CIVIL 369: “Structural Stability”

School of Architecture, Civil & Environmental Engineering
Civil Engineering Institute
Resilient Steel Structures Laboratory (RESSLab)

Stability of Axially Loaded Members

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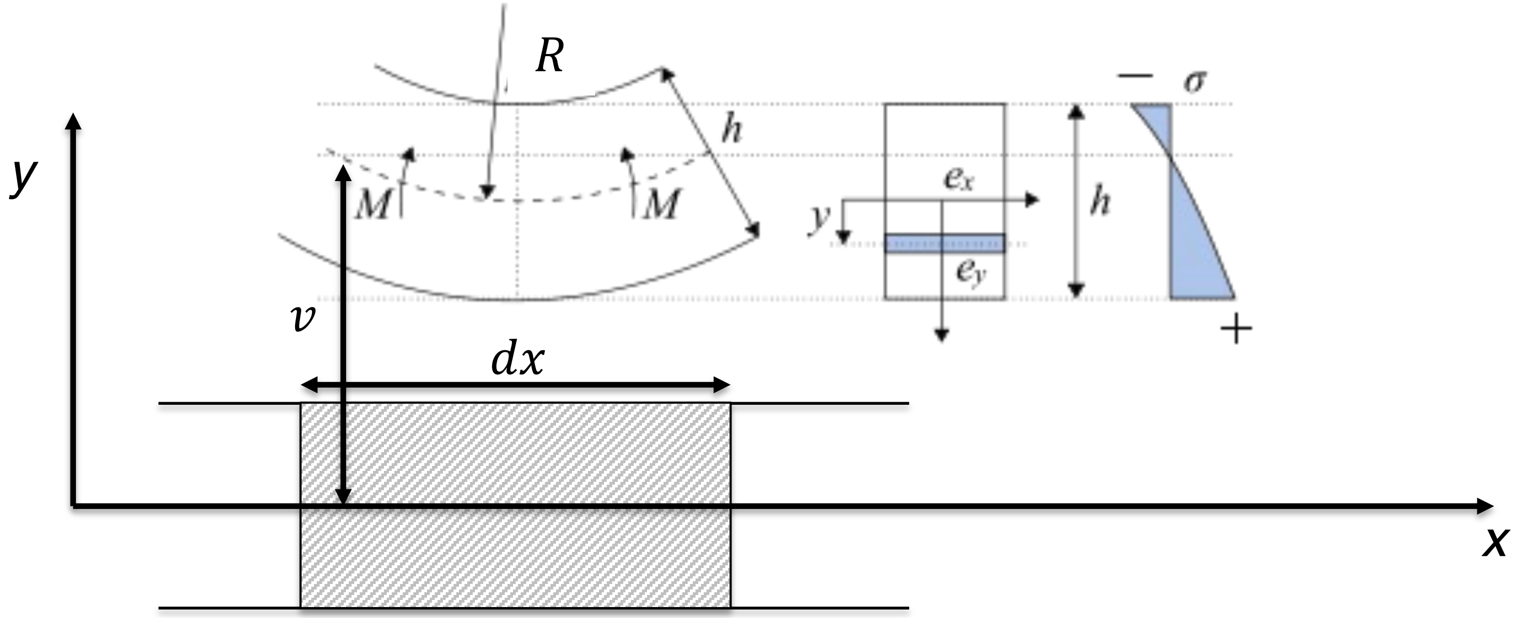
Objectives of This Week's Lecture

To introduce:

- ✧ General theory of beam-columns
  - ✧ Stability of continuous members
- ✧ Beam-column theory in axially loaded members
  - ✧ Buckling determinant
  - ✧ Effect of boundary conditions
  - ✧ Effect of residual stresses
  - ✧ Effect of imperfections
- ✧ Alignment charts for non-sway and sway frames
- ✧ Buckling resistance according to design standards

Theory of Flexure from Structural Mechanics

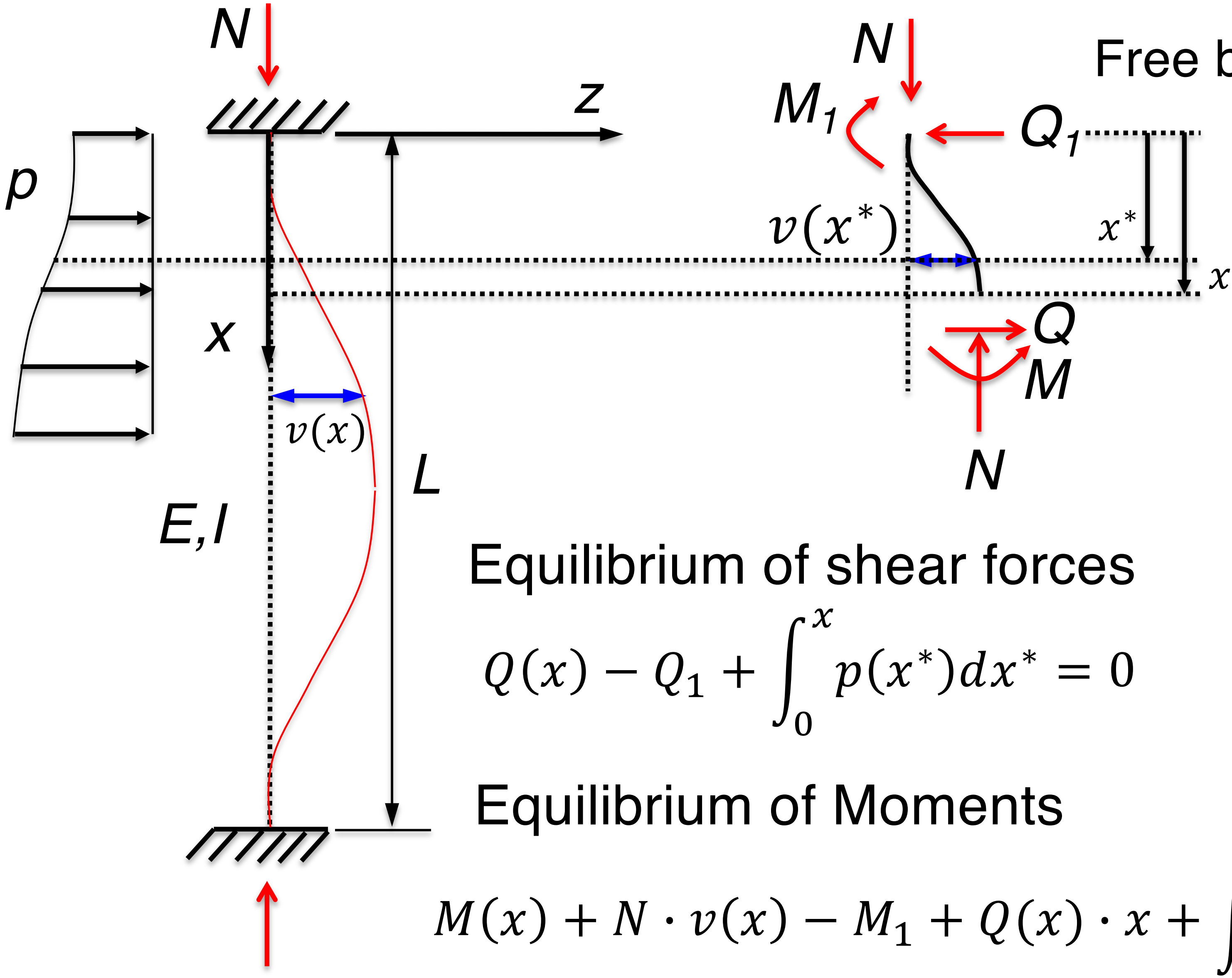


$$M = \int_A \sigma y dA = E \int_0^h y^2 dA/R \Longrightarrow M = E \cdot I/R$$

$$\mathrm{I} = \int_0^h y^2 dA \qquad \frac{1}{R} = \frac{v''}{(1+v'^2)^{\frac{3}{2}}} \cong v''$$

$$\Bigg\} \; M = E \cdot I \cdot v''$$

EPFL Equations of Equilibrium in Beam-Columns



Equilibrium of shear forces

$$Q(x) - Q_1 + \int_0^x p(x^*)dx^* = 0$$

Equilibrium of Moments

$$M(x) + N \cdot v(x) - M_1 + Q(x) \cdot x + \int_0^x p(x^*)x^*dx^* = 0$$

Equations of Equilibrium in Beam-Columns

Equilibrium of a segment of the deflected column for shear & flexure,

$$Q(x) - Q_1 + \int_0^x p(x^*)dx^* = 0$$

$$M(x) + N \cdot v(x) - M_1 + Q \cdot x + \int_0^x p(x^*)x^* dx^* = 0$$

Differentiating the equations above,

$$\frac{\partial Q}{\partial x} + p(x) = 0 \qquad \frac{\partial M}{\partial x} + N\frac{\partial v}{\partial x} + Q = 0 \quad (1)$$

Equilibrium equations of beam-columns in terms of internal forces *M* & *Q*

Equations of Equilibrium in Beam-Columns

The term $N\dfrac{\partial v}{\partial x}$ can be neglected if $N \ll N_{cr}$ *(first order analysis)*

Differentiating Equation (1) from previous slide,

$$\left.\begin{aligned}&\frac{\partial^2 M}{\partial x^2}+\frac{\partial(N\partial v)}{\partial x^2}+\frac{\partial Q}{\partial x}=0\\&\frac{\partial Q}{\partial x}+p(x)=0\end{aligned}\right\}\quad \left.\begin{aligned}&\frac{\partial^2 M}{\partial x^2}+\frac{\partial(N\partial v)}{\partial x^2}=p\\&M=EI\frac{\partial^2 v}{\partial x^2}\end{aligned}\right\}$$

$$\frac{\partial^2(EI\partial^2 v)}{\partial x^4}+\frac{\partial(N\partial v)}{\partial x^2}=p$$

Equations of Equilibrium in Beam-Columns

Assume EI = Constant and N = Constant

$$EI\frac{\partial^4 v}{\partial x^4} + N\frac{\partial^2 v}{\partial x^2} = p$$

for $p = 0$ We have a homogeneous problem with solution,

$$v = e^{\lambda x}$$

$$(EI\lambda^4 + N\lambda^2)e^{\lambda x} = 0$$

Homogeneous problem with solution,

$$\left(\lambda^2 + \frac{N}{EI}\right)\lambda^2 = 0 \quad \left(assume, k^2 = \frac{N}{EI}\right)$$

$$(\lambda^2 + k^2)\lambda^2 = 0$$

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Differential Equation of Planar Flexure

If $N > 0$ (compression) $\lambda = \pm ik \ or \ \lambda = 0$

General solution is,

$$v(x) = A + Bx + Csinkx + Dcoskx + v_p(x) \quad (N > 0)$$

A, B, C, D are constants

$v_p(x)$ is a particular solution corresponding to the transverse distributed load *p(x)*

If $N < 0$ (tension) $\lambda = \pm k \ or \ \lambda = 0, k^2 = \dfrac{|N|}{EI}$

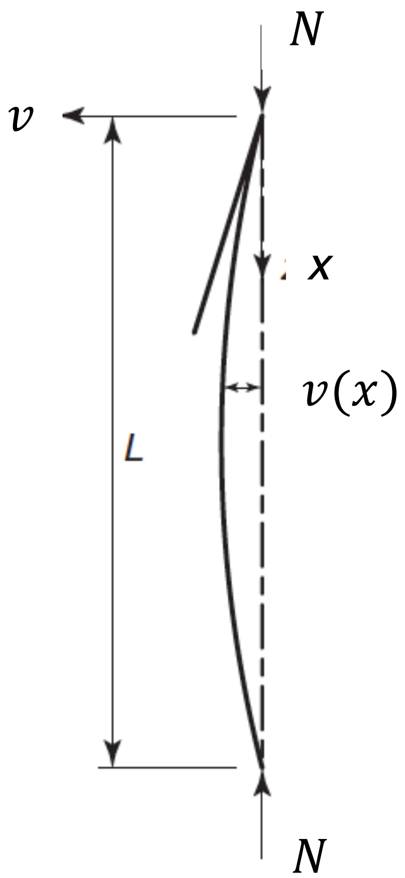
General solution is,

$$v(x) = A + Bx + Csinhkx + Dcoshkx + v_p(x) \quad (N < 0)$$

Boundary Conditions for Beam-Column Equation

Type of Boundary	Initial Conditions to be Imposed	
Fixed End	$v = 0$	$\frac{\partial v}{\partial x} = v' = 0$
Hinge	$v = 0$	$M = 0$ or $\frac{\partial^2 v}{\partial x^2} = v'' = 0$
Free End	$M = 0$	$V = 0$ or $\frac{\partial(EI\partial^2 v)}{\partial x^3} + \frac{N\partial \mathrm{v}}{\partial x} = 0$
Sliding Restraint (pin)	$\frac{\partial v}{\partial x} = v' = 0$	$V = 0$ or $\frac{\partial(EI\partial^2 v)}{\partial x^3} + \frac{N\partial \mathrm{v}}{\partial x} = 0$

Pin-Ended Column with Axial Load ($p = 0$)



$$EI\frac{\partial^4 v}{\partial x^4} + N\frac{\partial^2 v}{\partial x^2} = 0$$

$$v(0) = v(L) = 0$$

$$M(0) = M(L) = 0$$

General solution

$$v(x) = A + Bx + Csinkx + Dcoskx \qquad (N > 0)$$

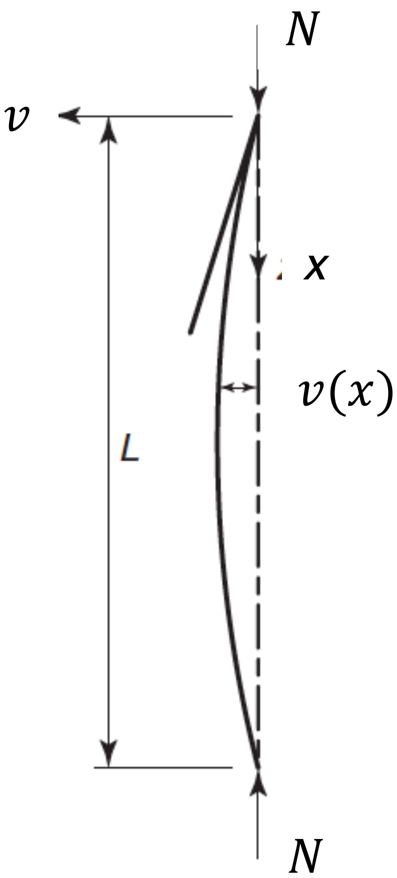
$$v(0) = A \cdot 1 + B \cdot 0 + C \cdot 0 + D = 0$$

$$v''(0) = A \cdot 0 + B \cdot 0 + C \cdot 0 + D(-k^2) = 0$$

$$v(L) = A \cdot 1 + B \cdot L + C \cdot sinkL + DcoskL = 0$$

$$v''(L) = A \cdot 0 + B \cdot 0 + C \cdot (-k^2 sinkL) + D(-k^2 coskL) = 0$$

Pin-Ended Column – Buckling Determinant



$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 0 & -k^2 \sin kL & -k^2 \cos kL \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

Homogeneous system

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 0 & -k^2 \sin kL & -k^2 \cos kL \end{vmatrix} = 0$$

Buckling Determinant

Pin-Ended Column – Buckling Determinant

-Determination of Critical Load, N_{cr}

$$1 \times \begin{vmatrix} 0 & 0 & -k^2 \\ L & \sin kL & \cos kL \\ 0 & -k^2 \sin kL & -k^2 \cos kL \end{vmatrix} - 0 \times \begin{vmatrix} 0 & 0 & -k^2 \\ 1 & \sin kL & \cos kL \\ 0 & -k^2 \sin kL & -k^2 \cos kL \end{vmatrix}$$

$$+0 \times \begin{vmatrix} 0 & 0 & -k^2 \\ 1 & L & \cos kL \\ 0 & 0 & -k^2 \cos kL \end{vmatrix} - 1 \times \begin{vmatrix} 0 & 0 & 0 \\ 1 & L & \sin kL \\ 0 & 0 & -k^2 \sin kL \end{vmatrix} = 0$$

$$sinkL = 0 \Longrightarrow kL = n\pi \Longrightarrow N_{cr} = \frac{n^2 \pi^2 EI}{L^2}, n = 1,2,3, \ldots$$

Pin-Ended Column – Buckling Determinant
-Determination of Buckling Shape (Eigenvector)

Substitute $kL = n\pi$ into the original simultaneous equations,

$$v = C sin kL = 0 \Longrightarrow v(x) = C sin \frac{n\pi x}{L}, n = 1,2,3, \ldots$$

The exact value *C* cannot be determined (remember we are solving a homogeneous system).

C is the unknown amplitude of the sinusoidal deflected shape of the column subjected to compressive load *N*.

This is called *Eigenvector* or *Eigen mode* of the column. The first three *eigen modes* are shown in the next slide.

Pin-Ended Column – Buckling Determinant

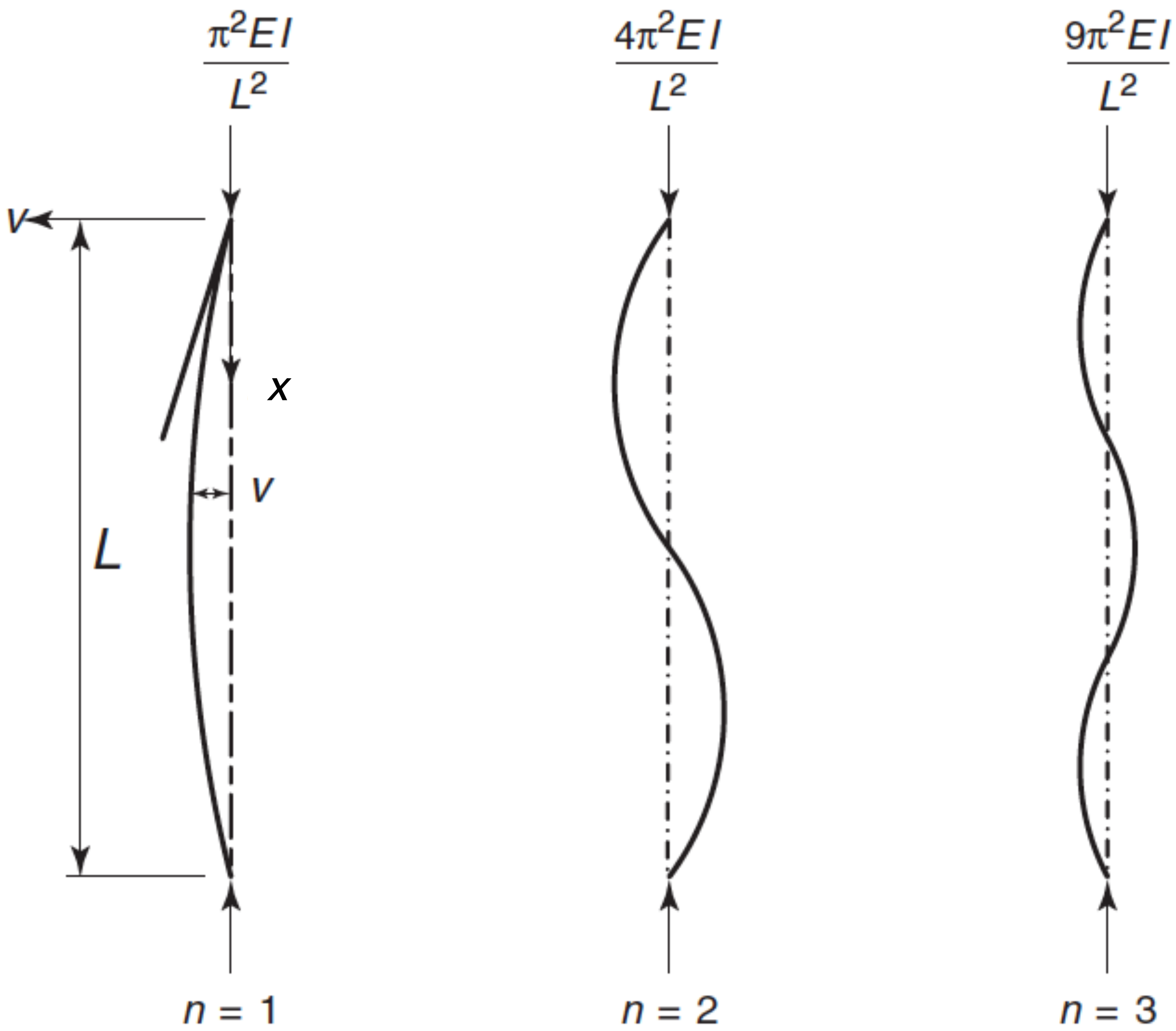
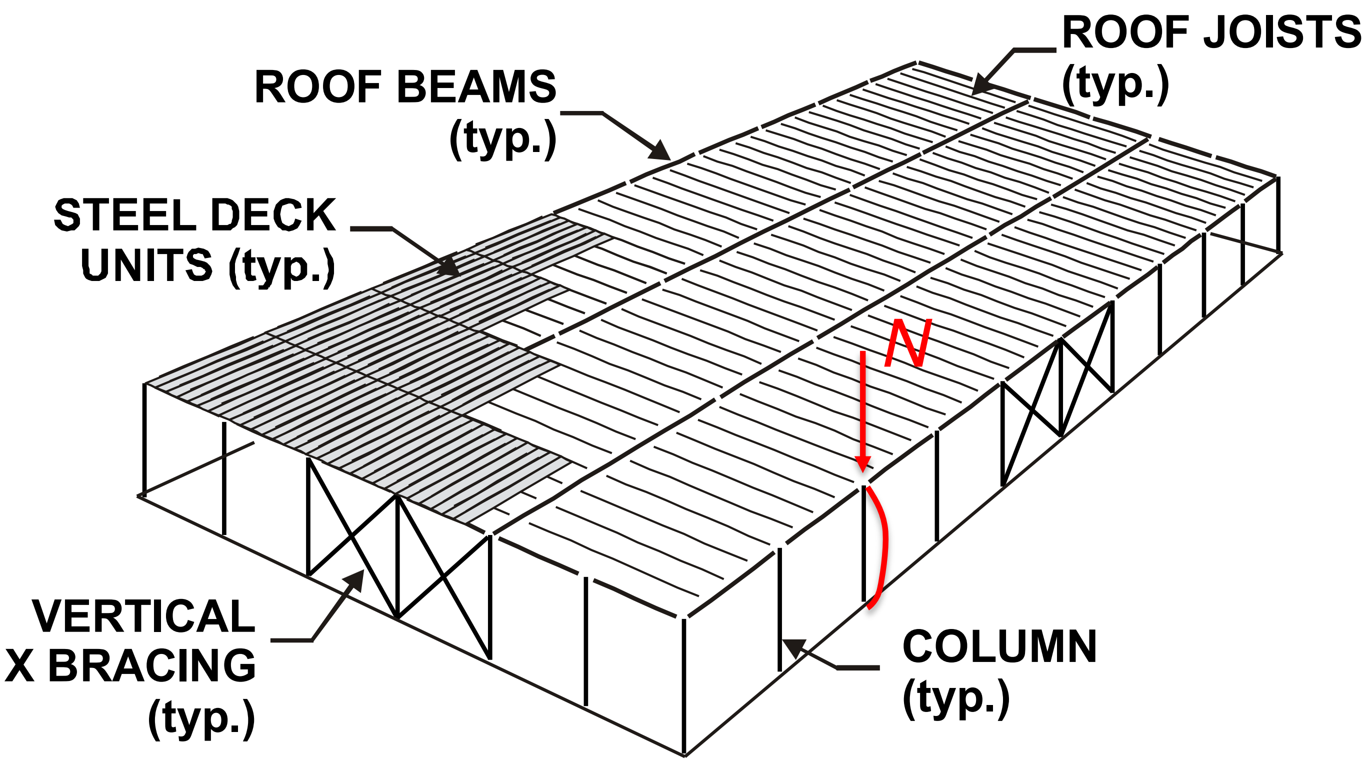


