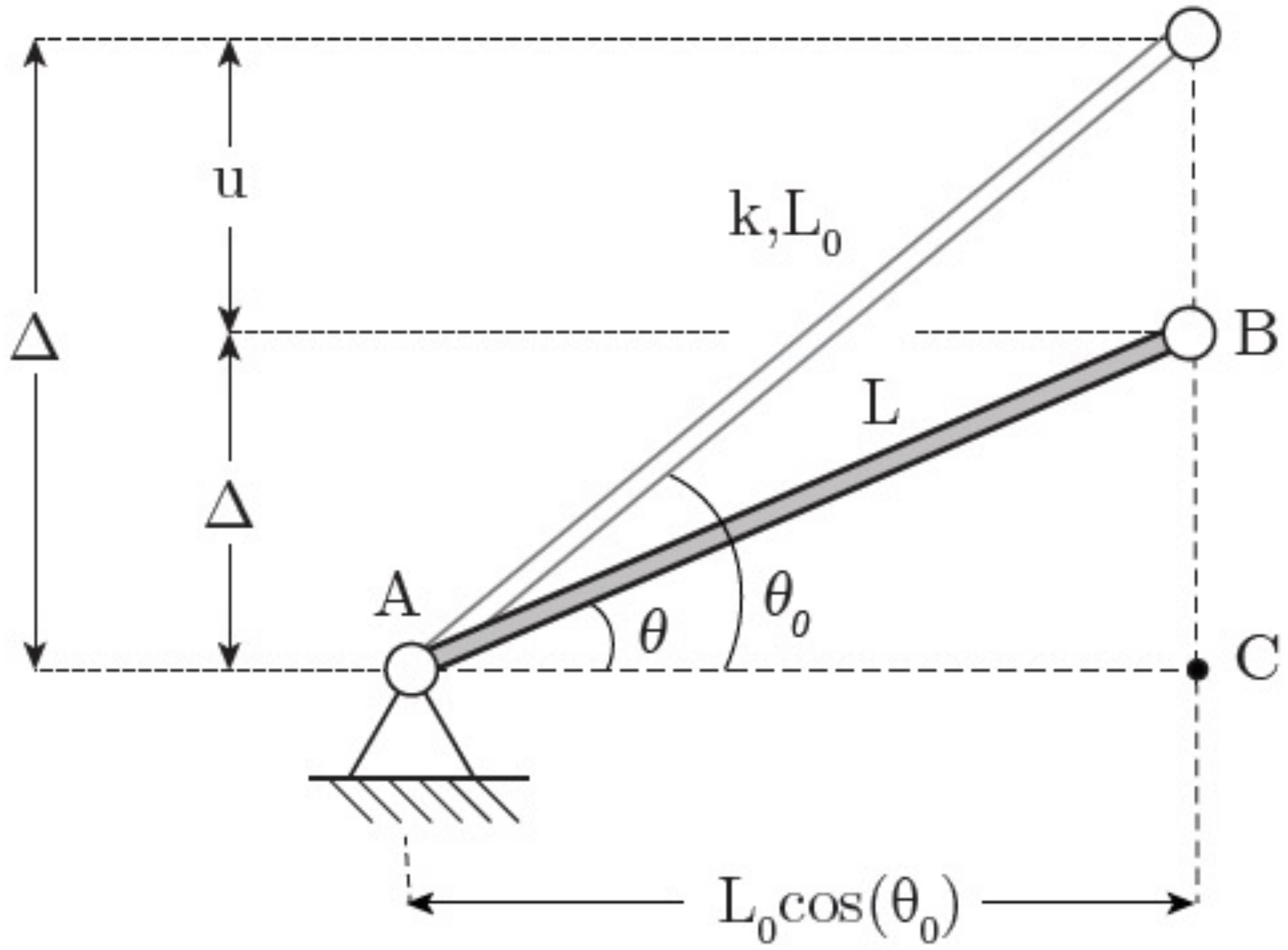
