Supply and demand

Fundamentals

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Introduction to transportation systems

Example: airline

Flight GVA-TFS

- Price of the seat depends on the popularity:

$$p = 200 + 0.02q.$$

- Popularity depends on the price of the seat:

$$q = 5000 - 20p.$$

Discussion: What price will be the ticket? How many passengers will fly?

Examples inspired by [?]

Easyjet example

$$p = 200 + 0.02q$$
$$q = 5000 - 20p$$

$q = 5000 - 20(200 + 0.02q)$

$q = 5000 - 4000 - 0.4q$

$1.4q = 1000$

$q = 714 \; p = 200 + 0.02 \cdot 714 = 214.3$

or $p = 200 + 0.02(5000 - 20p)$

$p = 200 + 100 - 0.4p$

$1.4p = 300$

$p = 214.3$

$q = 5000 - 20 \cdot 214.3 = 714$

Example: airline

Supply function

Price of the seat depends on the popularity:

$$p = 200 + 0.02q.$$

Airline strategy

Demand function

Popularity depends on the price of the seat:

$$q = 5000 - 20p.$$

Behavior of the travelers

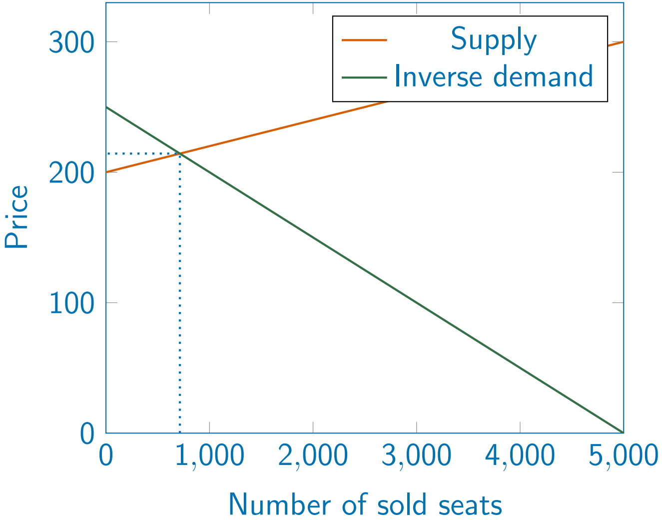
Inverse demand function

$$p = 250 - \frac{q}{20}.$$

To be compared with the supply function

Easyjet example

Supply and demand



Example: highway

Highway A1: Morges — Rolle

- Travel time depends on traffic:

$$t = 15 + 0.02x \text{ (minutes)}.$$

- Traffic depends on travel time:

$$x = 4000 - 120t \text{ (veh/hour)}.$$

Discussion: What is the travel time? What is the traffic?

Morges-Rolle example

$$t = 15 + 0.02x$$
$$x = 4000 - 120t$$

$x = 4000 - 120(15 + 0.02x)$

$x = 4000 - 1800 - 2.4x$

$3.4x = 2200$

$x = 647$ veh/hour.

$t = 27.94$ minutes or $t = 15 + 0.02(4000 - 120t)$

$t = 15 + 80 - 2.4t$

$3.4t = 95$

$t = 27.94$

$x = 647$ veh/hour.

Example: highway

Supply function

Travel time depends on traffic:

$$t = 15 + 0.02x \text{ (minutes)}.$$

System performance

Demand function

Traffic depends on travel time:

$$x = 4000 - 120t \text{ (veh/hour)}.$$

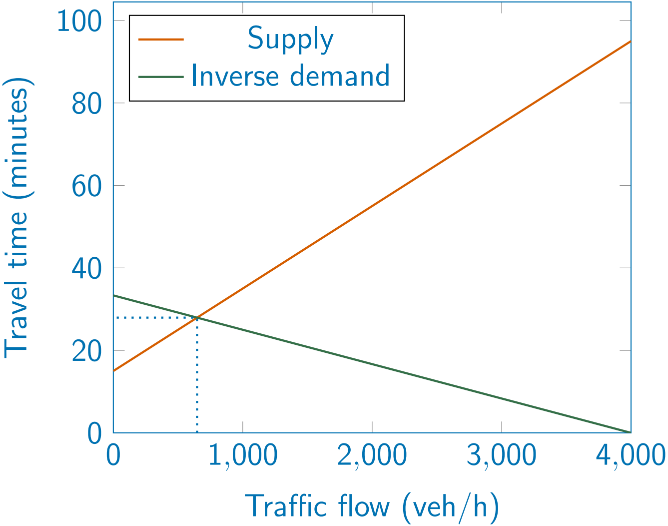
Behavior of the travelers

Inverse demand function

$$t = \frac{100}{3} - \frac{x}{120} \text{ (veh/hour)}.$$

Morges-Rolle example

Supply and demand



Example: highway

Morges — Rolle: adding one lane

- Improvement of the supply
- Travel time depends on traffic:

$$t = 15 + 0.01x \text{ (minutes)}.$$

Discussion: How does it improve the situation?

Morges-Rolle example

Before

- $t = 27.94$ minutes.
- $x = 647$ veh/hour.

After

- $t = 15 + 0.01 \cdot 647 = 21.5$ minutes.

Wrong: traffic will change!

Example: highway

Morges — Rolle: adding one lane

- Improvement of the supply
- Travel time depends on traffic:

$$t = 15 + 0.01x \text{ (minutes)}.$$

- Traffic depends on travel time:

$$x = 4000 - 120t \text{ (veh/hour)}.$$

Discussion: How does it improve the situation?

Morges-Rolle example

Before

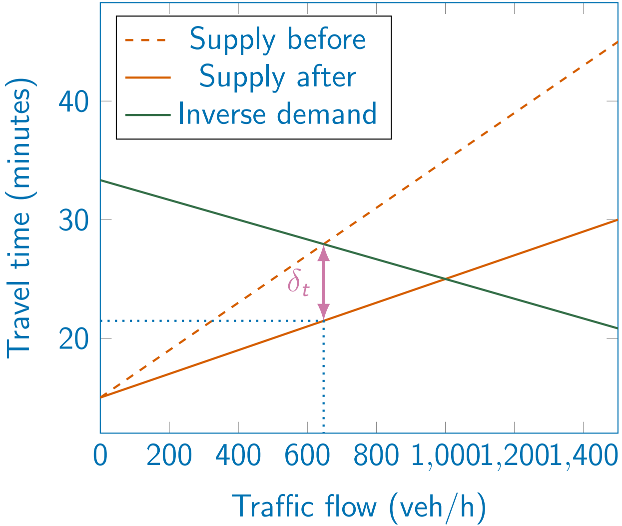
- $t = 27.94$ minutes.
- $x = 647$ veh/hour.

After

- $t = 25$ minutes.
- $x = 1000$ veh/hour.

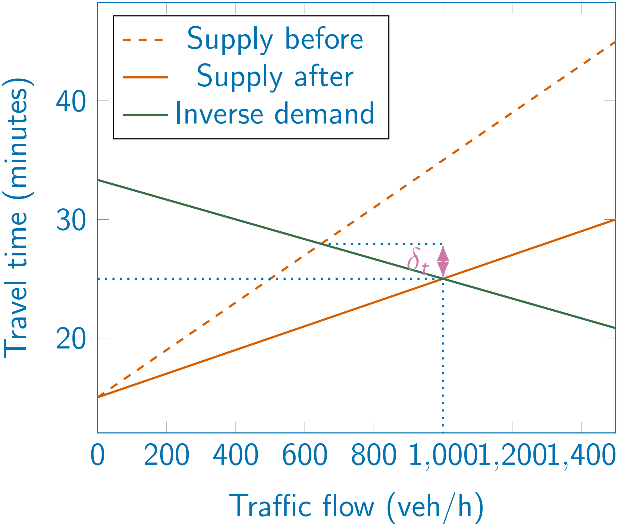
Morges-Rolle example

Wrong analysis



Morges-Rolle example

Correct analysis



Supply and demand

Lesson learned

- Engineers can modify the supply function.
- Supply and demand are strongly related.
- Understanding the demand is critical.