Illustration in Real Buildings



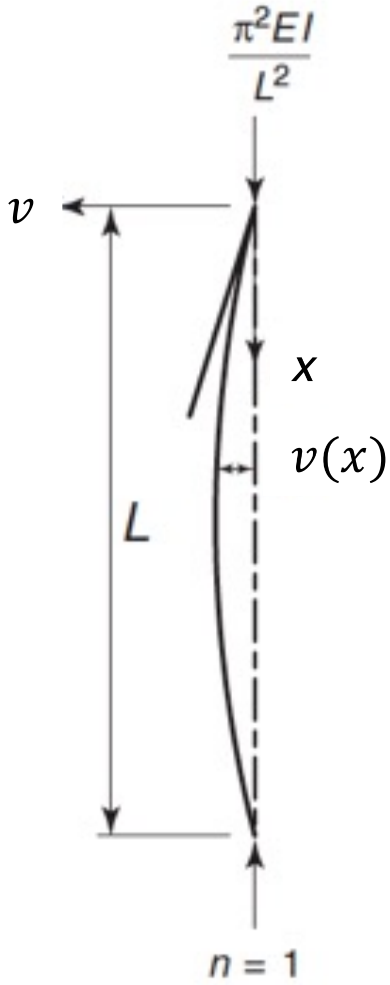
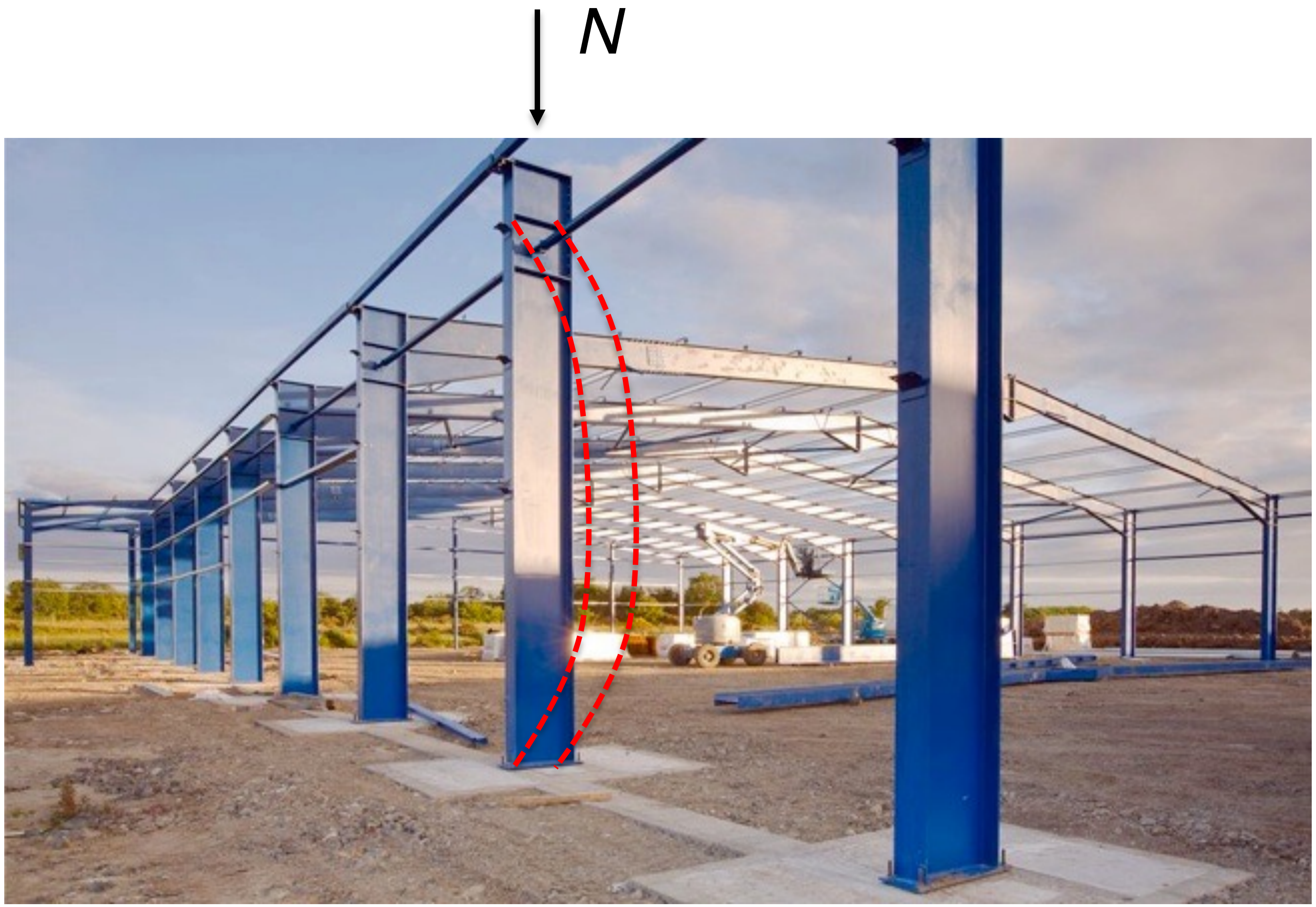
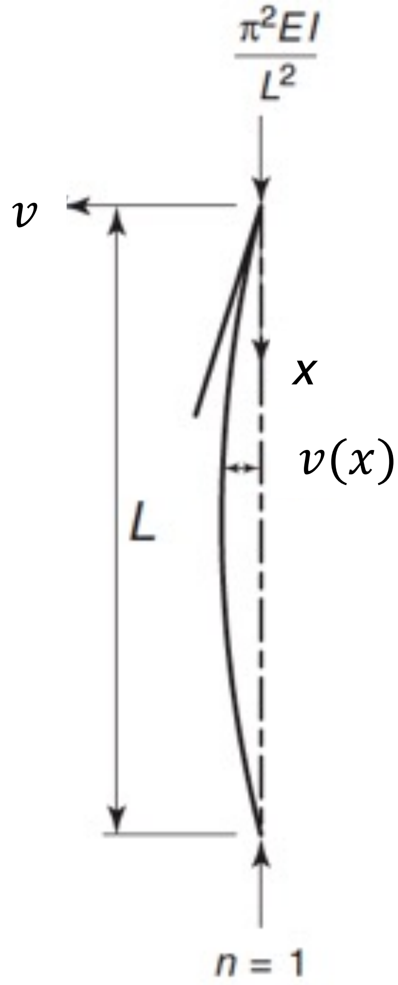
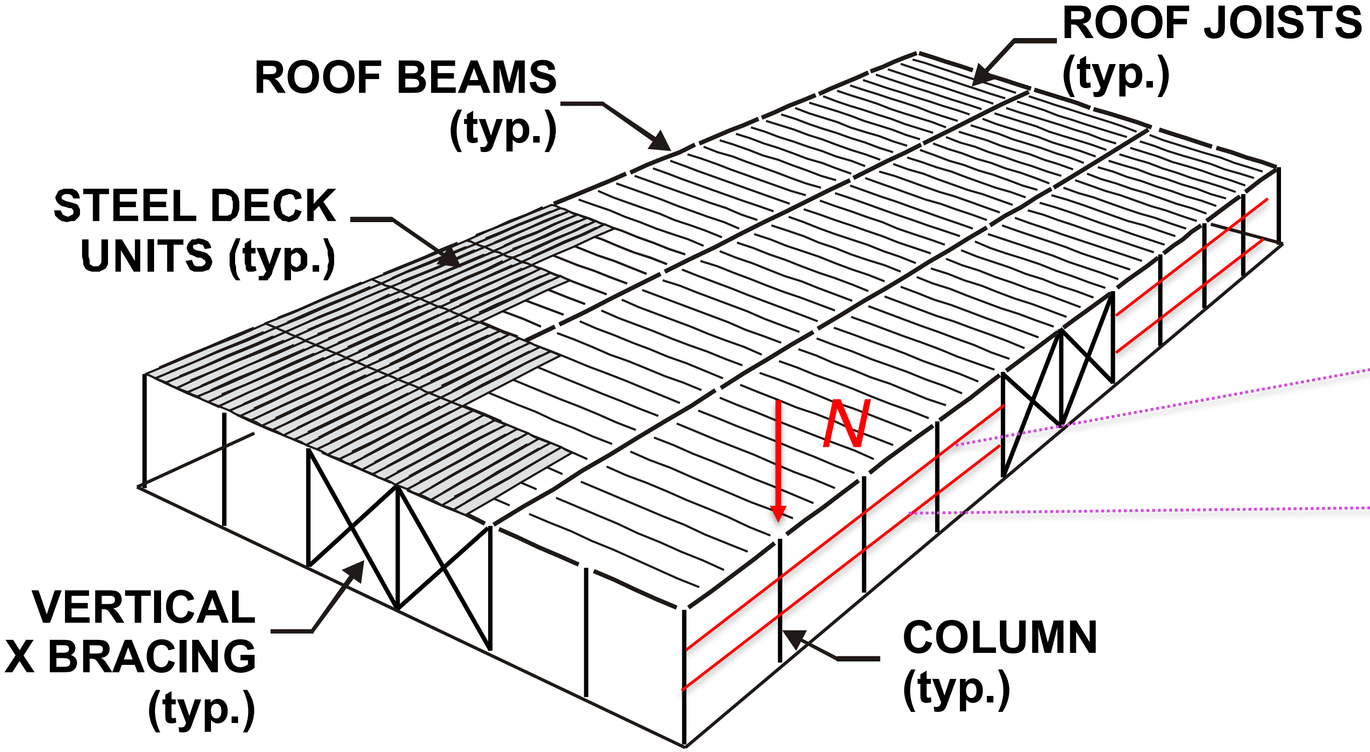


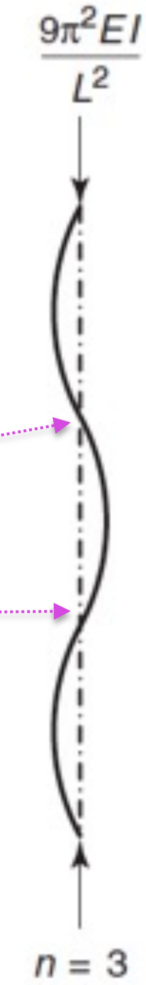
Illustration in Real Buildings





Increase of Critical Buckling Load

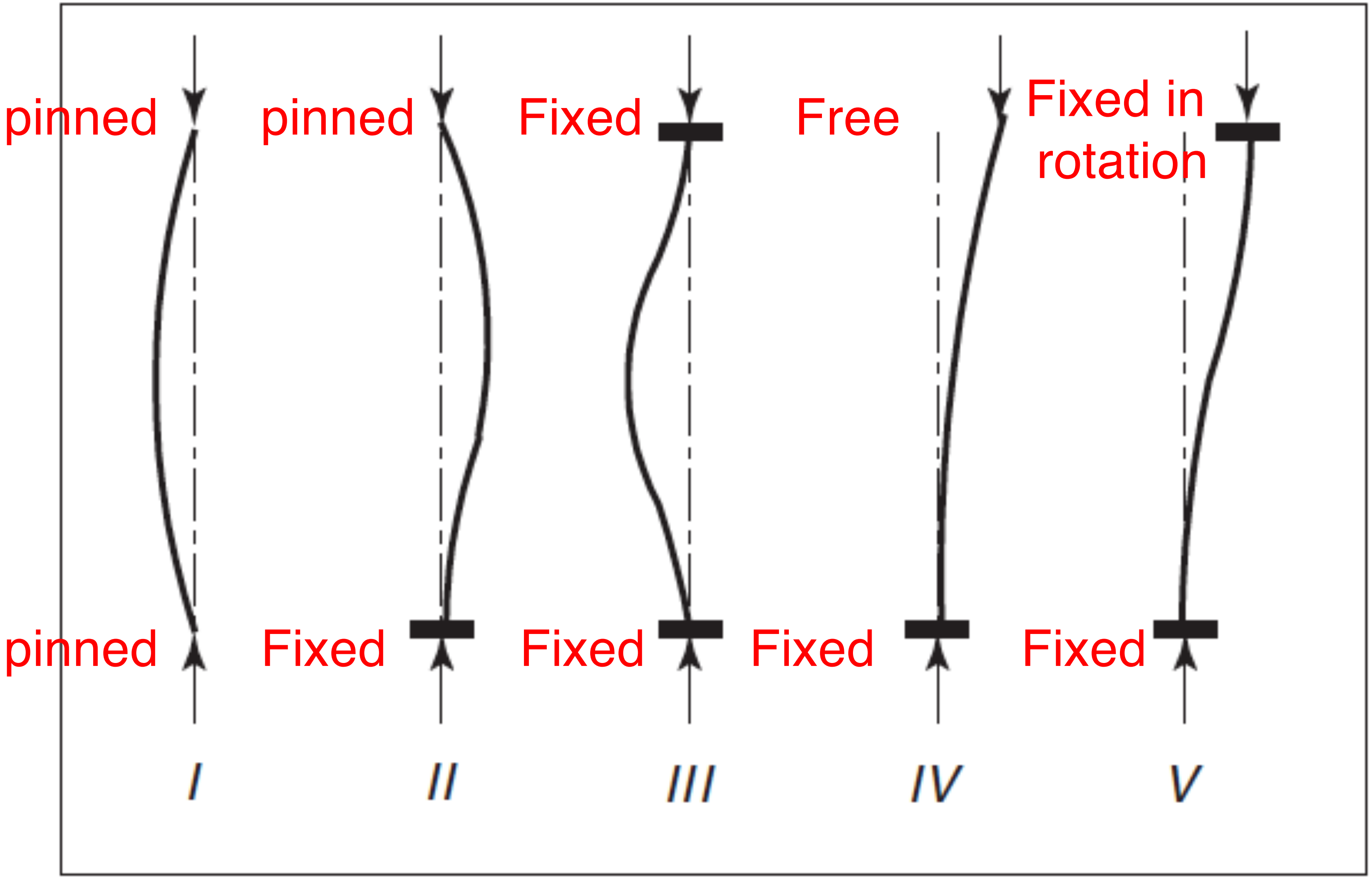




Increase of Critical Buckling Load

Source: http://www.timminsagricultural.co.uk/wp-content/uploads/2012/11/BUILDING-1.jpg

Critical Buckling Loads of Columns with Various End Restraints (no imperfections)



Critical Buckling Loads of Columns with Various End Restraints (no imperfections)

Case	Boundary Conditions	Buckling Determinant	Eigenfunction Eigenvalue Buckling Load	Effective Length Factor
I	$v(0)=v''(0)=0$ $v(L)=v''(L)=0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 0 & -k^2\sin kL & -k^2\cos kL \end{vmatrix}$	$\sin kL = 0$ $kL = \pi$ $N_{cr} = N_E$	1.0
II	$v(0)=v''(0)=0$ $v(L)=v'(L)=0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$	$\tan kl = kl$ $kl = 4.493$ $N_{cr} = 2.045 \cdot N_E$	0.7
III	$v(0)=v'(0)=0$ $v(L)=v'(L)=0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & k & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$	$\sin \frac{kL}{2} = 0$ $kL = 2\pi$ $N_{cr} = 4 \cdot N_E$	0.5
IV	$v'''(0)+k^2v'=v''(0)=0$ $v(L)=v'(L)=0$	$\begin{vmatrix} 0 & 0 & 0 & -k^2 \\ 0 & k^2 & 0 & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$	$\cos kL = 0$ $kL = \frac{\pi}{2}$ $N_{cr} = \frac{N_E}{4}$	2.0
V	$v'''(0)+k^2v'=v'(0)=0$ $v(L)=v'(L)=0$	$\begin{vmatrix} 0 & 1 & k & 0 \\ 0 & k^2 & 0 & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$	$\sin kL = 0$ $kL = \pi$ $N_{cr} = N_E$	1.0

$$N_E = \frac{\pi^2 EI}{L^2}$$

Solution by trial and error (Newton-Raphson)