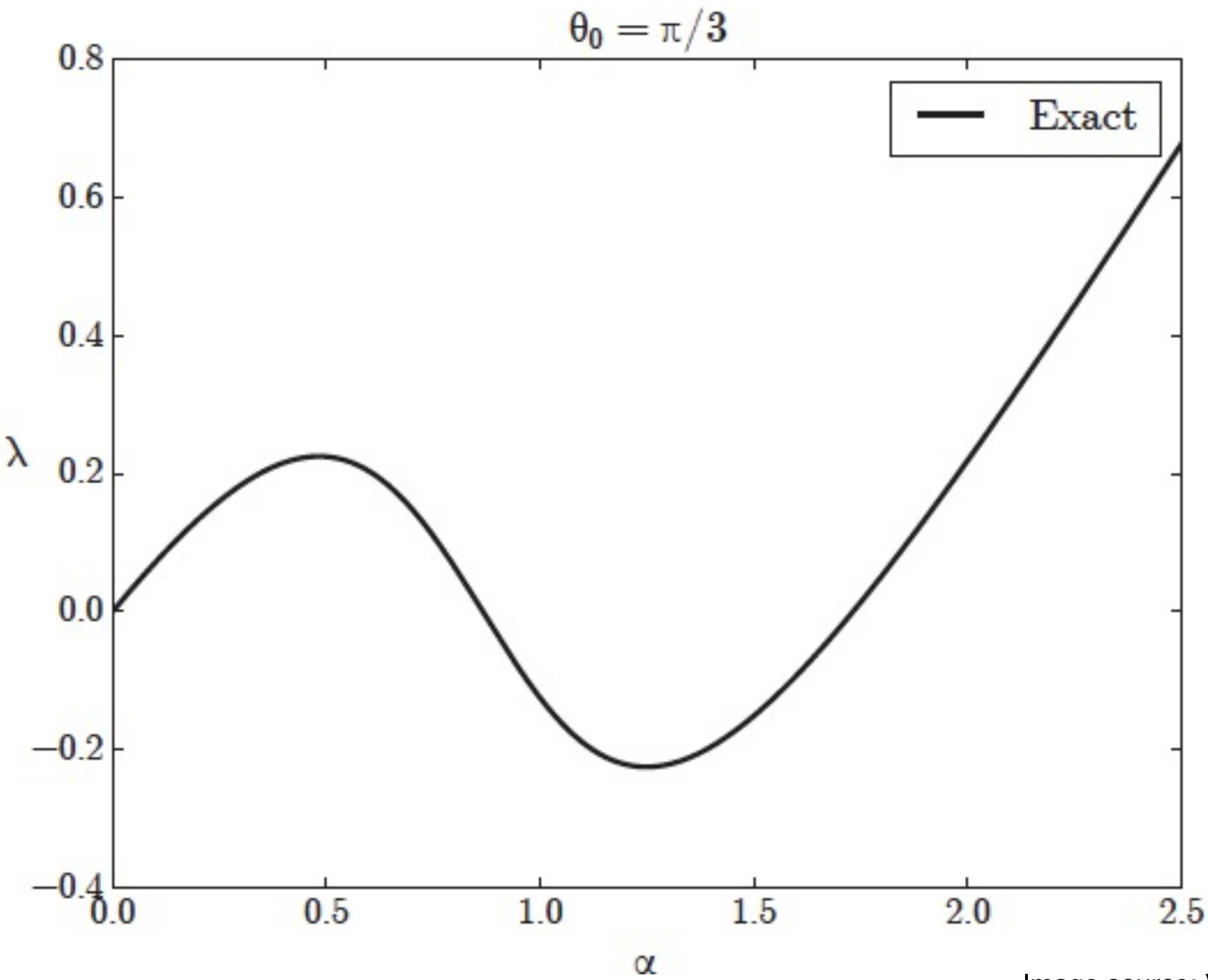