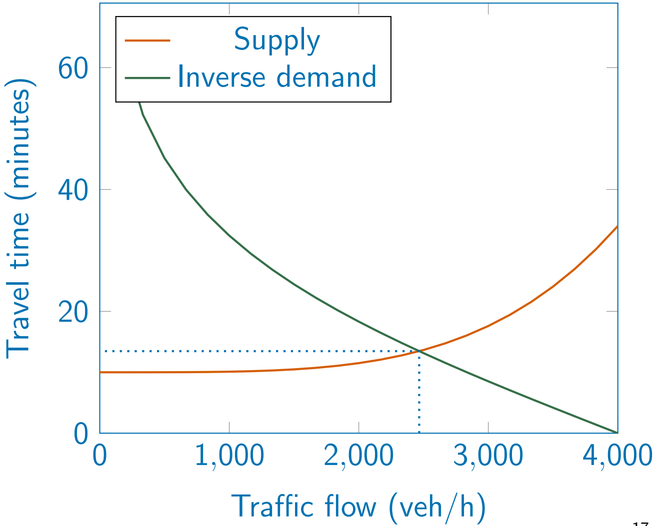
Note: real functions are nonlinear

Supply function

$$t = 10\left(1 + 0.15\left(\frac{x}{2000}\right)^4\right)$$

Demand function

$$x = 8000\frac{1}{1 + e^{0.06t}}$$



Elasticities

Demand function

$$x = 4000 - 120t$$

Modify t by 1%

	Before	After	Abs. diff	Rel. diff
t	27.94	28.2194	0.2794	1%
x	647.2	613.672	-33.528	-5.18%

	Before	After	Abs. diff	Rel. diff
t	25	25.25	0.25	1%
x	1000	970	-30	-3%

Definition

Point elasticity

$$e_t = \frac{dx/x}{dt/t} = \frac{dx}{dt}\frac{t}{x}$$

Example

$x = 4000 - 120t$

$e_t =$
$-120\frac{t}{x} = -120\frac{4000-x}{120x} = 1 - \frac{4000}{x}$

$x = 1000$

$e_t = 1 - \frac{4000}{1000} = -3$

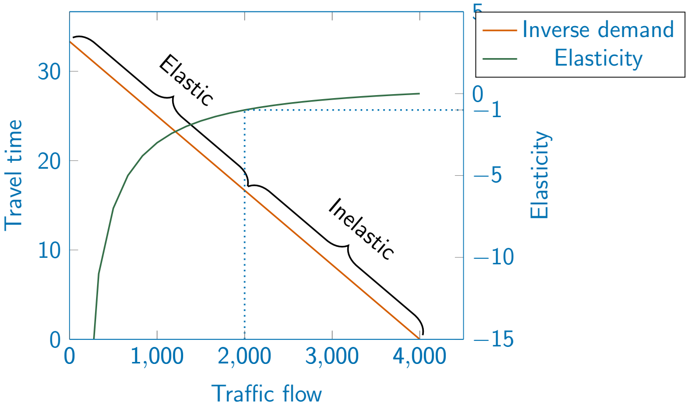
Elasticity

Elastic demand

$$e_t < -1$$

Inelastic demand

$$e_t > -1$$



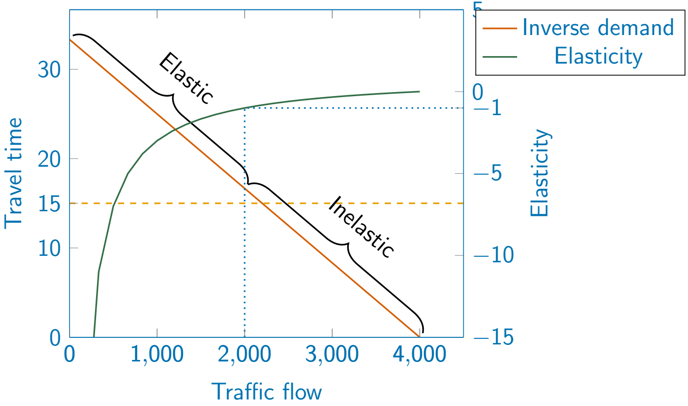
Elasticity

In this example

Minimum travel time: 15 minutes.

Observation

Demand is almost always elastic.