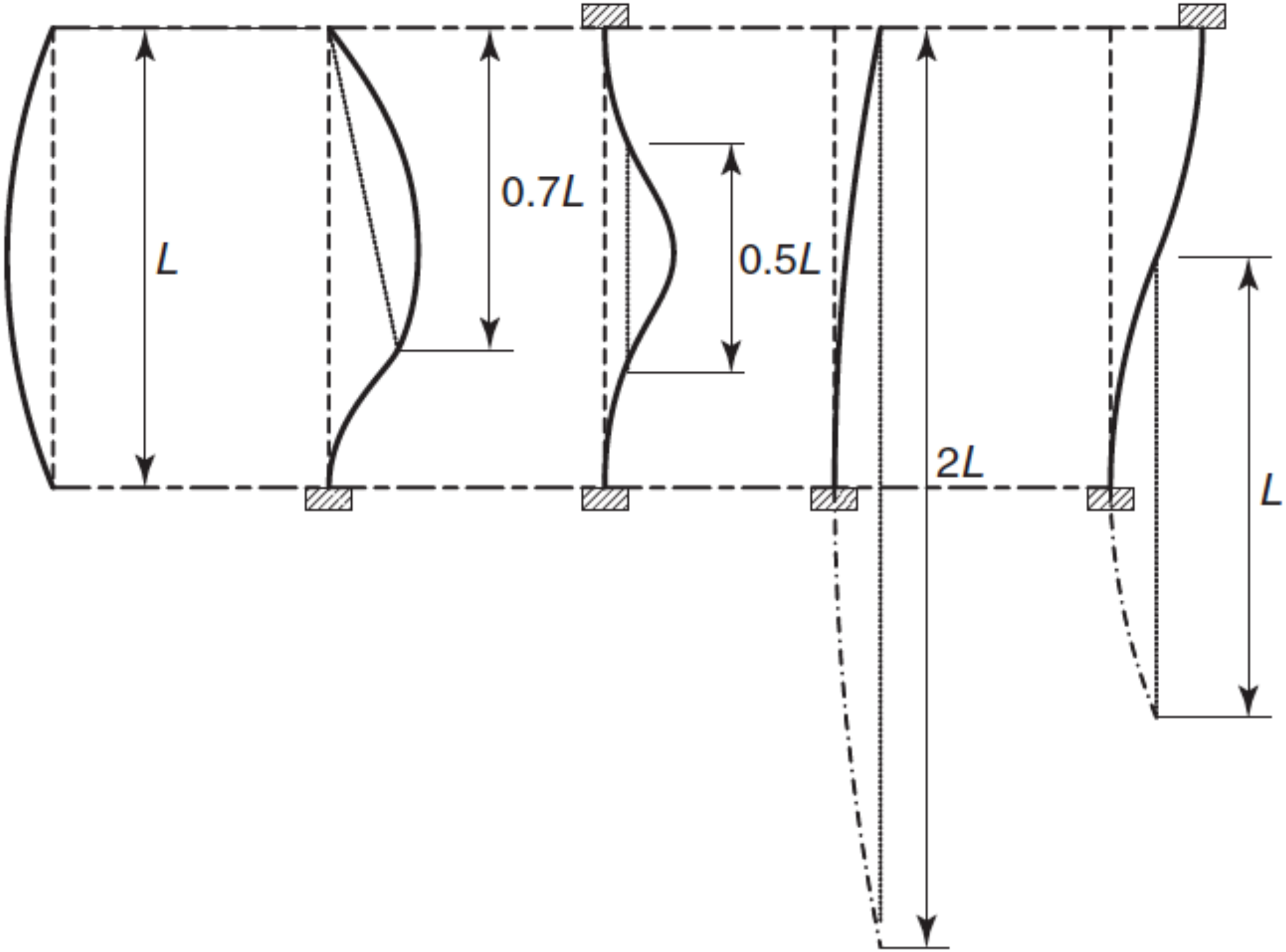
The Concept of the Effective Length Factor

Popular artifice that connects any buckling load to the basic pin-pin case (pinned column) that we just solved:

$$N_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

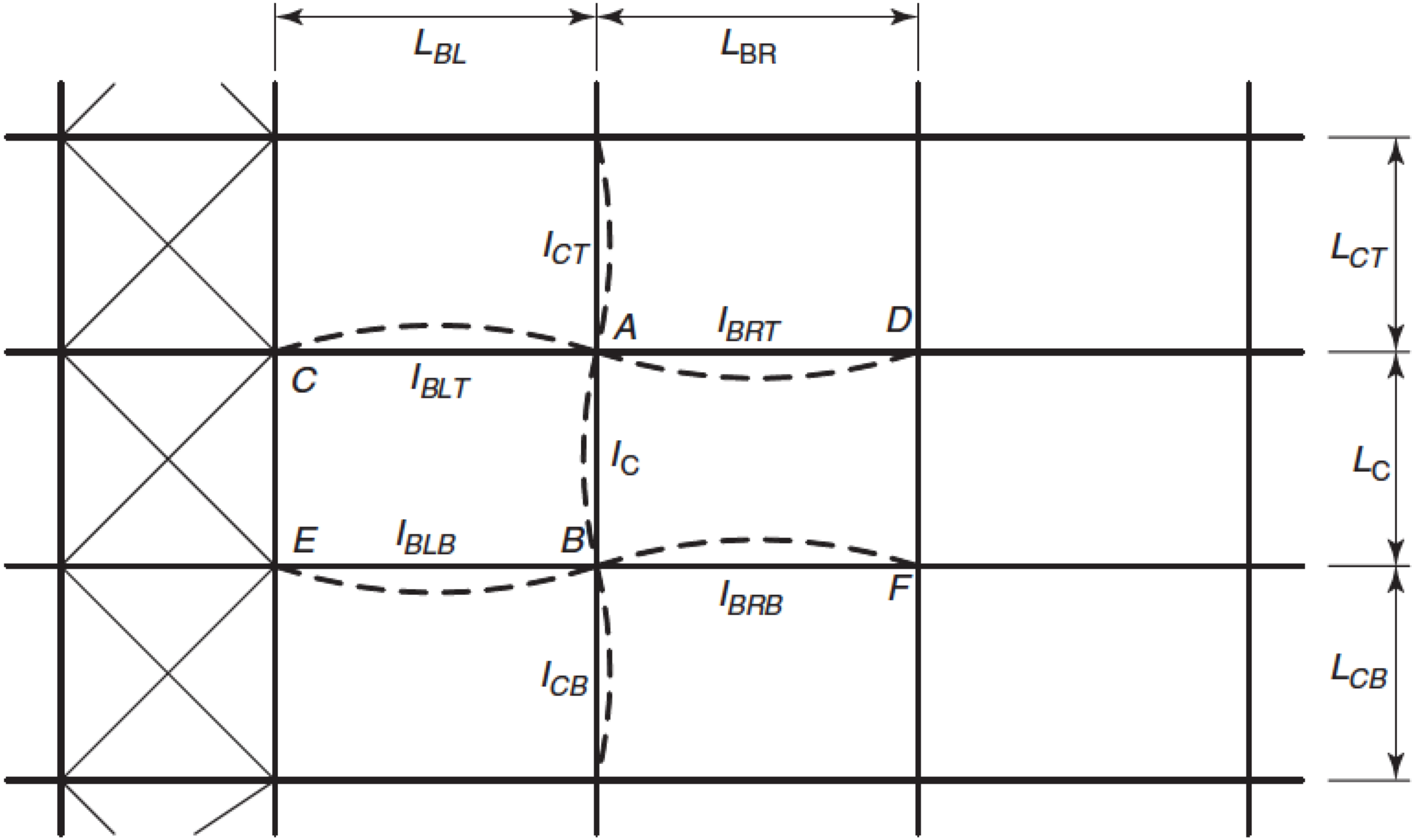
For the 5 elementary cases discussed above, one can visualize the effective length as the distance between points of inflection on the buckled shape of the column.

Effective Length – Geometric Interpretation

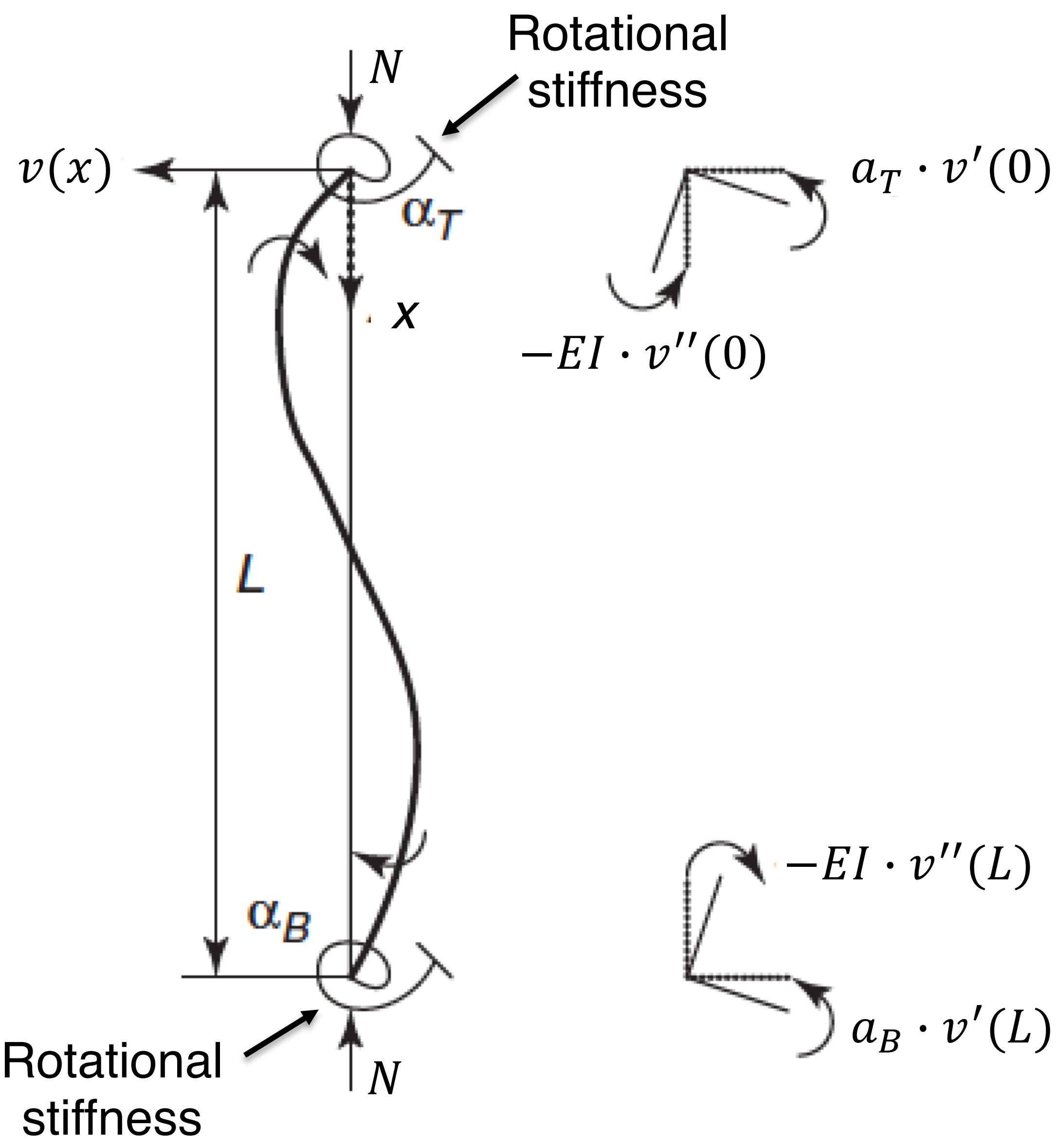


End-Restrained Columns in Multi-Storey Frames

Characteristic in frames with bracings (often called non-sway)



End-Restrained Columns in Multi-Storey Frames



Boundary conditions:

$$v(0) = 0$$

$$a_T \cdot v'(0) - EI \cdot v''(0) = 0$$

$$v(L) = 0$$

$$-a_B \cdot v'(L) - EI \cdot v''(L) = 0$$

End-Restrained Columns in Multi-Storey Frames
-General Solution and Boundary Conditions

General solution,

$$v(x) = A + Bx + Csinkx + Dcoskx \qquad (N > 0)$$

Buckling Determinant

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & L & \sin kL & \cos kL \\ 0 & \alpha_T & \alpha_T k & EIk^2 \\ 0 & -\alpha_B & -\alpha_B k \cos kL + EIk^2 \sin kL & \alpha_B k \sin kL + EIk^2 \cos kL \end{vmatrix} = 0$$

Introduce non-dimensional rigidities,

$$R_T = \frac{a_T L}{EI} \qquad R_B = \frac{a_B L}{EI}$$

End-Restrained Columns in Multi-Storey Frames

-General Solution and Boundary Conditions

$$-2R_T R_B + \sin kL[R_T R_B kL - kL(R_T + R_B) - (kL)^3] + \cos kL[2R_T R_B + (kL)^2(R_T + R_B)] = 0 \quad (1)$$

Pinned Column

$$a_T = a_B = 0 \rightarrow R_T = R_B = 0$$

$$sinkL = 0$$

Fixed Column

$$a_T = a_B = \infty \rightarrow R_T = R_B = \infty$$

$$sin\frac{kL}{2} = 0$$

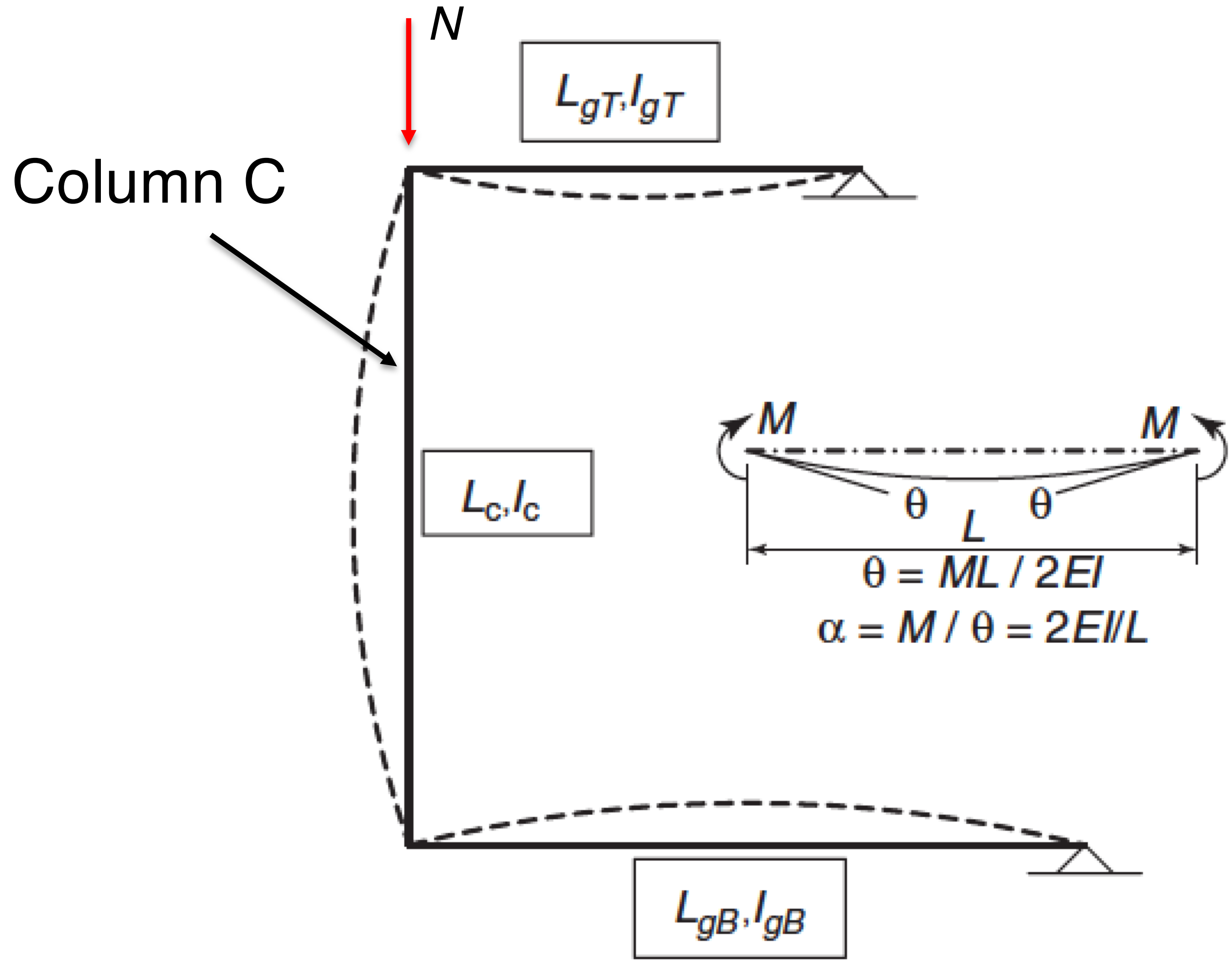
Eq. (1) encloses all the intermediate conditions between totally pinned and fixed ends. Therefore,

$$N_E \leq N_{Cr} \leq 4 \cdot N_E \qquad 0.5 \leq k \leq 1.0$$

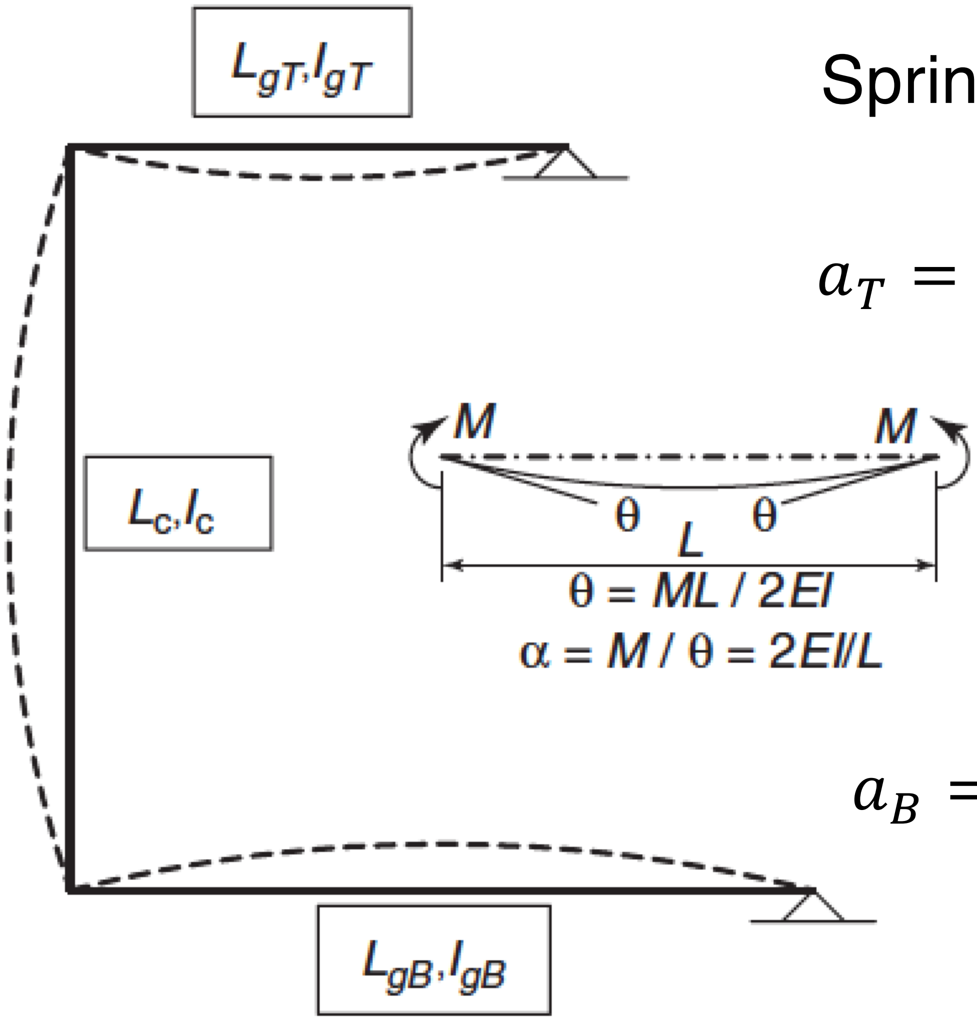
If the elastic rotational spring constants a_T and a_B are known, then the buckling condition of Eq. (1) is directly applicable.