Image source: Vasios (2015)

Going Back to the Arch Problem Now…



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Going Back to the Arch Problem Now…

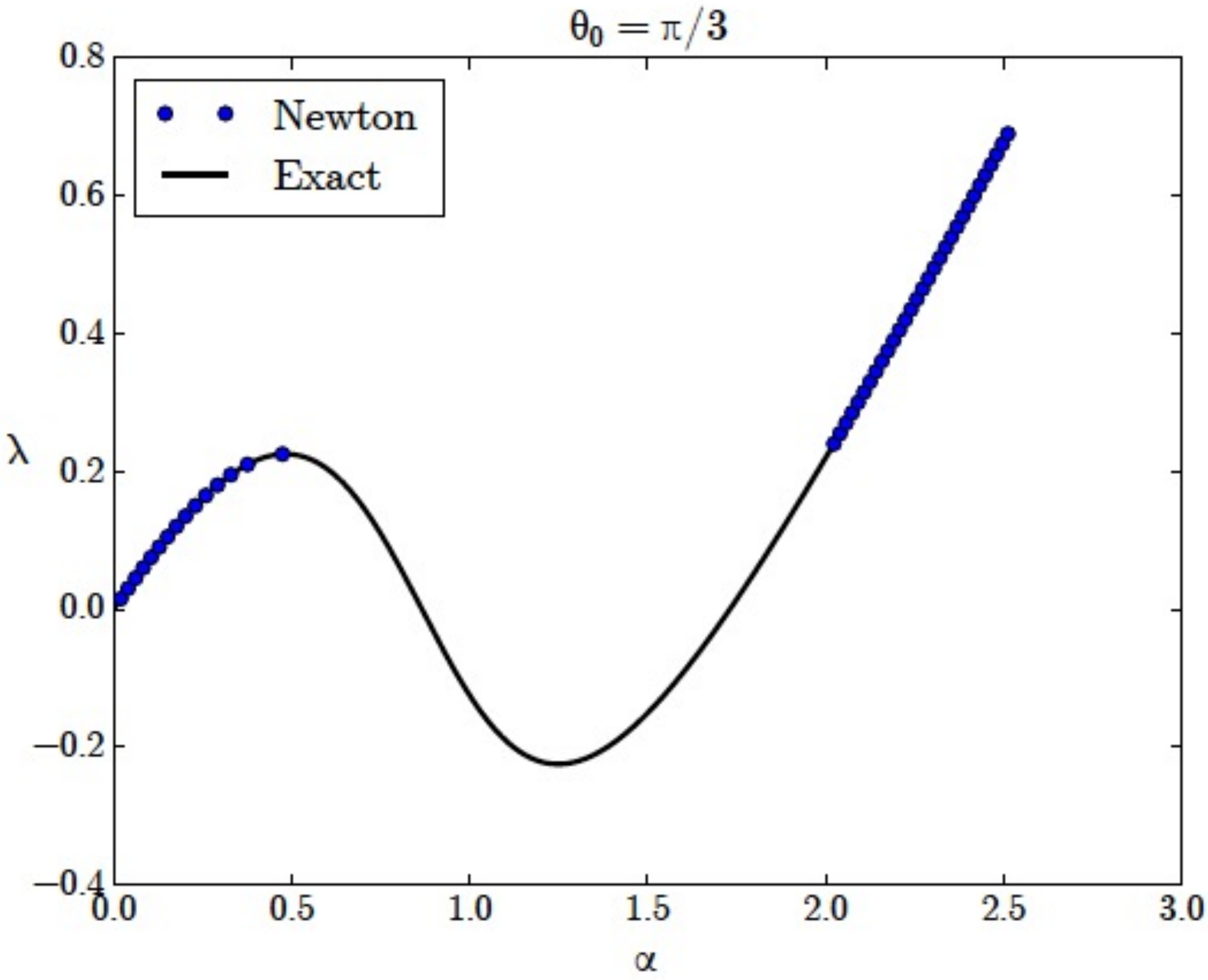

Image source: Vasios (2015)