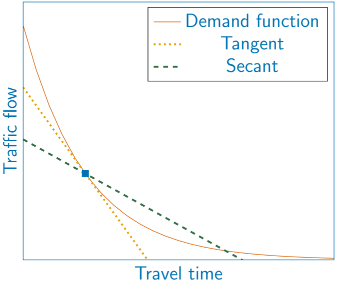
Definition

Arc elasticity

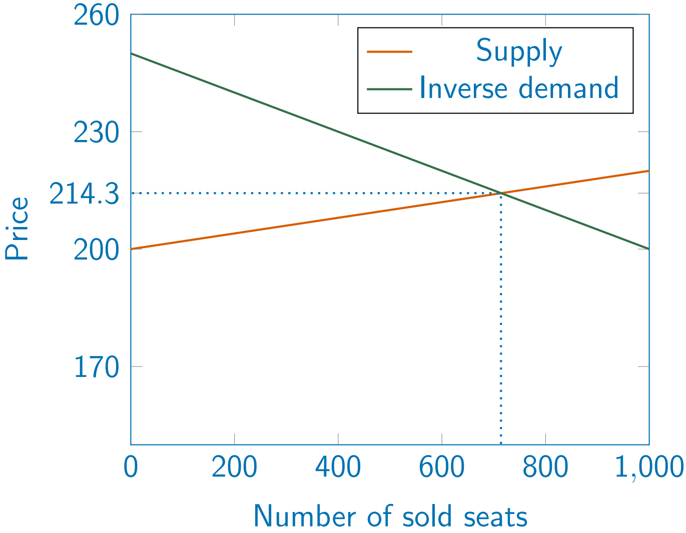
$$e_{\Delta t} = \frac{\Delta x / x}{\Delta t / t} = \frac{\Delta x}{\Delta t} \frac{t}{x}$$

Note

- Linear demand function:
  $e_t = e_{\Delta t}$.
- Nonlinear demand function:
  $e_t = \lim_{\Delta_t \to 0} e_{\Delta t}$.



Consumer surplus: Easyjet



Discussion: Who is happy?

Consumer surplus

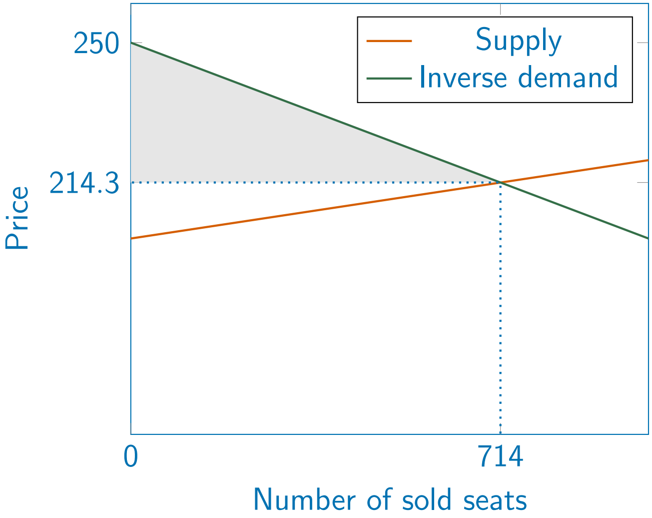
Definition

Difference between what consumers might be willing to pay for a service and what they actually pay.

Concept

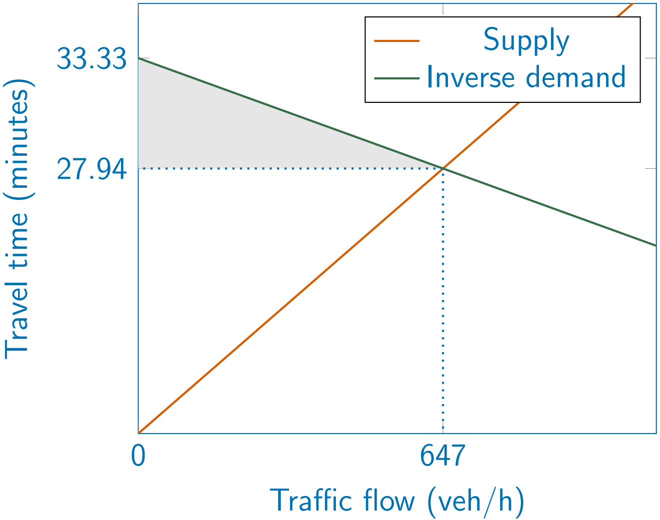
- Measure of the monetary value made available to consumers by the existence of a facility.
- Measure of social welfare.