End-Restrained Columns
-Example: Frame with Rigid Connections

Effective length determination for column C



End-Restrained Columns
-Example: Frame with Rigid Connections



Spring Constants

$$a_T = \frac{2EI_{gT}}{L_{gT}}$$

$$a_B = \frac{2EI_{gB}}{L_{gB}}$$

Joint Rigidities

$$n_{sup} = \frac{I_C/L_C}{I_C/L_C + I_{gT}/L_{gT}}$$

$$n_{inf} = \frac{I_C/L_C}{I_C/L_C + I_{gB}/L_{gB}}$$

End-Restrained Columns
-Example: Frame with Rigid Connections

Eq. (1) Becomes

$$\frac{(kL)^2 n_{sup} \cdot n_{inf}}{4} - 1 + \frac{n_{sup} + n_{inf}}{2}\left(1 - \frac{kL}{tankL}\right) + 2tan\frac{kL}{kL} = 0 \ (2)$$

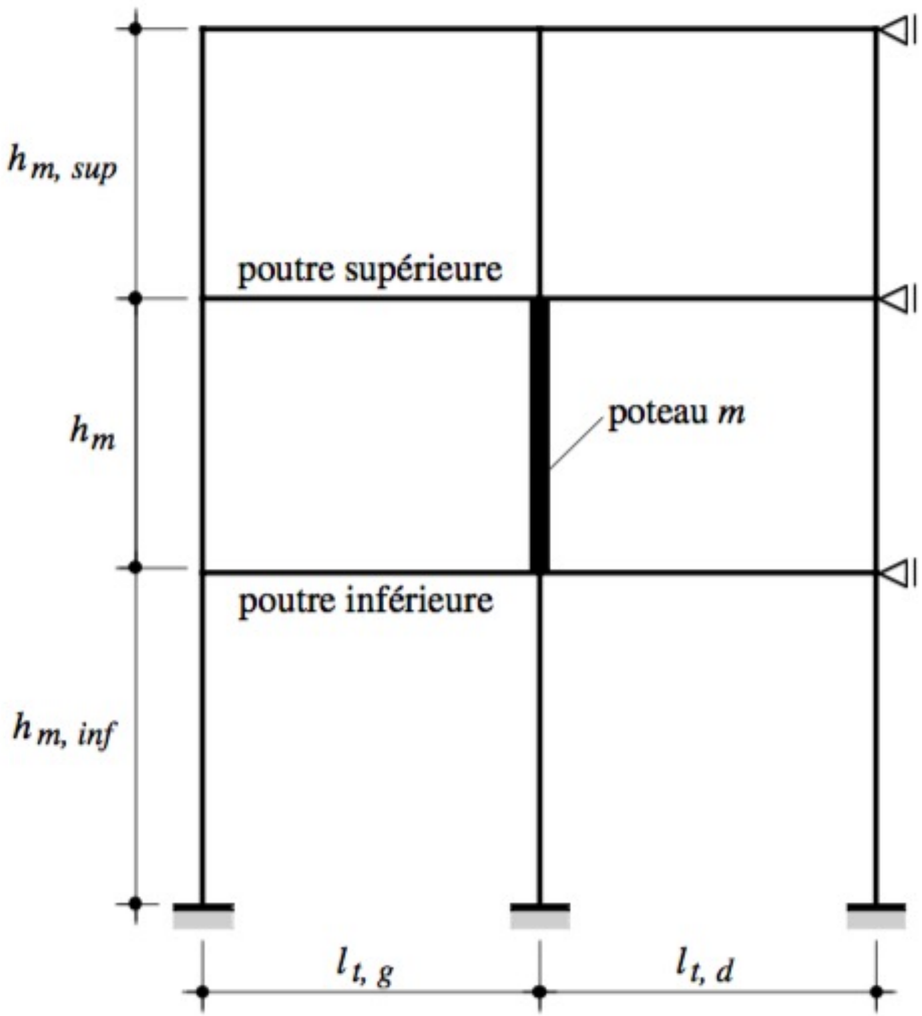
Assume that the effective factor is, $K = \frac{\pi}{kL}$, then Eq (2) becomes,

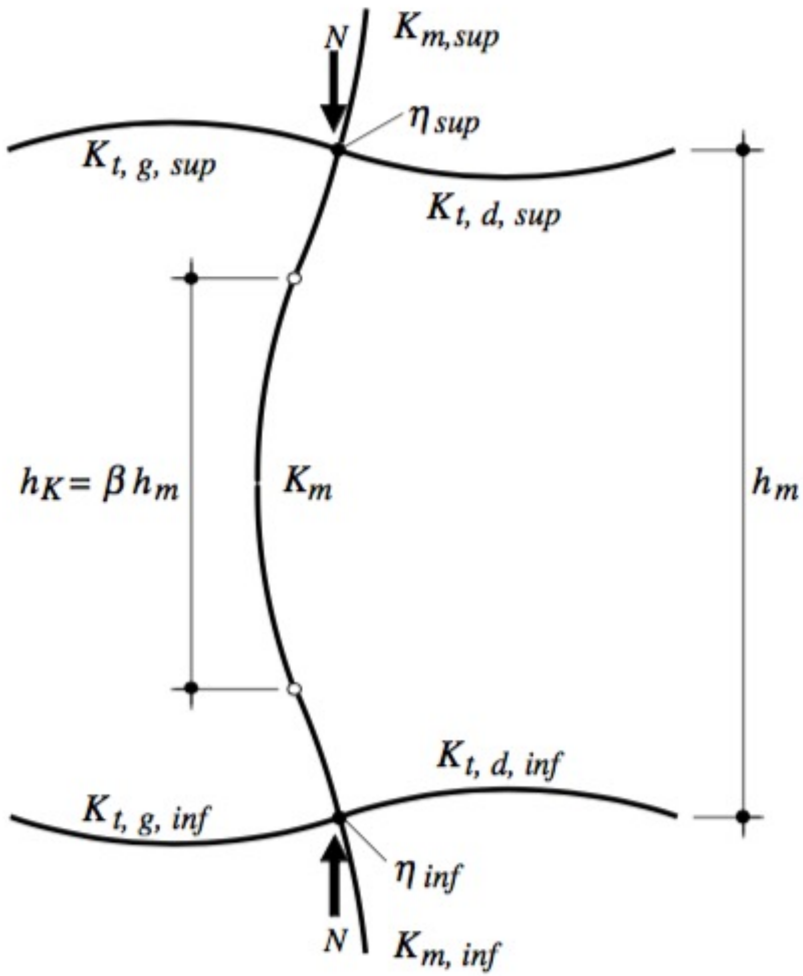
$$\frac{\left(\frac{\pi}{K}\right)^2 n_{sup} \cdot n_{inf}}{4} - 1 + \frac{n_{sup} + n_{inf}}{2}\left(1 - \frac{\frac{\pi}{K}}{tan\frac{\pi}{K}}\right) + \frac{2tan\frac{\pi}{2K}}{\frac{\pi}{K}} = 0 \ (3)$$

The equation above is known as the non-sway *nomograph*

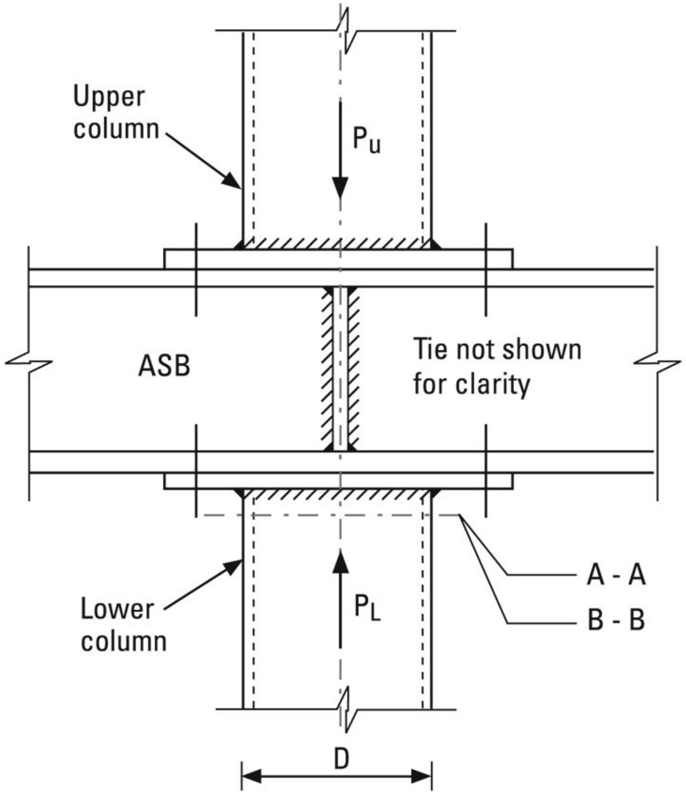
(*alignment chart*) in all the design standards.

End-Restrained Columns
(TGC11-Charpentes Métalliques, Ch.13, p 585)
-Example: Planar Rigid Frame



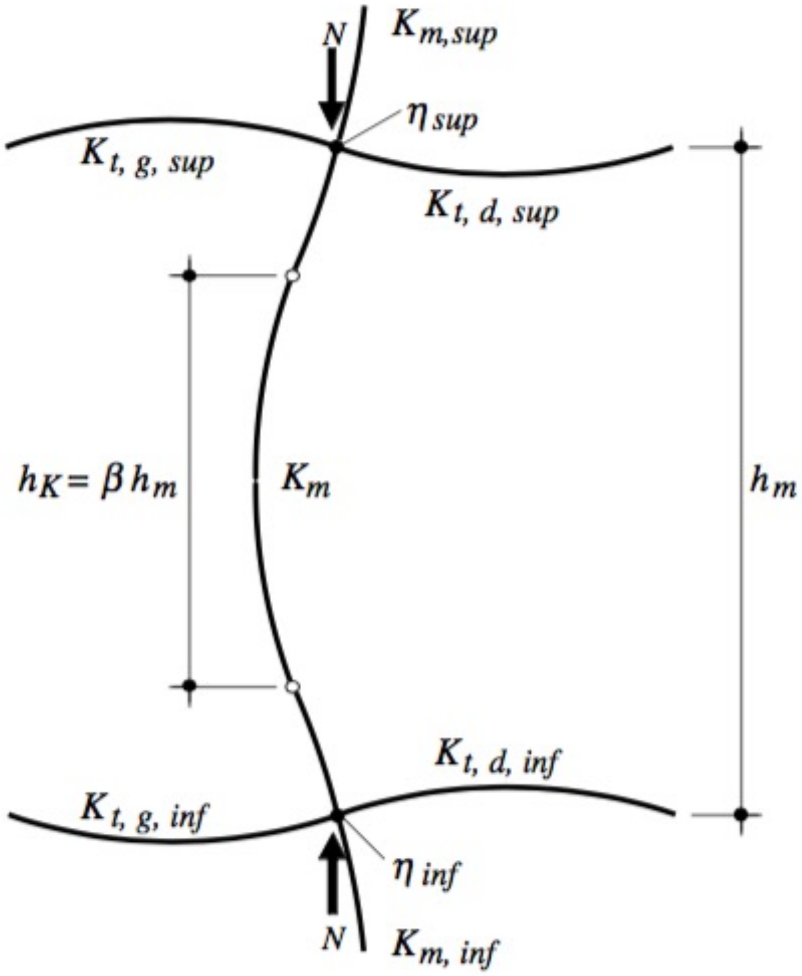


End-Restrained Columns – Not Continuous between Floors
-Example: Planar Rigid Frame



End-Restrained Columns – Not Continuous between Floors
-Example: Planar Rigid Frame

By using the Cross method:



$$n_{sup} = \frac{K_m}{K_m + \sum K_{t,sup}} \qquad n_{inf} = \frac{K_m}{K_m + \sum K_{t,inf}}$$

$K_m = EI_m/h_m$ Rigidity of considered column

$K_t = EI_t/l_t$ Rigidity of beam with moment of inertia, I_t, and length l_t

$$\sum K_{t,sup} = K_{t,g,sup} + K_{t,d,sup}$$

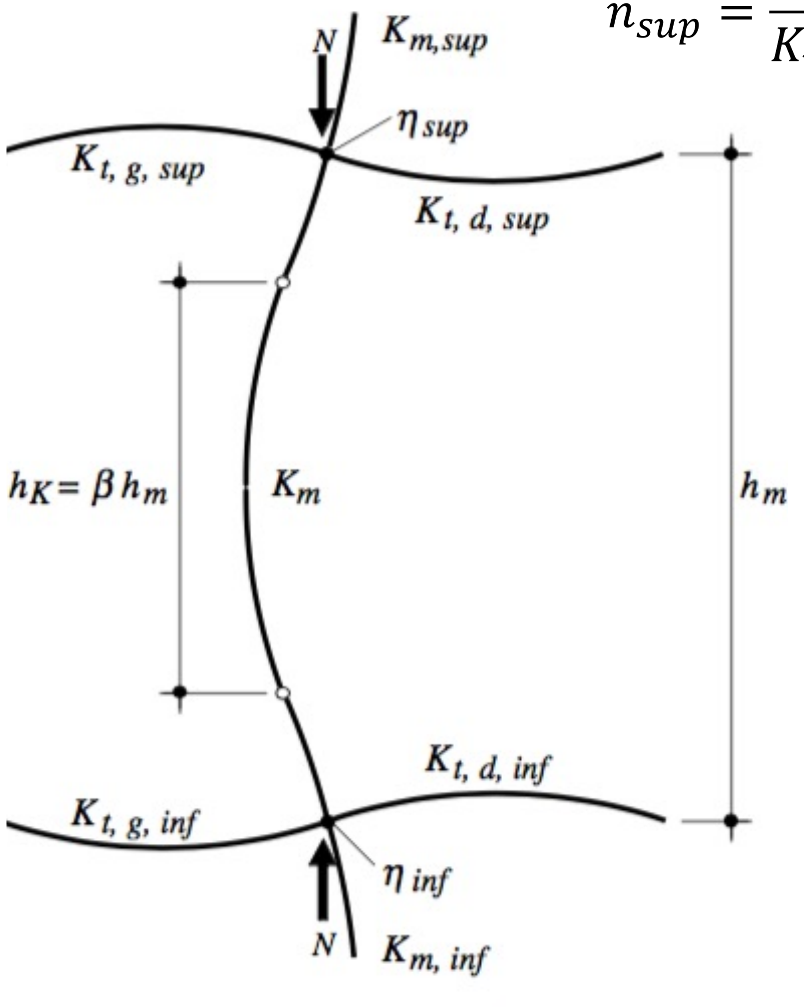
$$\sum K_{t,inf} = K_{t,g,inf} + K_{t,d,inf}$$

m : Column
t : Beam (cross beam)
sup : Top of the column
inf : Bottom of the column
d : Right of the column
g : Left of the column

End-Restrained Columns – Continuous between Floors
-Example: Planar Rigid Frame

End-Restrained Columns – Continuous between Floors
-Example: Planar Rigid Frame (General case)

By using the Cross method:



$$n_{sup} = \frac{K_m + K_{m,sup}}{K_m + K_{m,sup} + \sum K_{t,sup}} \qquad n_{inf} = \frac{K_m + K_{m,inf}}{K_m + K_{m,inf} + \sum K_{t,inf}}$$

$K_m = EI_m/h_m$ Rigidity of considered column

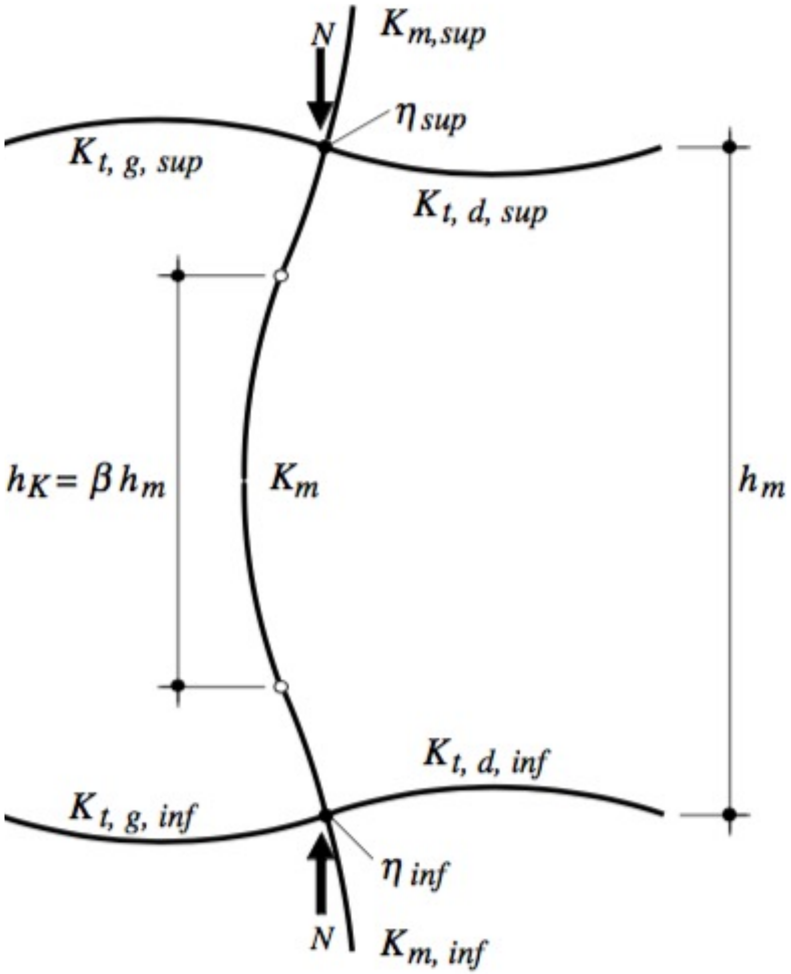
$K_t = EI_t/l_t$ Rigidity of beam with moment of inertia, I_t, and length l_t

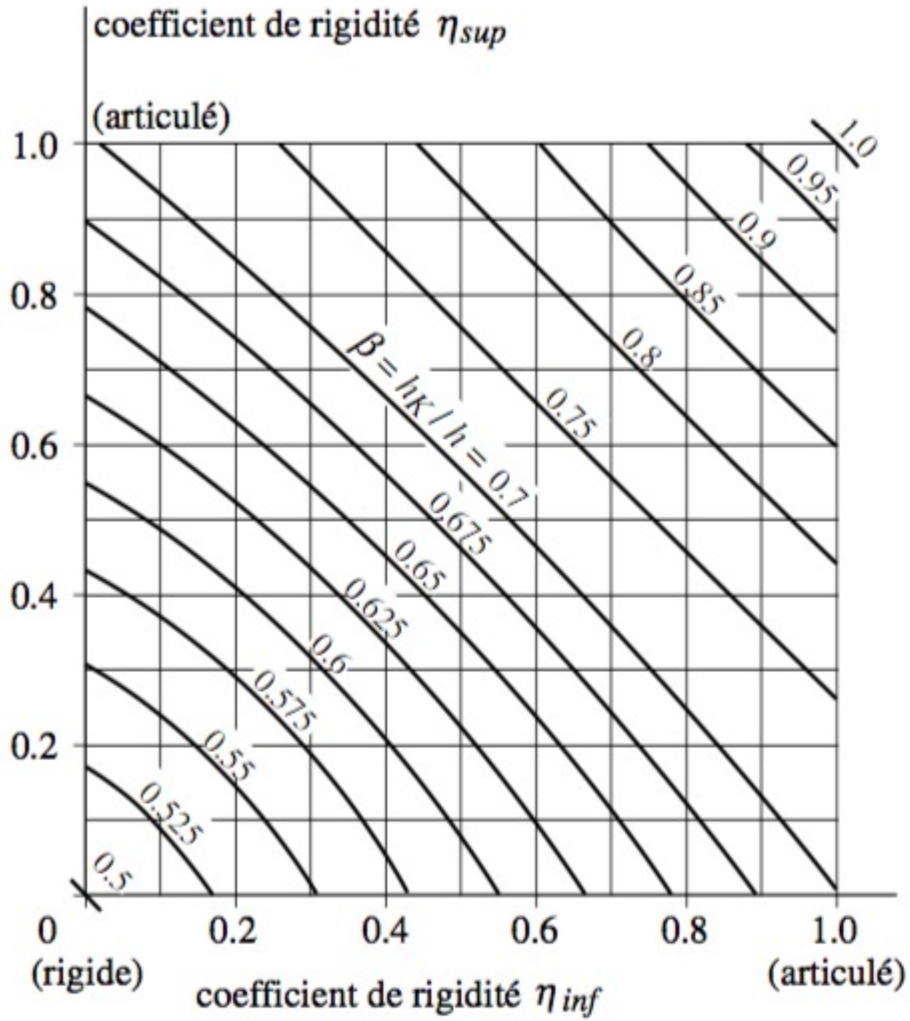
$$\sum K_{t,sup} = K_{t,g,sup} + K_{t,d,sup}$$

$$\sum K_{t,inf} = K_{t,g,inf} + K_{t,d,inf}$$

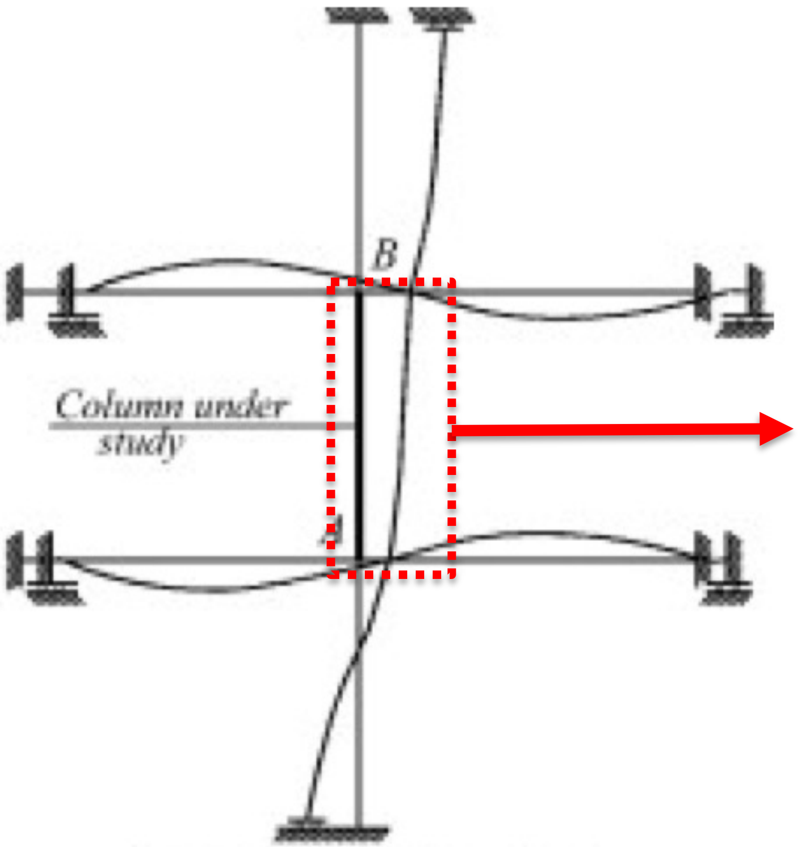
m	: Column
t	: Beam (cross beam)
sup	: Top of the column
inf	: Bottom of the column
d	: Right of the column
g	: Left of the column