Going Back to the Arch Problem Now…

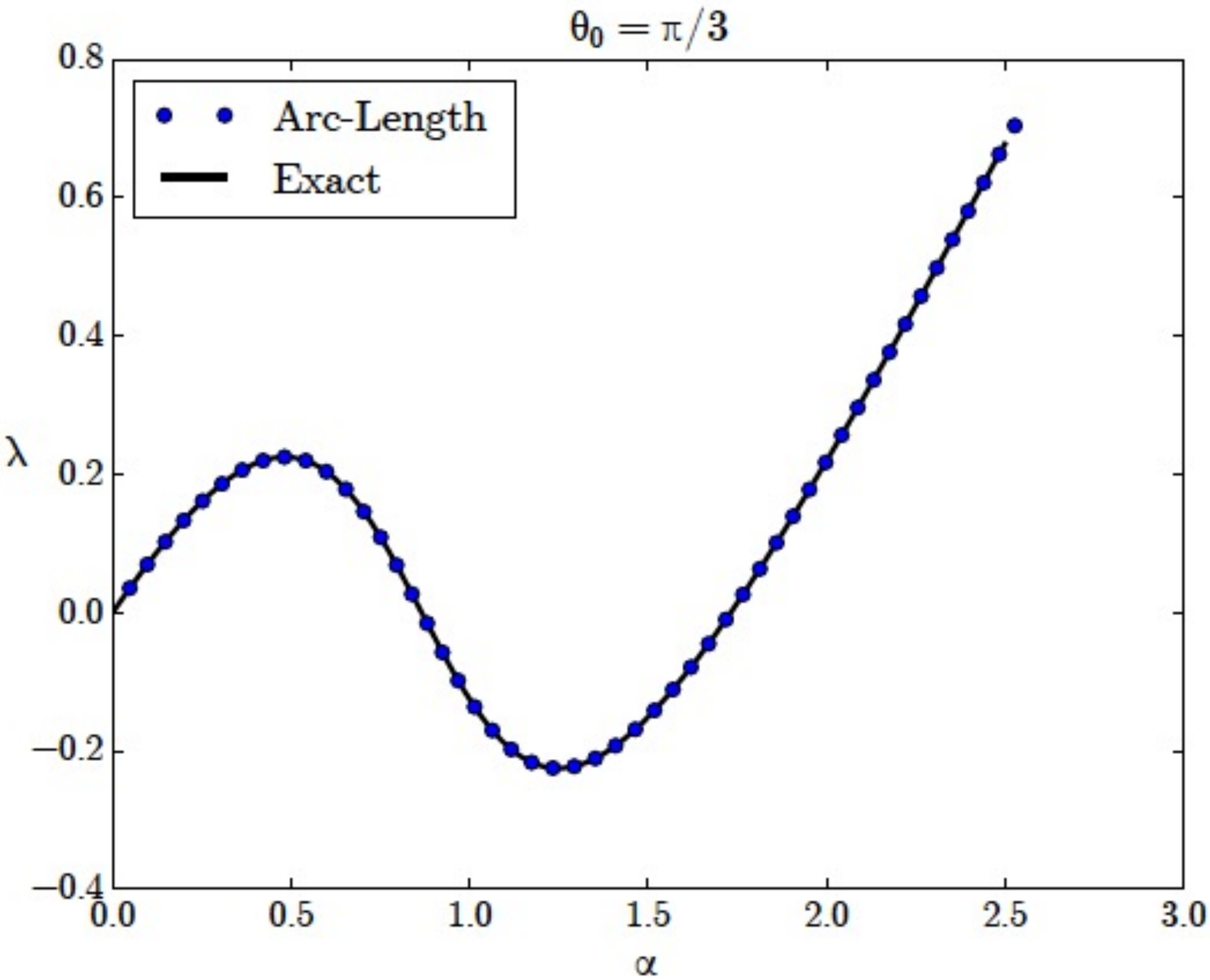

Image source: Vasios (2015)

The Arc Length Method (Crisfield 1983)

- This is an alternative formulation for the Arc-Length method, which could be readily implemented in any commercial finite element software that was able to solve nonlinear problems using Newton's method. Recall, Equation (15)

$$\delta\mathbf{u} = -[K_T]^{-1}_{\mathbf{u}_0+\Delta\mathbf{u}} \cdot \underbrace{\left[\mathbf{F}^{int}(\mathbf{u_0}+\Delta\mathbf{u}) - (\lambda_0+\Delta\lambda)\mathbf{q}\right]}_{\delta\bar{\mathbf{u}}} + \delta\lambda\underbrace{\left([K_T]^{-1}_{\mathbf{u}_0+\Delta\mathbf{u}} \cdot \mathbf{q}\right)}_{\delta\mathbf{u}_t} \quad (18)$$

- The above quantities can be calculated immediately because they only require known information. Once the displacement correction is expressed with the equation above, it can be substituted in the arc-length Equation (15).