Consumer surplus: Easyjet



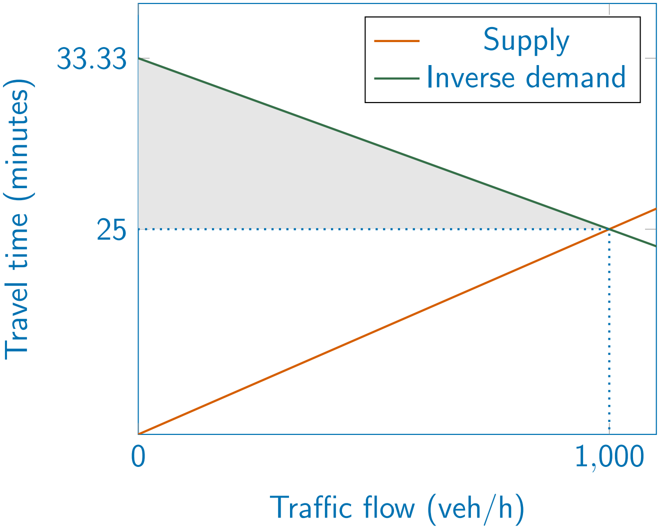
$$\text{Consumer surplus} = (250 - 214.3) \cdot 714/2 = 12744.9 \text{ CHF}$$

Consumer surplus: highway



Consumer surplus $= (33.33 - 27.94) \cdot 647/2 = 1744.5$ min.

Consumer surplus: highway with one more lane



Consumer surplus = $(33.33 - 25) \cdot 1000/2 = 4165$ min.

Consumer surplus

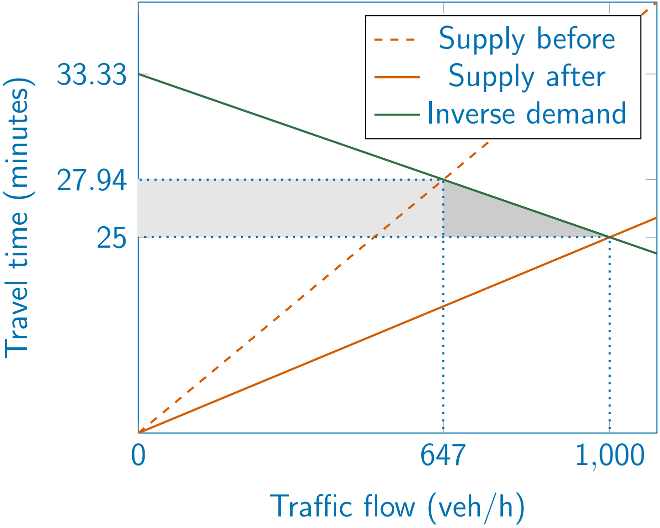
Impact of improvement

- Before: 1744.5 minutes.
- After: 4165 minutes.

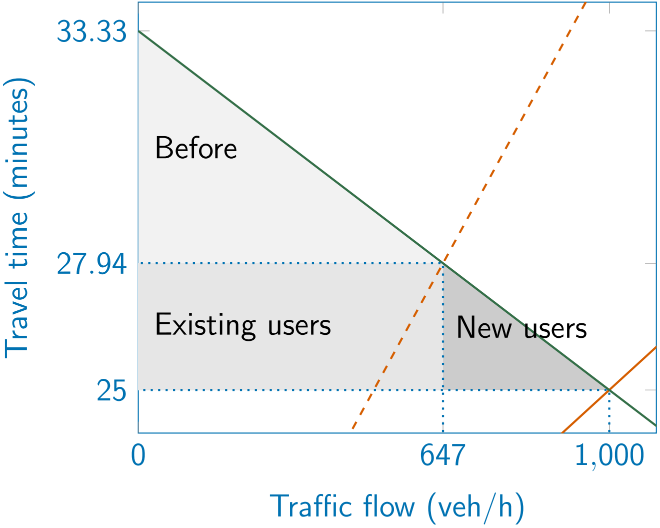
Question

Who enjoys these benefits?