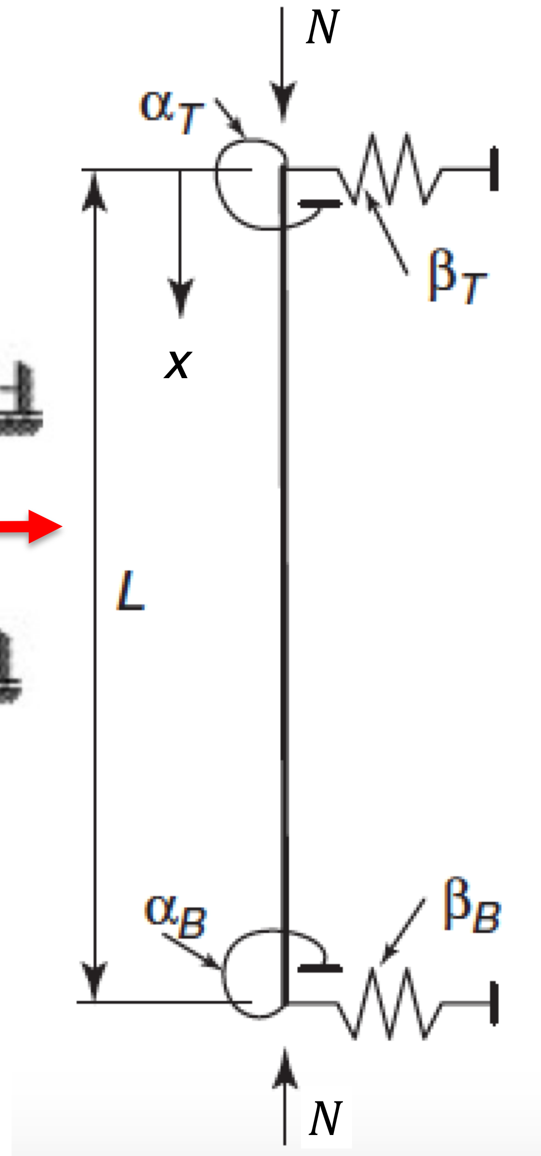
Alignment Chart for Non-Sway Frames





Expansion of Stability Problem to Sway Frames





Boundary conditions

$@x = 0$:

$$-EIv''' - Nv' = \beta_T v$$

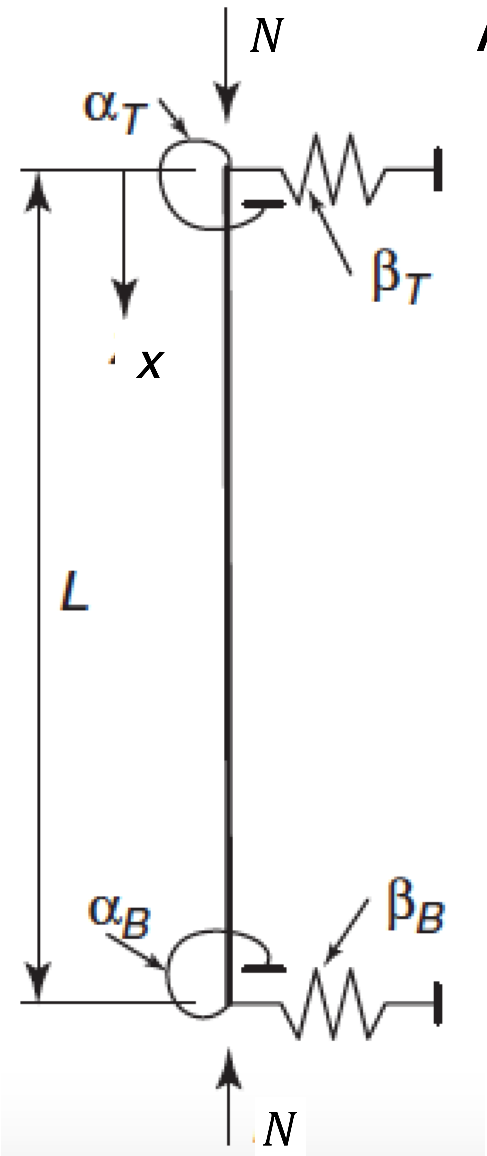
$$-EIv'' = -a_T v'$$

$@x = L$:

$$-EIv''' - Nv' = -\beta_B v$$

$$-EIv'' = a_B v'$$

Expansion of Stability Problem to Sway Frames



Assume the non-dimensional restraint factors

$$R_T = \frac{\alpha_T L}{EI} \quad R_B = \frac{\alpha_B L}{EI} \quad T_T = \frac{\beta_T L^3}{EI} \quad T_B = \frac{\beta_B L^3}{EI} \quad k = \sqrt{\frac{N}{EI}}$$

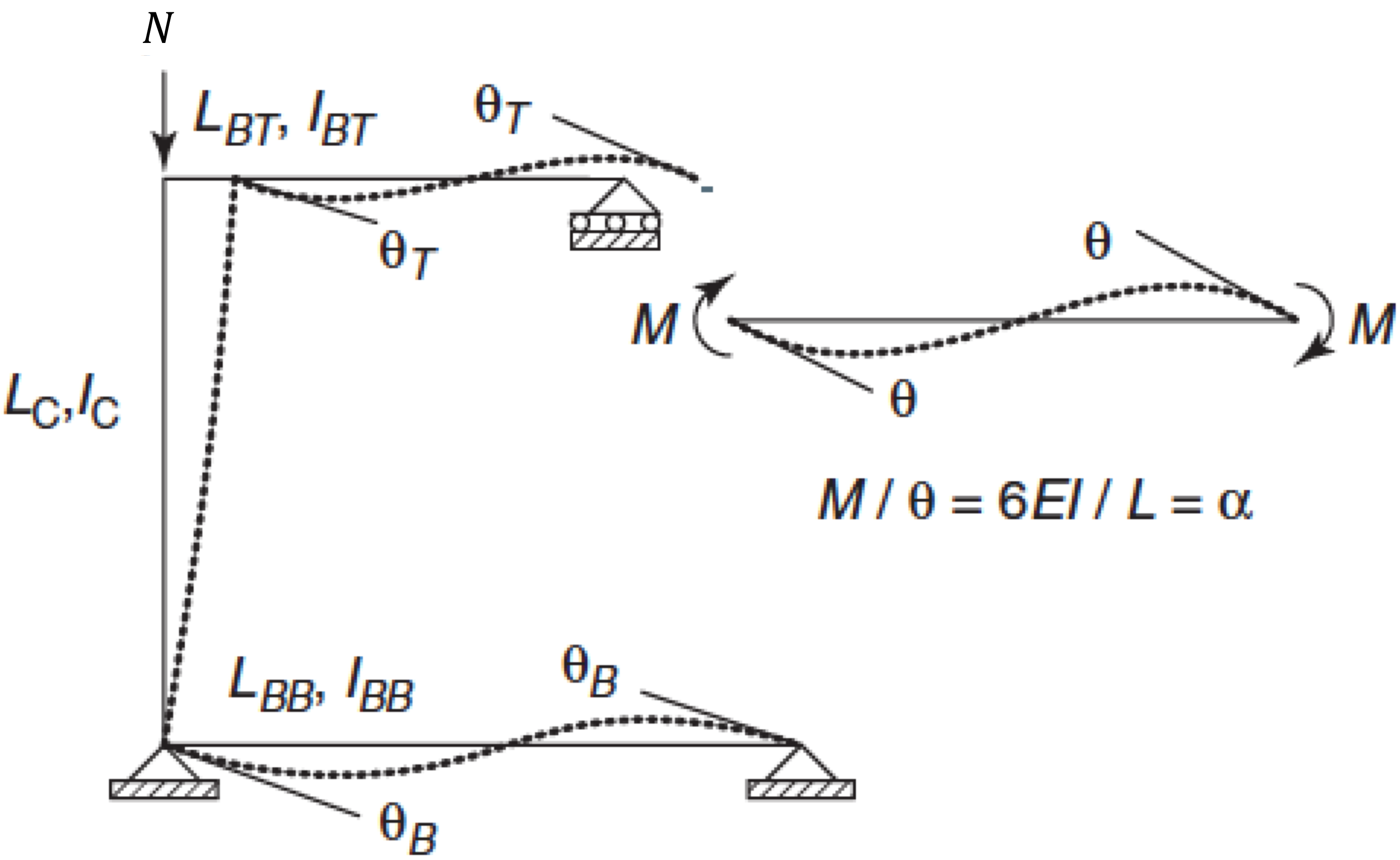
The general buckling determinant becomes

$$\begin{vmatrix} T_T & (kL)^2 & 0 & T_T \\ 0 & R_T & R_T kL & (kL)^2 \\ T_B & [T_B - (kL)^2] & T_B \sin kL & T_B \cos kL \\ 0 & R_B & \begin{matrix}[R_B kL \cos kL \\ - (kL)^2 \sin kL]\end{matrix} & \begin{matrix}[-R_B kL \sin kL \\ - (kL)^2 \cos kL]\end{matrix} \end{vmatrix} = 0$$

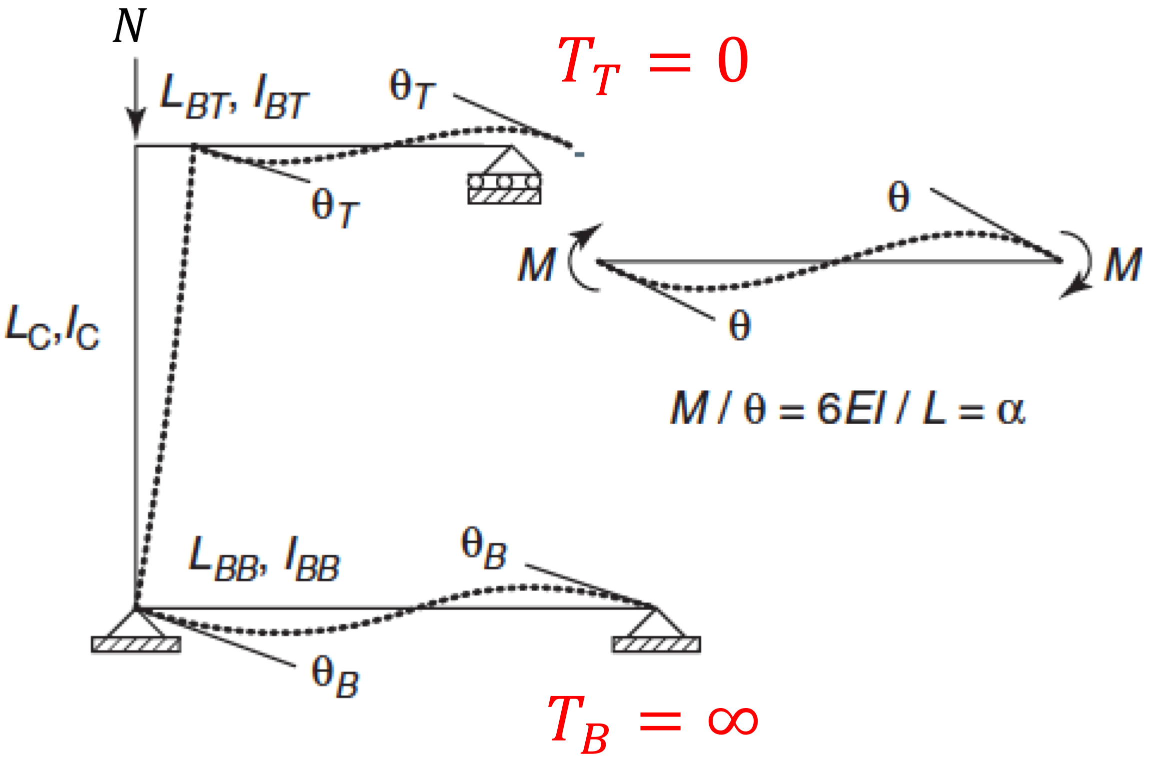
The buckling load (kL) can be solved numerically

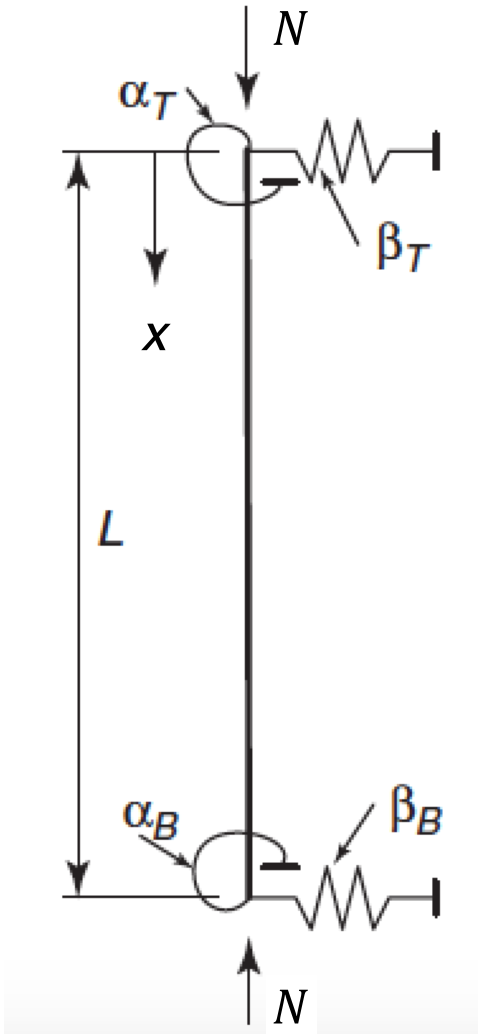
(e.g., with Newton-Raphson iteration)

End-Restrained Columns
-Example: Sway Permitted Subassembly



End-Restrained Columns
-Example: Sway-Permitted Subassembly





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End-Restrained Columns
-Example: Sway-Permitted Subassembly

In this case, the buckling determinant becomes:

$$\begin{vmatrix} 0 & (kL)^2 & 0 & 0 \\ 0 & R_T & R_T kL & (kL)^2 \\ 1 & 1 & \sin kL & \cos kL \\ 0 & R_B & \begin{matrix}[R_B kL \cos kL \\ -(kL)^2 \sin kL]\end{matrix} & \begin{matrix}[-R_B kL \sin kL \\ -(kL)^2 \cos kL]\end{matrix} \end{vmatrix} = 0$$

Assume

$$a_T = \frac{6EI_{BT}}{L_{BT}} \rightarrow R_T = \frac{a_T L_C}{EI_C} = \frac{6EI_{BT}}{L_{BT}} \cdot \frac{L_C}{EI_C} = 6\left(\frac{I_{BT}/L_{BT}}{I_C/L_C}\right) = \frac{6}{n_{sup}}$$

$$a_B = \frac{6EI_{BB}}{L_{BB}} \rightarrow R_B = \frac{a_B L_C}{EI_C} = \frac{6EI_{BB}}{L_{BB}} \cdot \frac{L_C}{EI_C} = 6\left(\frac{I_{BB}/L_{BB}}{I_C/L_C}\right) = \frac{6}{n_{inf}}$$

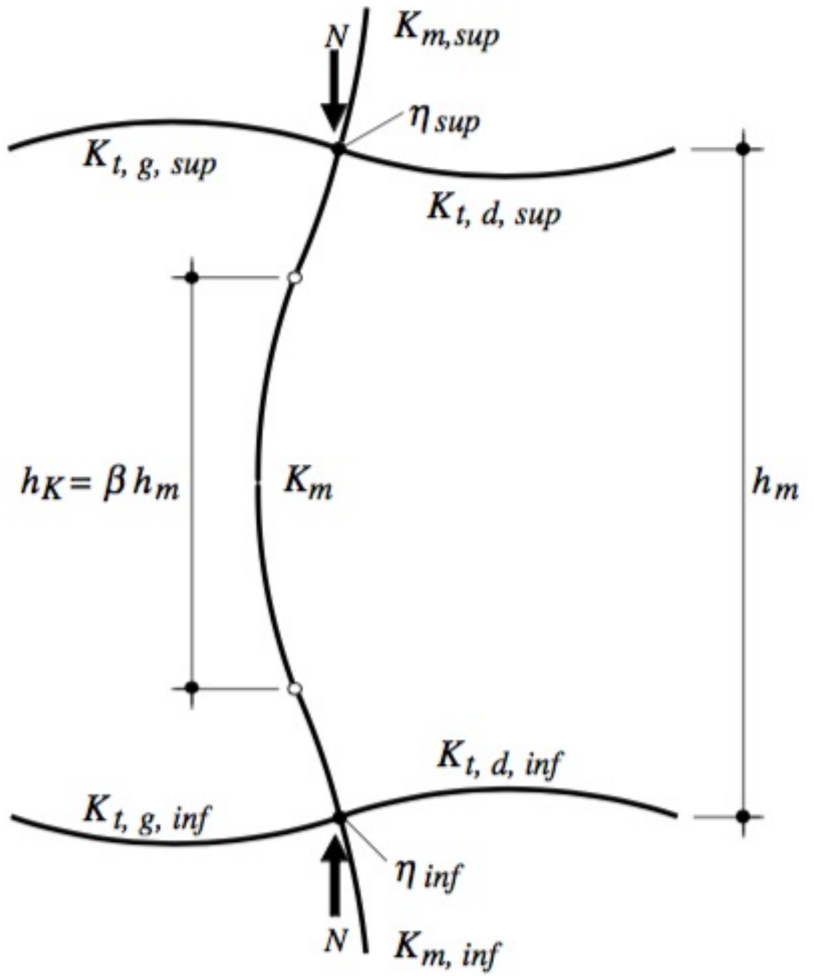
End-Restrained Columns
-Example: Sway Permitted Subassembly

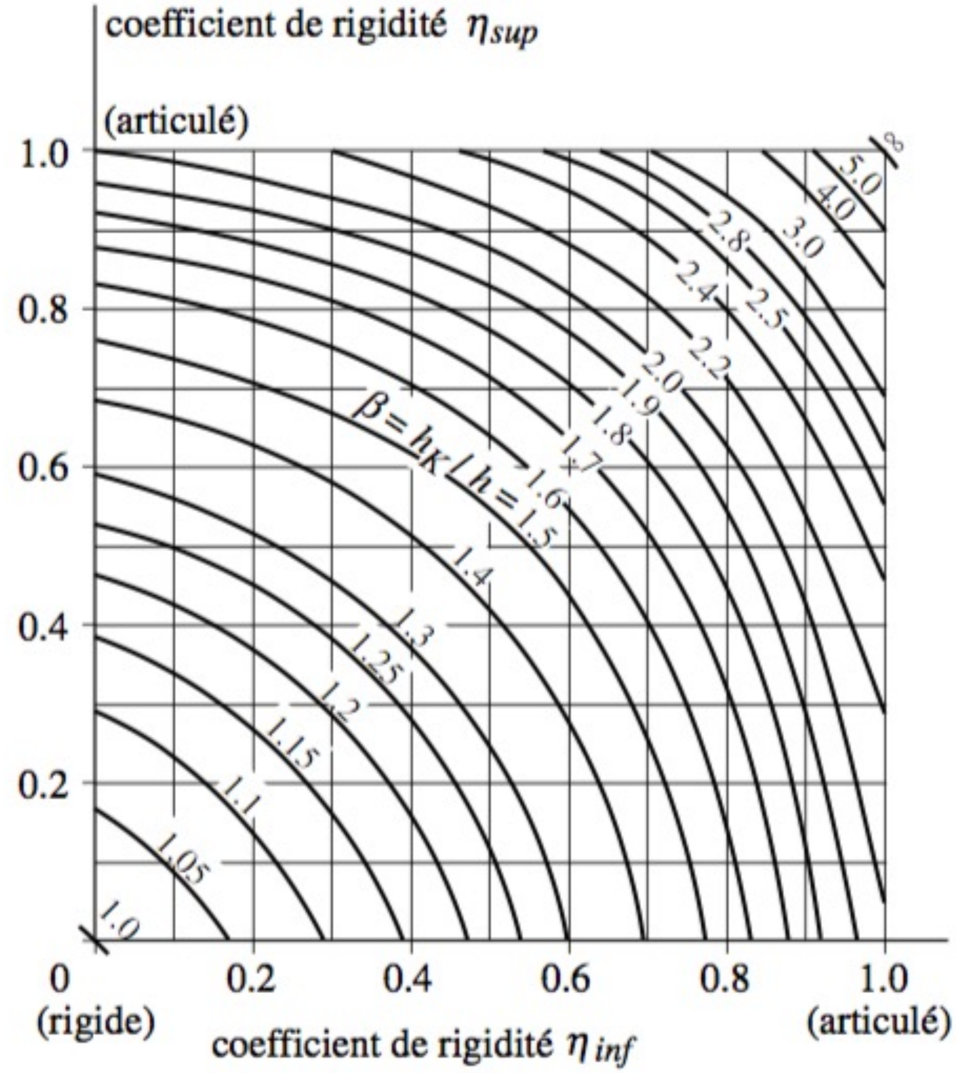
The decomposed determinant becomes

$$\frac{kL}{\tan KL} - \frac{(kL)^2 n_{sup} \cdot n_{inf} - 36}{6\left(n_{sup} + n_{inf}\right)} = 0 \qquad (4)$$