The Arc Length Method (Crisfield 1983)

✧ Doing so, would ultimately lead to:

$$a_1 \delta\lambda^2 + a_2 \delta\lambda + a_3 = 0 \tag{19}$$

✧ Where the coefficients a_1, a_2 and a_3 are given by,

$$a_1 = \delta\mathbf{u}^T \cdot \delta\mathbf{u} + \psi^2(\mathbf{q}^T \cdot \mathbf{q})$$

$$a_2 = 2(\Delta\mathbf{u} + \delta\bar{\mathbf{u}})^T \cdot \delta\mathbf{u}_t + 2\psi^2\Delta\lambda(\mathbf{q}^T \cdot \mathbf{q})$$

$$a_3 = (\Delta\mathbf{u} + \delta\bar{\mathbf{u}})^T \cdot (\Delta\mathbf{u} + \delta\bar{\mathbf{u}}) + \psi^2\Delta\lambda^2(\mathbf{q}^T \cdot \mathbf{q}) - \Delta l^2$$

✧ Equation (19) can be easily solved to find $\delta\lambda$. Once this is known, it can be substituted in Equation (18) to update the displacement variation and complete the iteration.

The Arc Length Method (Crisfield 1983)

✧ The method's drawback is related to the two distinct solutions that we obtain from Equation (19). The solver will determine two sets of solutions, namely $(\delta\mathbf{u}_1, \delta\lambda_1)$ and $(\delta\mathbf{u}_2, \delta\lambda_2)$.

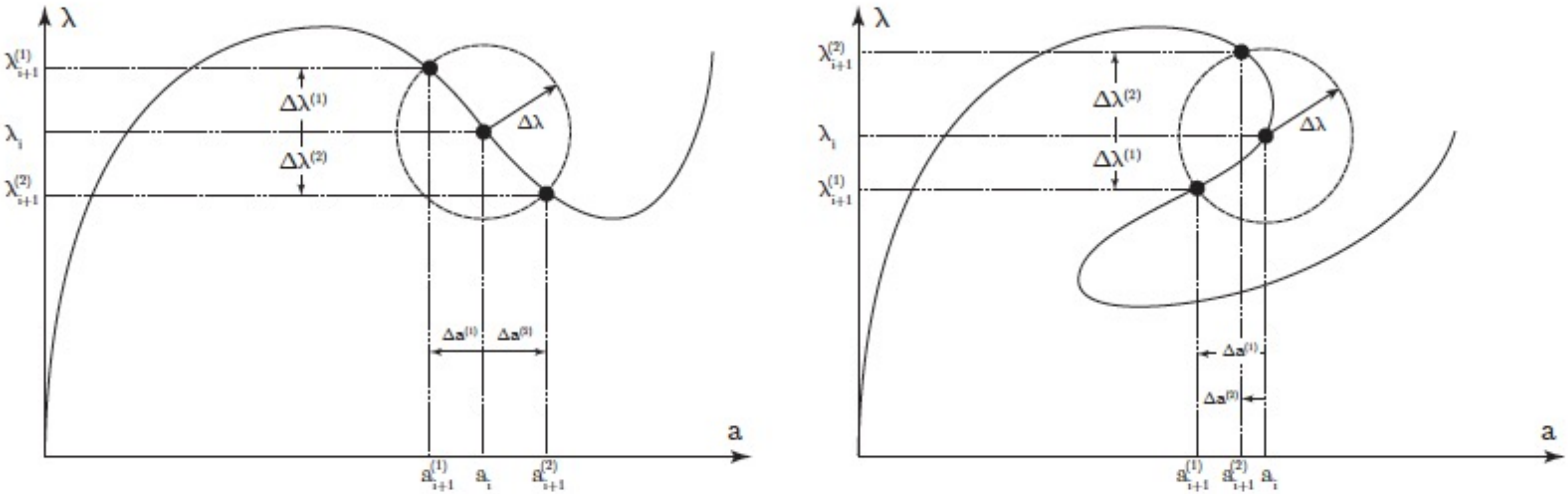

Image source: Vasios (2015)

The Arc Length Method (Crisfield 1983)

✧ The issue that arises then, is the develop a robust algorithm that would be able to accurately determine the correct set of $(\delta \mathbf{u}, \delta\lambda)$ to update the solution. We would like to choose the solution that the algorithm evolves forward. This is very challenging in snap-back problems.