Eq. (4) provides the basis of the sway alignment chart in, design standards (see next page)

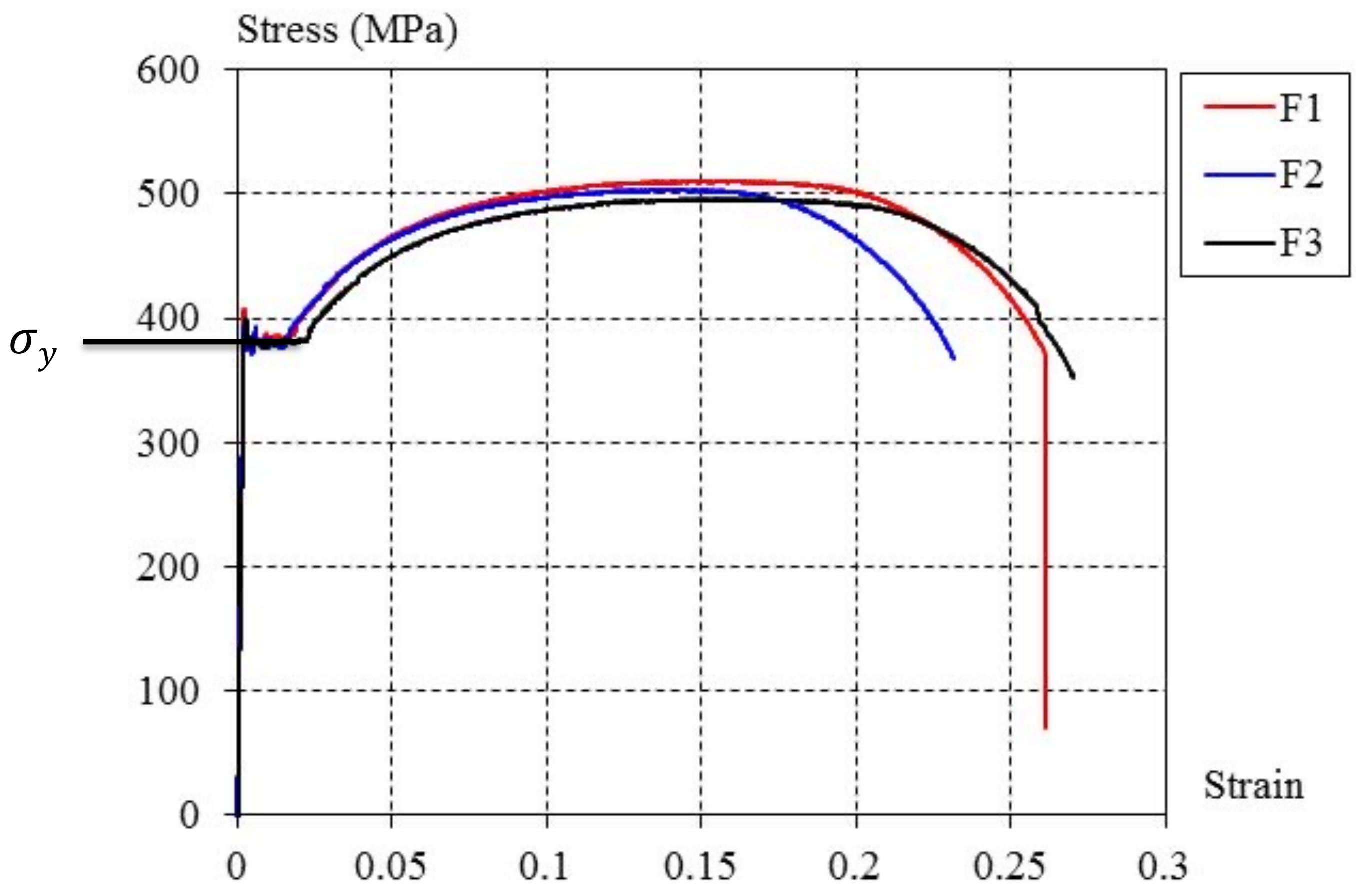
Alignment Chart for Effective Length
-Sway-Permitted Planar Rigid Frames





Combining Yield and Buckling Resistance

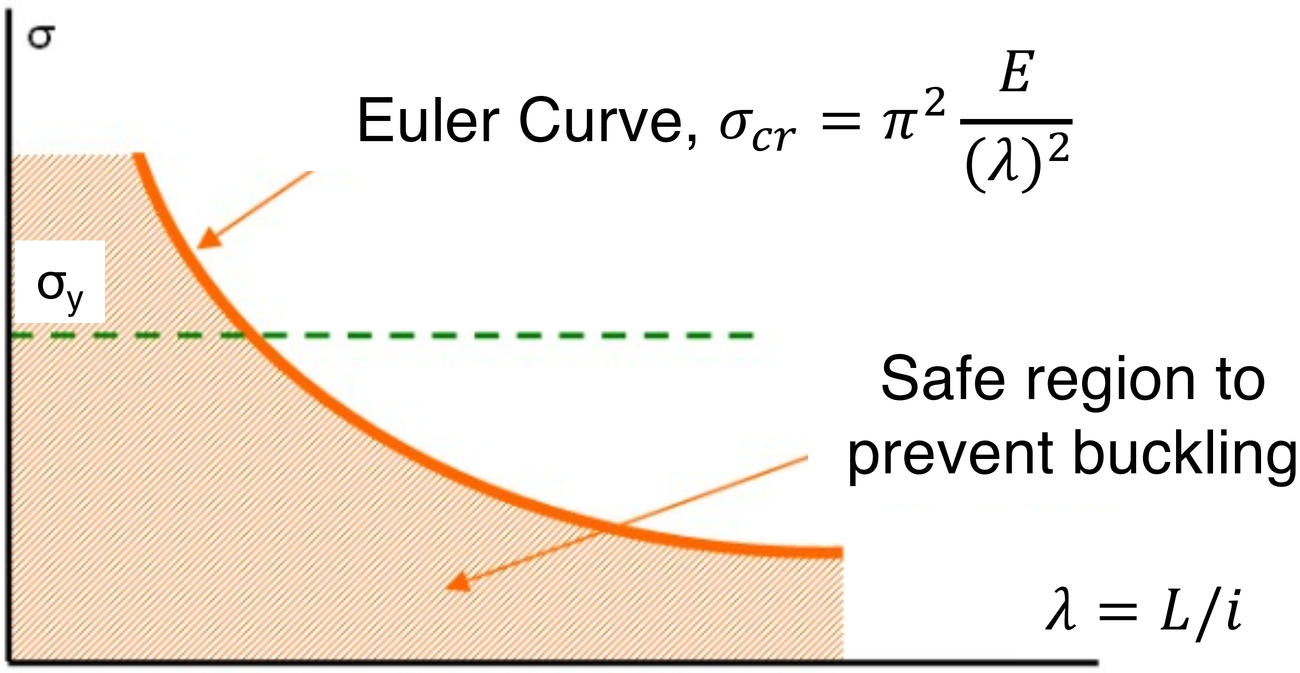
Depending on the member slenderness, the critical stress may be restricted by σ_y rather than $\sigma_{cr} = N_{cr} / A$



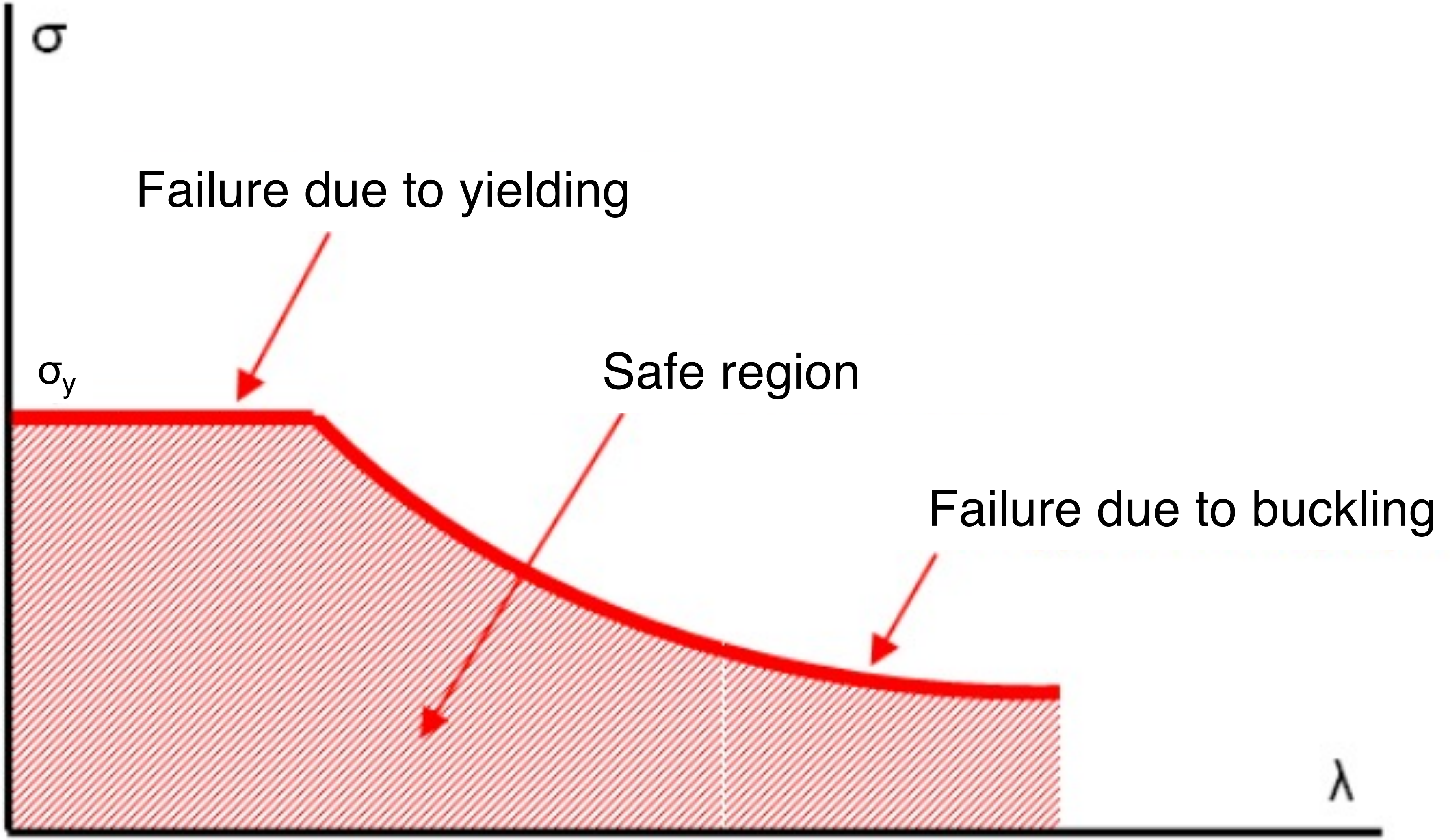
Buckling Resistance of Steel Columns

$$N_{cr} = \frac{\pi^2 EI}{L^2} \quad \Rightarrow \sigma_{cr} = \frac{N_{cr}}{A} = \pi^2 \frac{E \cdot I/A}{L^2} = \pi^2 \frac{E \cdot i^2}{L^2} = \pi^2 \frac{E}{(L/i)^2}$$

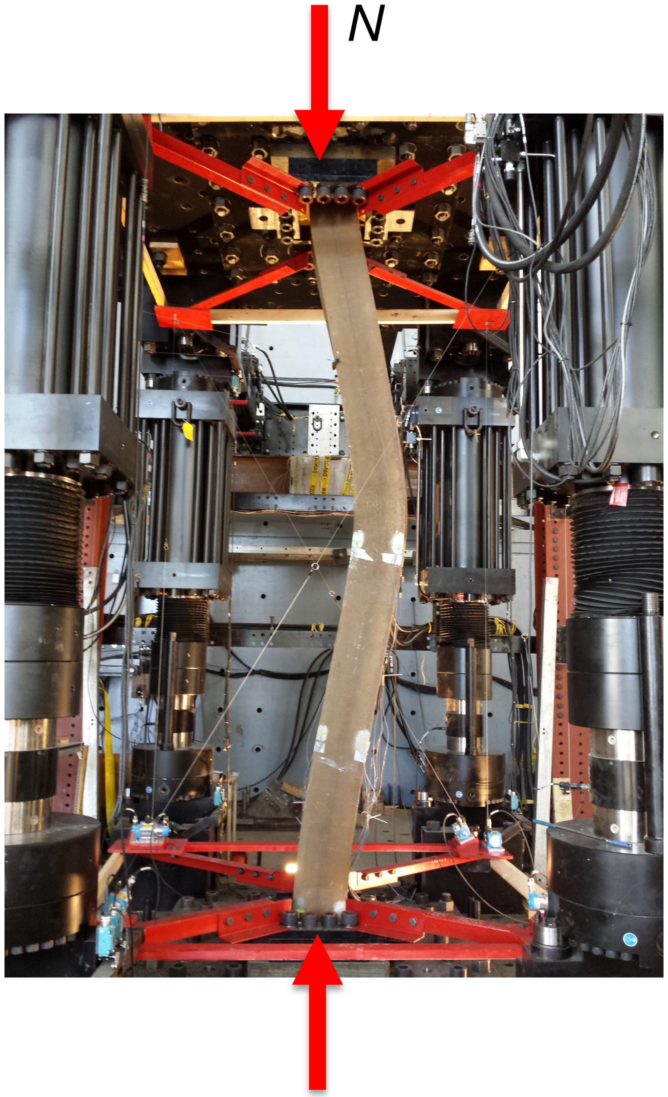
Member slenderness: $\lambda = L/i$



Interaction Curve for Elastic & Inelastic Buckling



Buckling Experiments on Steel Columns



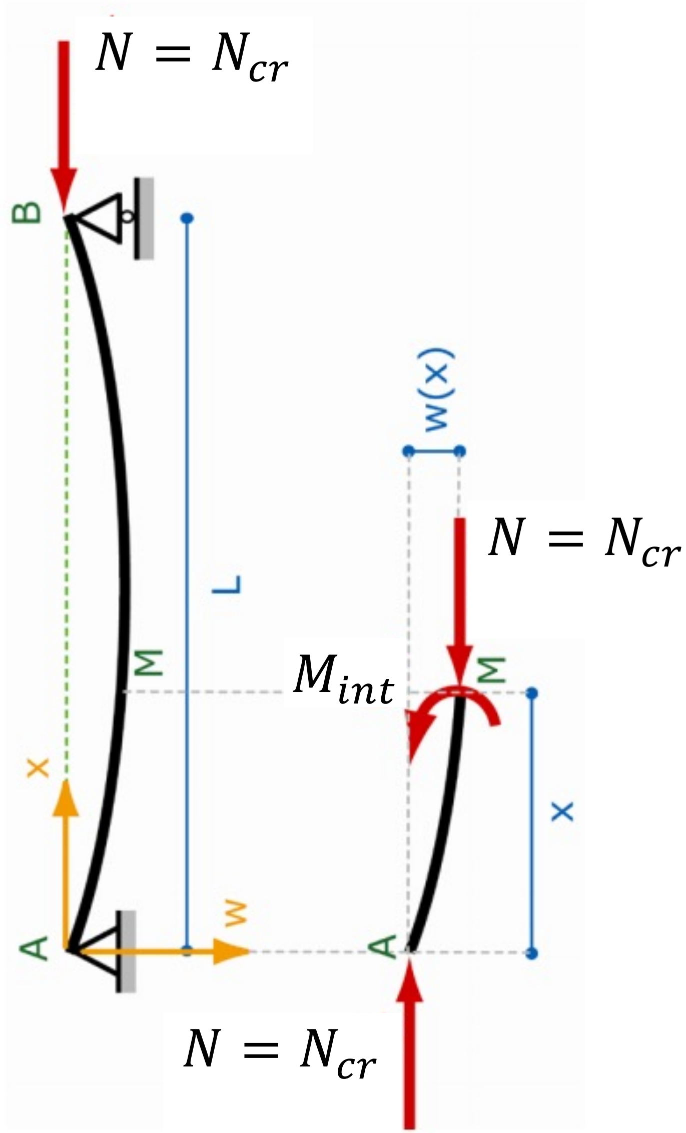
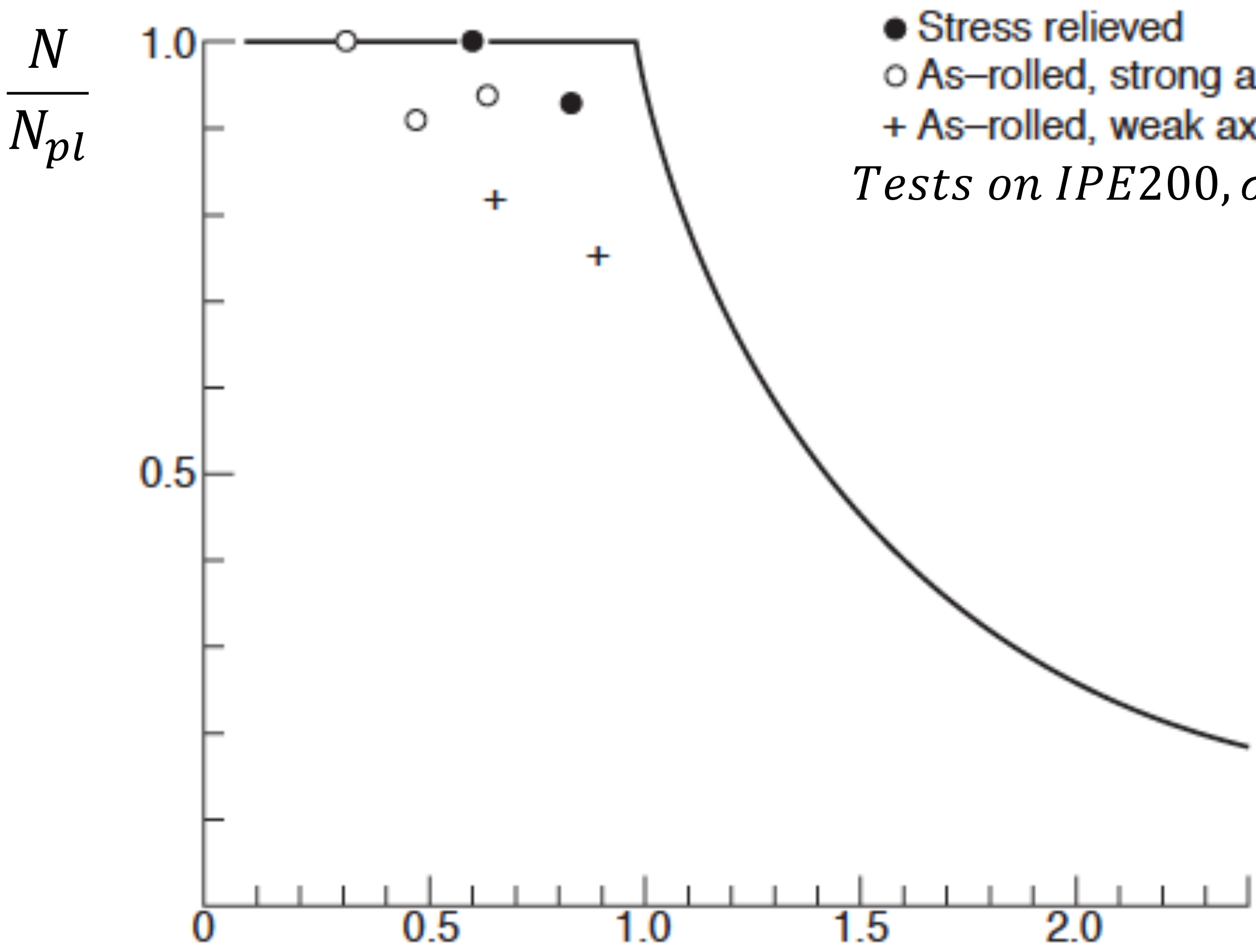


Image courtesy of Prof. Tremblay

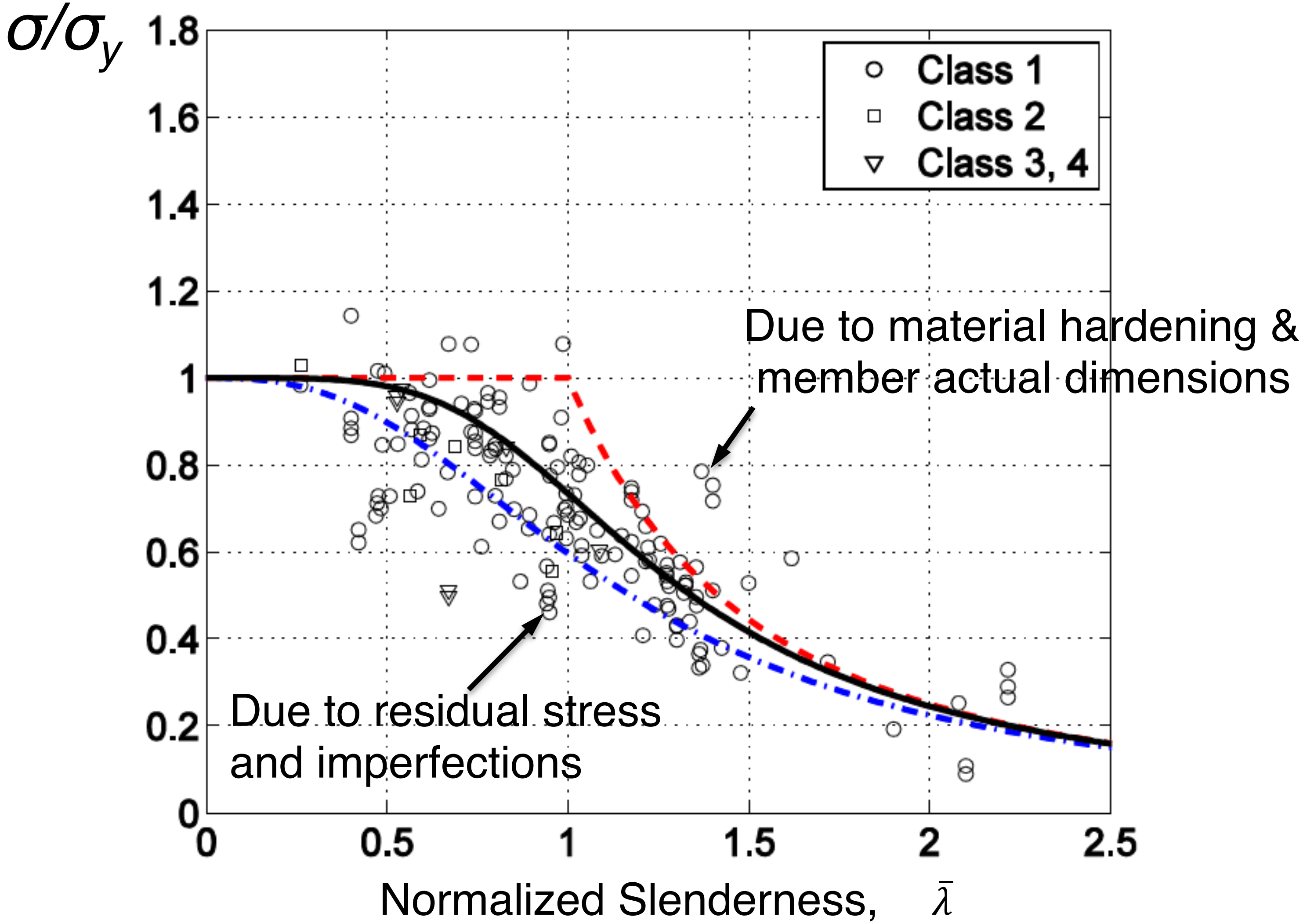
Buckling Experiments on Steel Columns -Comparison with Theoretical Solution



$Tests\ on\ IPE200, \sigma_y = 275MPa$

$$\bar{\lambda} = \sqrt{\frac{A \cdot f_y}{N_{cr}}}$$

Buckling Experiments on Steel Columns -Comparison with Theoretical Solution



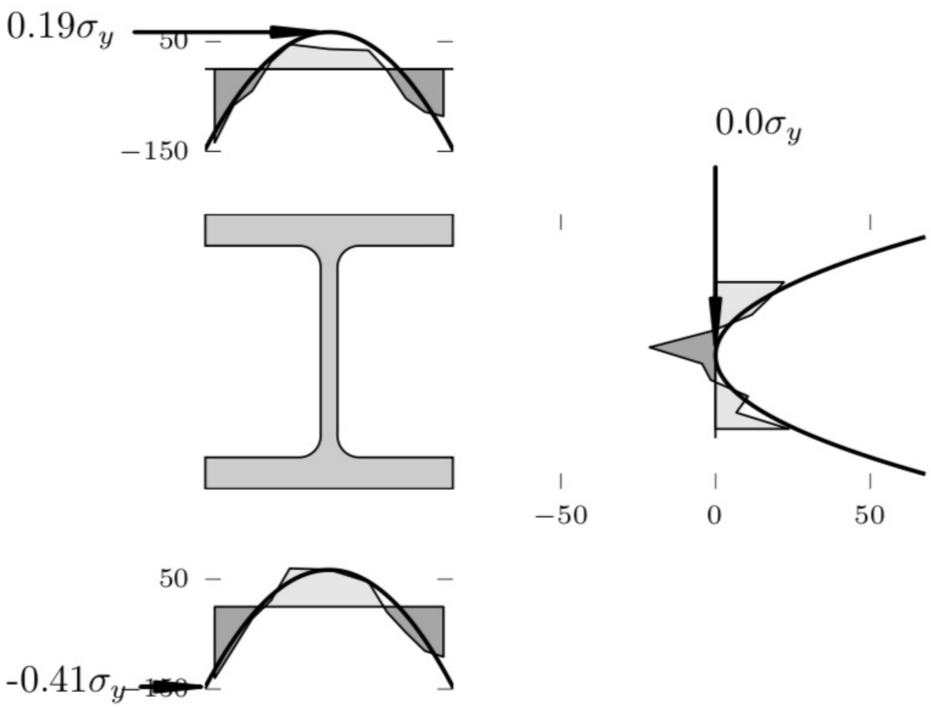
Source: Karamanci and Lignos (2011)

Effect of Residual Stresses on Buckling Resistance

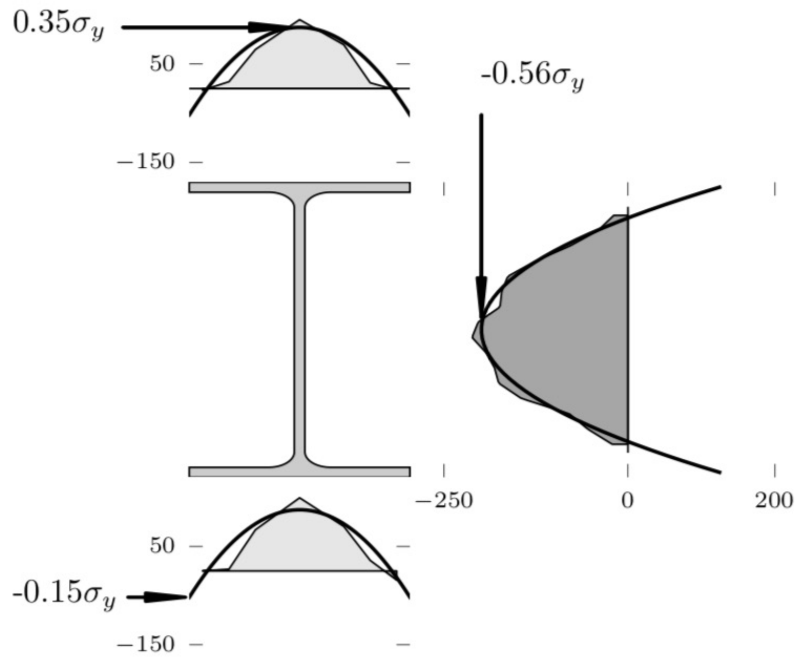
- ☆ Residual stresses could be held accountable for the lower resistance of steel members with intermediate length
- ☆ Confirmed by research at Lehigh University on residual stress measurements (Beedle and Tall 1960)
- ☆ Residual stresses are caused by:
  - ☆ Manufacturing process of steel profiles
  - ☆ Welding (common in built-up sections)
- ☆ For stress-relieved columns, deviations are only due to initial out-of-straightness of the members (see slide 52)

Residual Stresses due to Manufacturing

Residual Stress Distribution HEM300

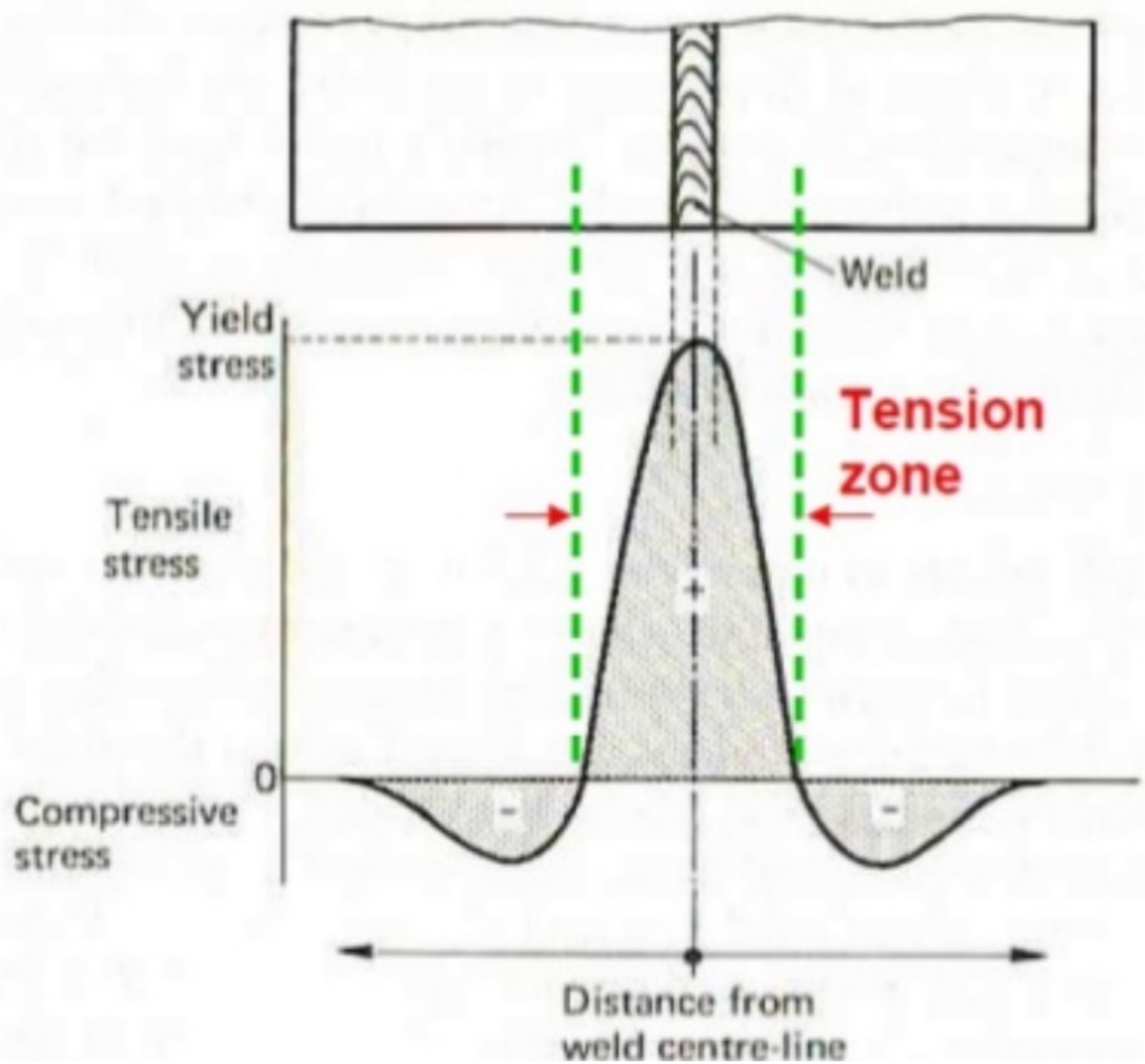


Residual Stress Distribution IPE400



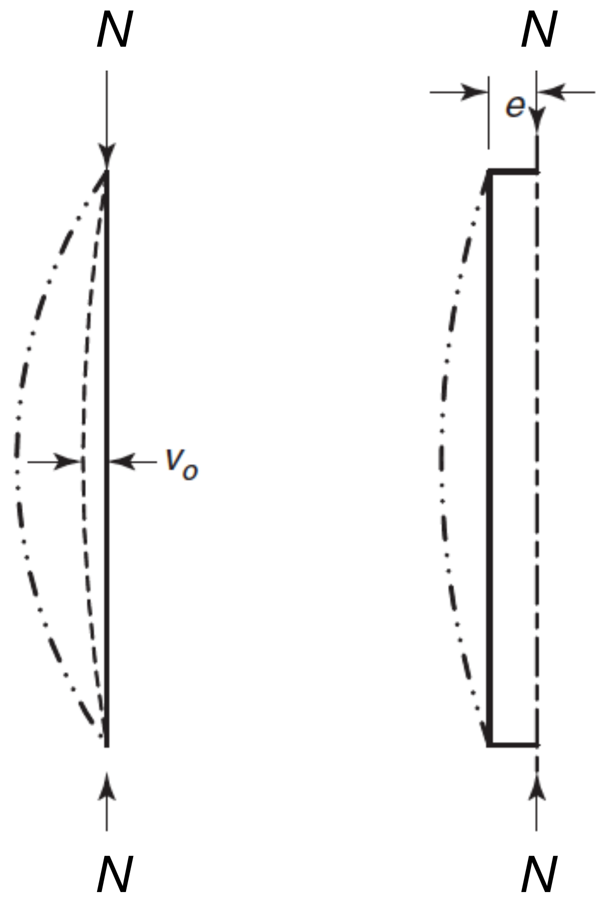
Source: Sousa and Lignos 2017

Residual Stresses due to Welding



The Effect of Imperfections

In a real column, imperfections affect the behaviour near N_{cr}. Imperfections may be due to out-of-straightness of the column axis or small load eccentricity.



The Effect of Imperfections
-Out-of-Straightness

Assumed initial shape due to out-of-straightness:

$$v_i = v_0 \sin\left(\frac{\pi x}{L}\right)$$

Internal and external moments at location *x*,

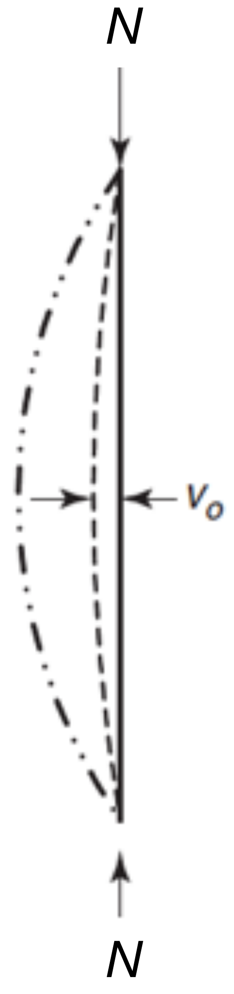
$$M_{int} = -EIv'' \qquad M_{ext} = N(v_i + v)$$

$$EIv'' + Nv = -Nv_i \quad \left(assume\ k^2 = \frac{N}{EI}\right)$$

$$v'' + k^2 v = -k^2 v_0 \sin\left(\frac{\pi x}{L}\right) \ (2)$$

Homogeneous solution $v_H = Asinkx + Bcoskx$

Particular solution $v_P = Csin\left(\frac{\pi x}{L}\right) + Dcos\left(\frac{\pi x}{L}\right)$



The Effect of Imperfections
-Out-of-Straightness

Substituting to Eq. (2):

$$-C\frac{\pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right)-D\frac{\pi^2}{L^2}\cos\left(\frac{\pi x}{L}\right)+Ck^2\sin\left(\frac{\pi x}{L}\right)+Dk^2\cos\left(\frac{\pi x}{L}\right)=-k^2 v_0\sin\left(\frac{\pi x}{L}\right)$$

$$C\left[k^2-\frac{\pi^2}{L^2}\right]=-v_0, D=0$$

Total deflection due to axial load N,

$$v=v_H+v_P=Asinkx+Bcoskx+\frac{N/N_E}{1-N/N_E}v_0\sin\left(\frac{\pi x}{L}\right)$$

Initial Conditions: $v(0)=v(L)=0 \Longrightarrow A=B=0$

Therefore, $$v=\frac{N/N_E}{1-N/N_E}v_0\sin\left(\frac{\pi x}{L}\right)$$

EPFL

The Effect of Imperfections

-Out-of-Straightness

$$v_{total} = v_i + v = \frac{v_0 \overbrace{\sin\left(\frac{\pi x}{L}\right)}^{1}}{1 - N/N_E} \leq \frac{v_0}{1 - N/N_E}$$

At the center :

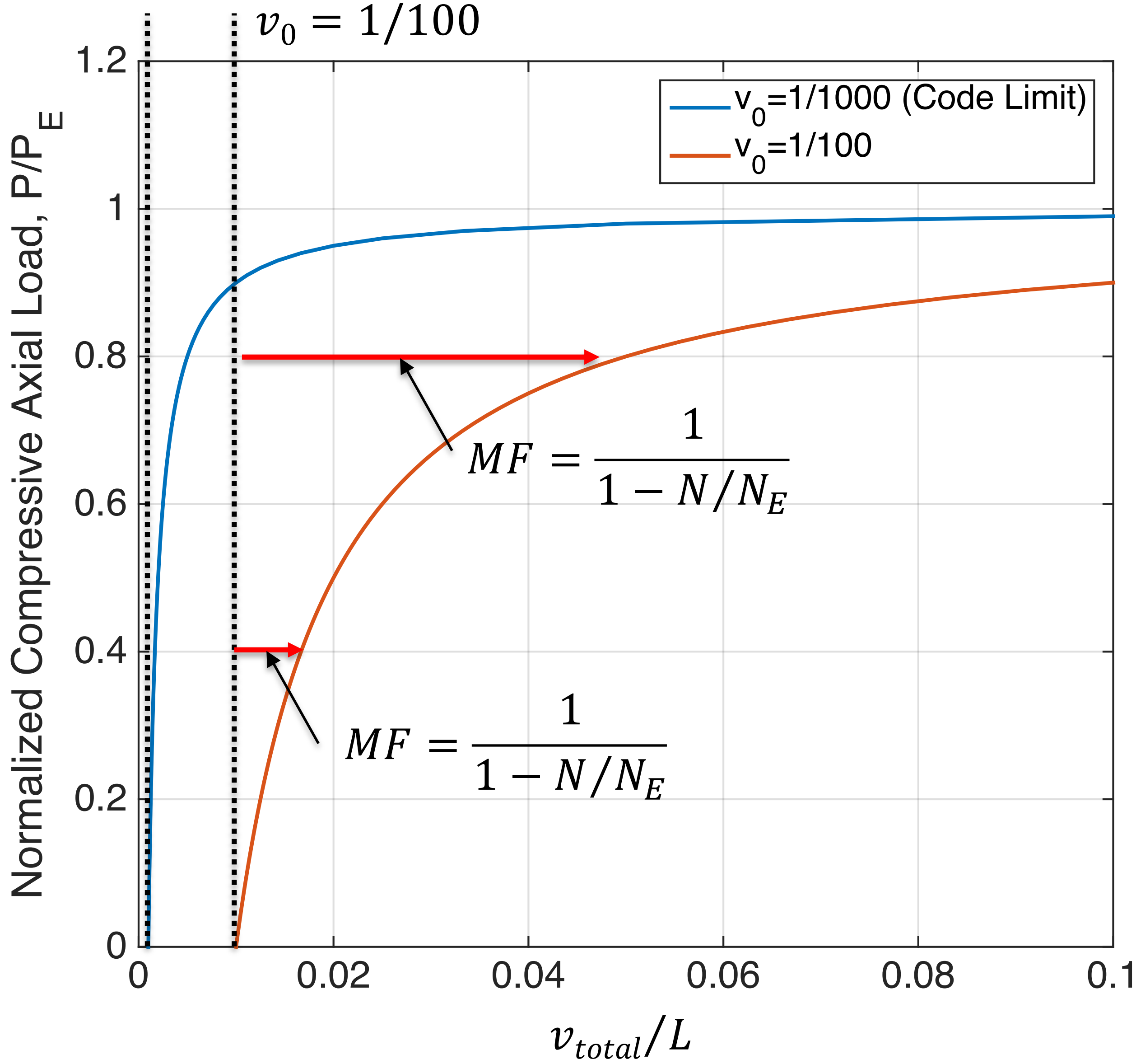
$$\frac{v_{total}}{L} = \frac{v_0/L}{1 - N/N_E} = v_0/L \boxed{\frac{1}{1 - N/N_E}}$$

Magnification Factor (MF)

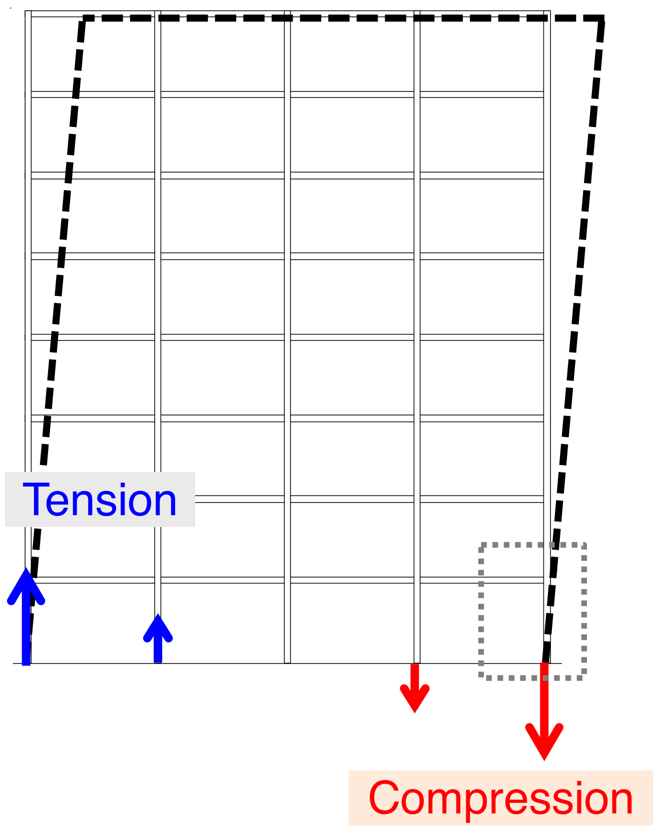
The initial out-of-straightness v_0 is the fabrication tolerance for straightness in the rolling mill, and in steel design practice it is usually 1/1000 of the respective column length. It is not detectable by eye. If it is then you have a big problem!