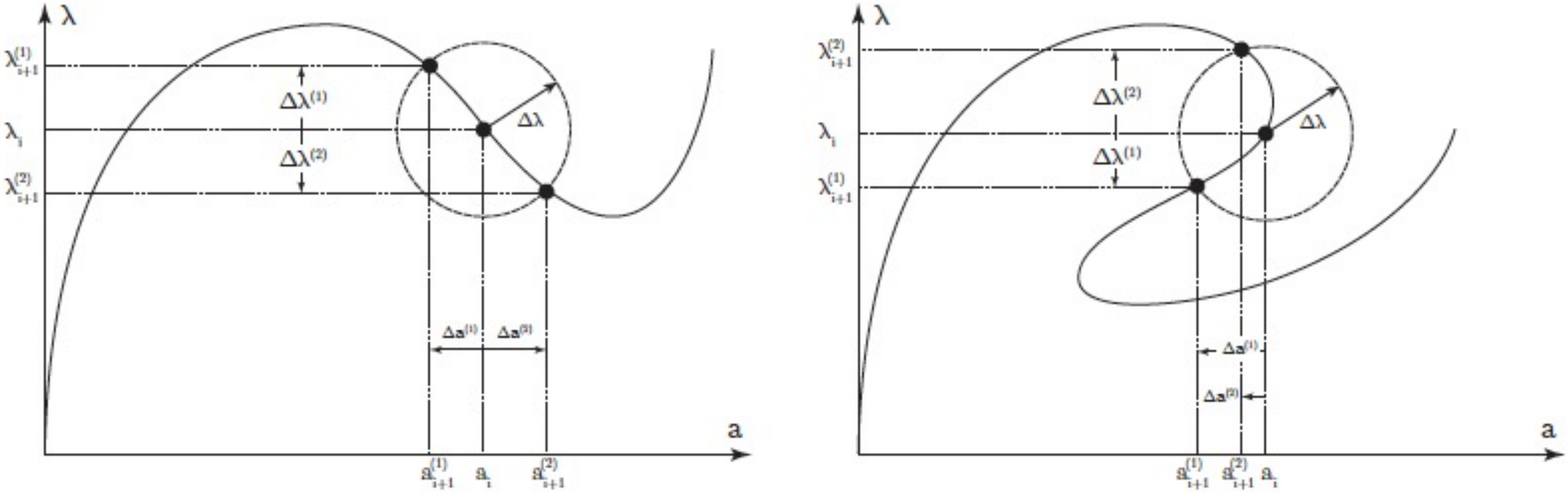

Image source: Vasios (2015)

The Arc Length Method (Crisfield 1983)

- An efficient rule to follow in order to choose the next point correctly even when extreme 'snap-back' cases occur is to compute the two displacement corrections $\delta\mathbf{u}_1$ and $\delta\mathbf{u}_2$ corresponding to $\delta\lambda_1$ and $\delta\lambda_2$, respectively.
- Subsequently, we calculate the projections (dot-products) of these generalized correction vectors on the previous corrections.
- We eventually choose the $\delta\lambda$ and the corresponding $\delta\mathbf{u}$ that lead to the largest value for the dot product and thus form the closest correction to the previous one (hoping that it will be in the right direction).

The Arc Length Method (Crisfield 1983)

✧ In math form:

$$DOT^{(i)} = (\Delta\mathbf{u} + \delta\mathbf{u}^i, \lambda + \Delta\lambda + \delta\lambda_i) \cdot (\Delta\mathbf{u}, \lambda + \Delta\lambda) \Longrightarrow$$

$$DOT^{(i)} = \left(\Delta\mathbf{u} + \delta\mathbf{u}^i\right)^T \cdot \Delta\mathbf{u} + \psi^2 \Delta\lambda(\Delta\lambda + \delta\lambda)(\mathrm{q}^T \cdot \mathrm{q}), i = 1{,}2$$

a) Every converged increment store the converged displacement and load correction as $(\Delta\mathbf{u}_n, \Delta\lambda_n)$.

b) Calculate the sign of the dot product.

c) We choose the $\delta\lambda$ that leads to the largest DOT product and thus is closer to the previous correction.

d) In the special case where the two solutions given the same dot products, then choose either one.