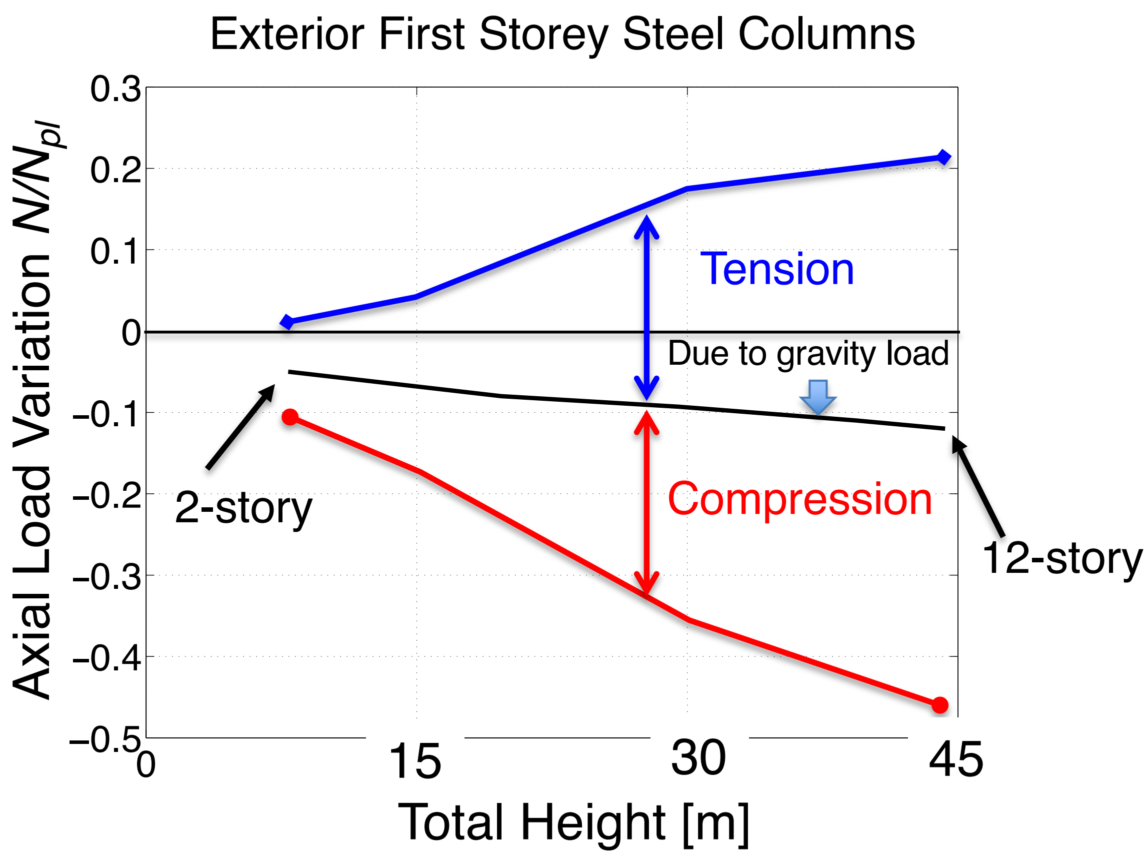
The Effect of Imperfections

-Out-of-Straightness



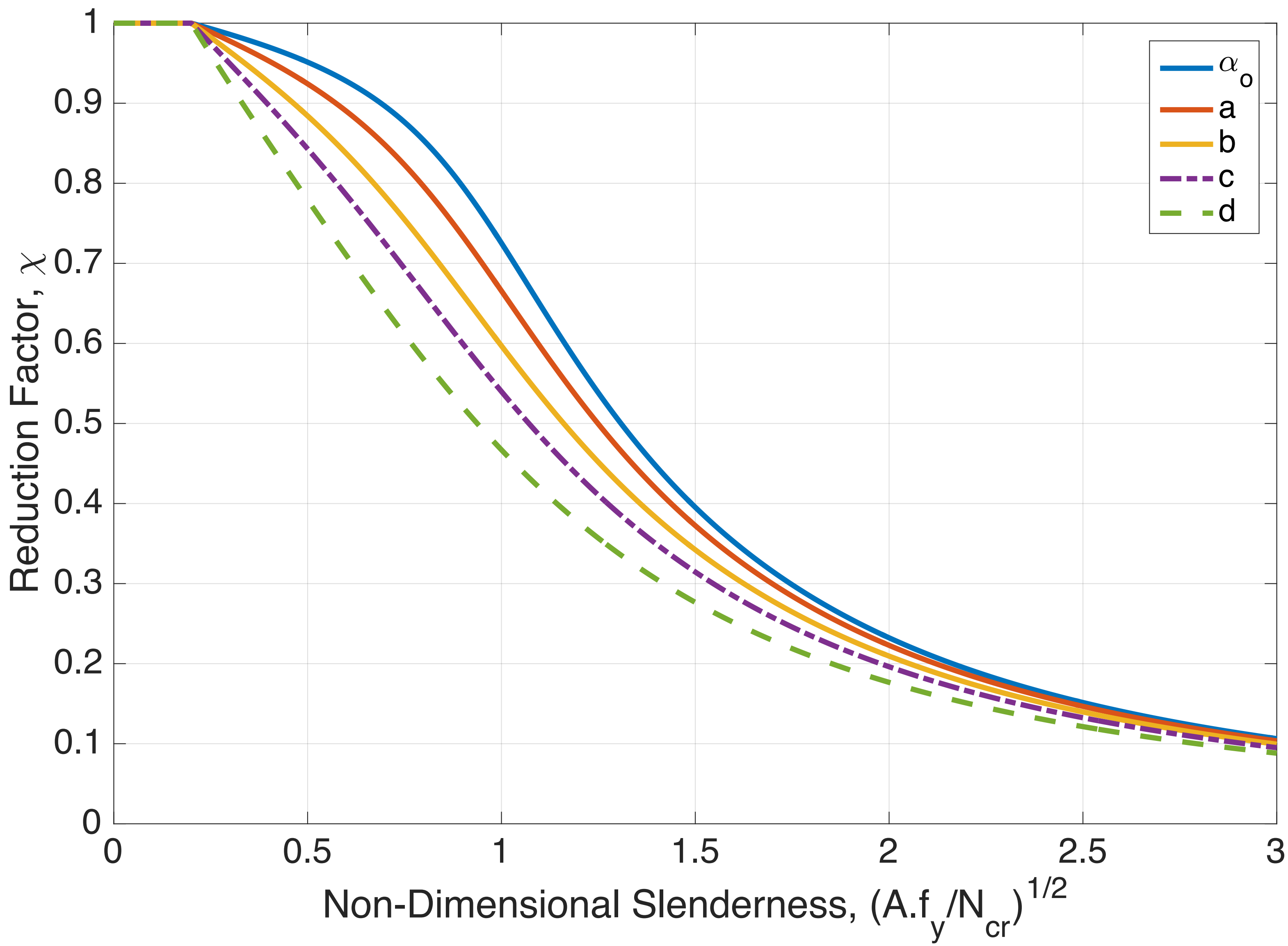
The Effect of Imperfections
-One Important Example (Frame under Seismic Load)





Source: Suzuki and Lignos (2014)

Buckling Curves in SIA 263 or EC3-1-1



Steel Column Design

-Buckling Resistance (EC3-Part 1-1 Section 6.3)

General Condition: $$\frac{N_{Ed}}{N_{b,Rd}} \leq 1$$

Class 1 & 2 members: Buckling resistance in compression:

$$N_{b,Rd} = \frac{\chi \cdot A \cdot f_y}{\gamma_{M1}}$$

Assume $\gamma_{M1} = 1.0$

Buckling factor:

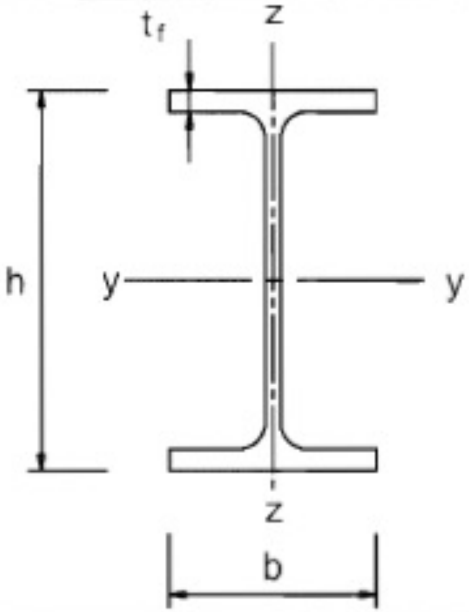
$$\chi = \frac{1}{\Phi + \sqrt{\Phi^2 - \bar{\lambda}^2}} \leq 1.0$$

$$\Phi = 0.5 \cdot \left[1 + a \cdot \left(\bar{\lambda} - 0.2\right) + \bar{\lambda}^2\right] \qquad \bar{\lambda} = \sqrt{\frac{A \cdot f_y}{N_{cr}}}$$

Imperfection factor
(depends on shape, steel material, thickness)

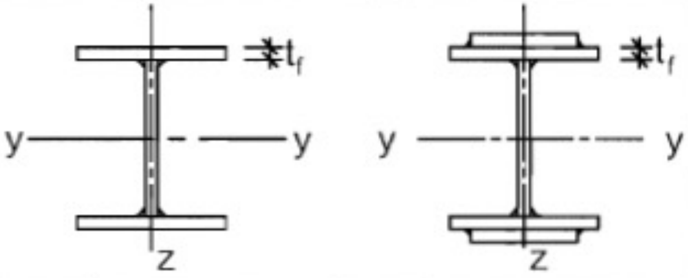
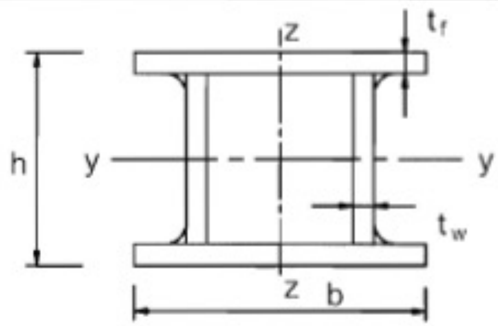
Steel Column Design

-Buckling Resistance (EC3-Part 1-1 Section 6.3)

<table>
<tr><th colspan="2" rowspan="2">Cross section</th><th colspan="2" rowspan="2">Limits</th><th rowspan="2">Buckling about axis</th><th colspan="2">Buckling curve</th></tr>
<tr><th>S 235
S 275
S 355
S 420</th><th>S 460</th></tr>
<tr><td rowspan="4">Rolled sections</td><td rowspan="4"></td><td rowspan="2">h/b > 1,2</td><td>$t_f \leq 40$ mm</td><td>y – y
z – z</td><td>a
b</td><td>a_0
a_0</td></tr>
<tr><td>40 mm < $t_f \leq 100$</td><td>y – y
z – z</td><td>b
c</td><td>a
a</td></tr>
<tr><td rowspan="2">h/b ≤ 1,2</td><td>$t_f \leq 100$ mm</td><td>y – y
z – z</td><td>b
c</td><td>a
a</td></tr>
<tr><td>$t_f > 100$ mm</td><td>y – y
z – z</td><td>d
d</td><td>c
c</td></tr>
</table>

Steel Column Design

-Buckling Resistance (EC3-Part 1-1 Section 6.3)

Cross section		Limits	Buckling about axis	Buckling curve: S 235, S 275, S 355, S 420	Buckling curve: S 460
Welded I-sections		$t_f \leq 40$ mm	y – y z – z	b c	b c
		$t_f > 40$ mm	y – y z – z	c d	c d
Hollow sections		hot finished	any	a	a_0
		cold formed	any	c	c
Welded box sections		generally (except as below)	any	b	b
		thick welds: $a > 0{,}5t_f$ $b/t_f < 30$ $h/t_w < 30$	any	c	c

Steel Column Design

-Buckling Resistance (EC3-Part 1-1 Section 6.3)

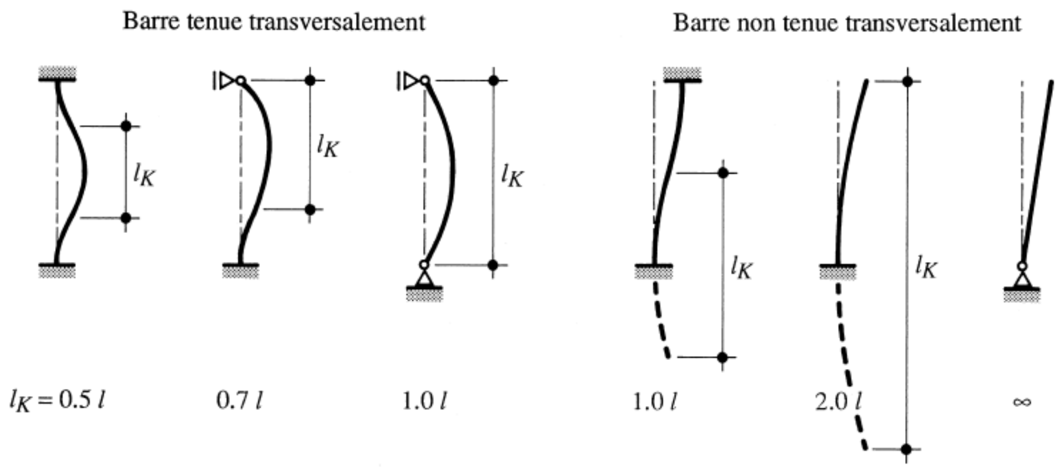
Normalized Member Slenderness

$$\bar{\lambda} = \sqrt{\frac{A \cdot f_y}{N_{cr}}}$$

Critical Load

$$N_{cr} = \pi^2 \cdot \frac{E \cdot I}{(l_k)^2}$$

l_k: Distance between two points that moment becomes zero within the column



Modern Codes – Buckling Resistance

Code Authority	Resistance Factor ϕ	Column formula $\frac{P_{cr}}{P_y}$	Comments
AISC, AISI, AASHTOUSA	0.9	0.658^{λ^2}	$\lambda \leq 1.5$
		$\frac{0.877}{\lambda^2}$	$\lambda > 1.5$
CSA, CANADA, SOUTH AFRICA	0.9	$\left[1+\lambda^{2n}\right]^{\frac{-1}{n}}$	SSRC Curve / n: I 2.24; II 1.34; II 1.00
EC, EUROPE	0.909	$\frac{1}{Q+\sqrt{Q^2-\lambda^2}} \leq 1.0$	$Q=0.5\left[1+\alpha(\lambda-0.2)+\lambda^2\right]$; European Column Curve / α: a 0.21; b 0.34; c 0.49; d 0.76
AS, AUSTRALIA	0.9	$\xi\left[1-\sqrt{1-\left(\frac{90}{\xi\bar{\lambda}}\right)^2}\right]$	$\bar{\lambda}=\pi\lambda\sqrt{800}$; $\eta=0.00326(\bar{\lambda}-13.5)\geq 0$; $\xi=\frac{(\bar{\lambda}/90)^2+1+\eta}{2(\bar{\lambda}/90)^2}$
AIJ, JAPAN	0.9	1.0	$\lambda \leq 0.15$
	$0.9-0.05\left[\frac{\lambda-0.15}{\frac{1}{\sqrt{0.6}}-0.15}\right]$	$1.0-0.5\left[\frac{\lambda-0.15}{\frac{1}{\sqrt{0.6}}-0.15}\right]$	$0.15<\lambda\leq\frac{1}{\sqrt{0.6}}$
	0.85	$\frac{1.0}{1.2\lambda^2}$	$\lambda\leq\frac{1}{\sqrt{0.6}}$

SSRC Curve	n
I	2.24
II	1.34
II	1.00

European Column Curve	α
a	0.21
b	0.34
c	0.49
d	0.76

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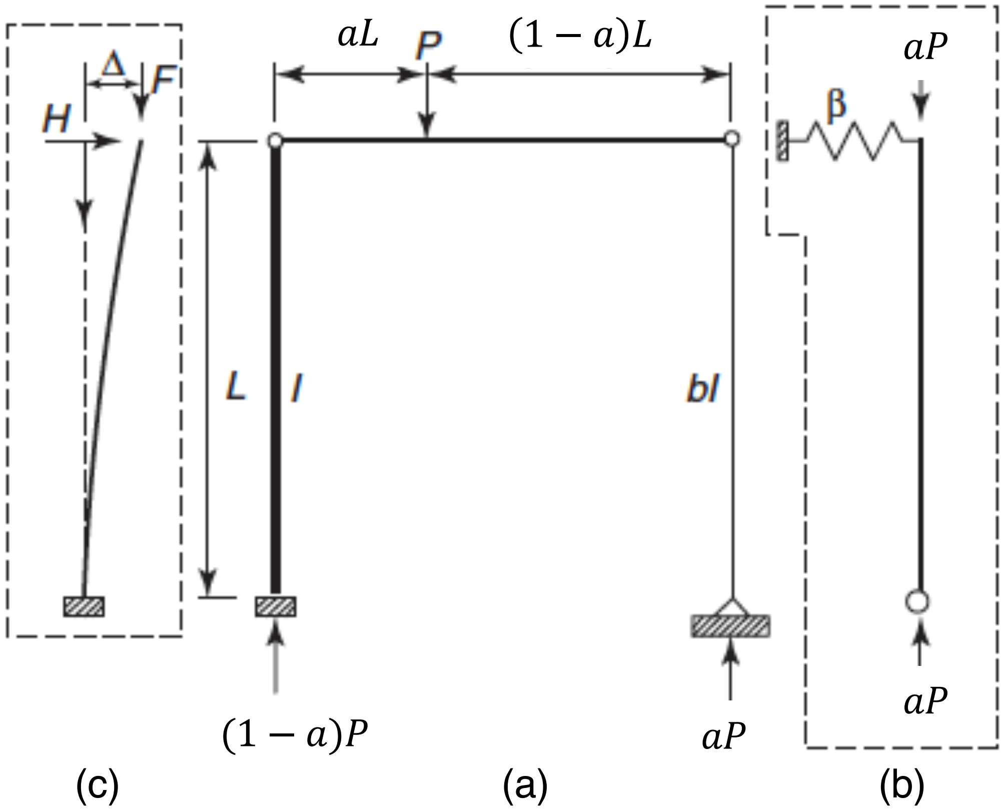
Exercise Figures



Δ
F
H
aL
P
$(1-a)L$
L
I
bI
aP
β
aP
$(1-a)P$
aP
(c)
(a)
(b)