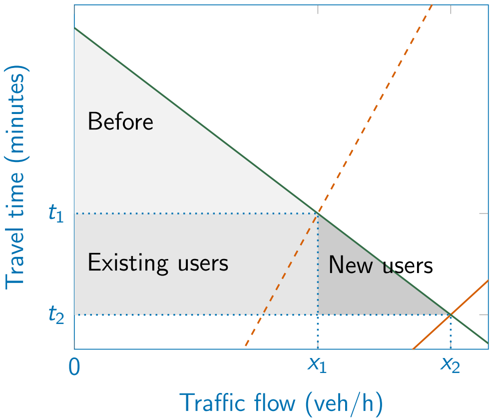
Additional consumer surplus



Additional consumer surplus



Rule of half



Additional consumer surplus

$$x_1(t_1 - t_2) + \frac{1}{2}(x_2 - x_1)(t_1 - t_2) = \frac{1}{2}(x_1 + x_2)(t_1 - t_2)$$

Highway: influencing the demand

Demand function

$$x = 4000 - 120t$$

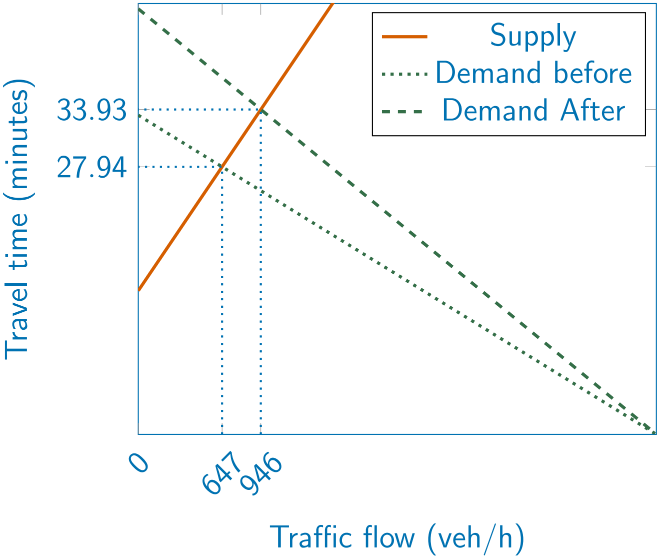
Modified demand function

Suppose that we are able to modify the demand function:

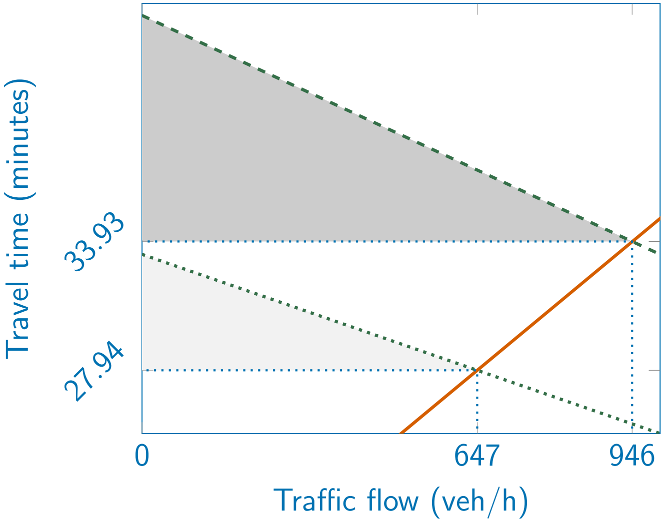
$$x = 4000 - 90t$$

What is the new consumer surplus?

New demand function: equilibrium



New demand function: consumer surplus



Consumer surplus: from 1744.5 to 4976.3

Comments

Change the supply function

- Main role of the engineer.
- Build / improve infrastructure.
- Provide new services.

Change the demand function

- Modify perception/behavior.
- Provide incentives/penalties.

Question: Where does the demand function come from?

Behavioral assumptions

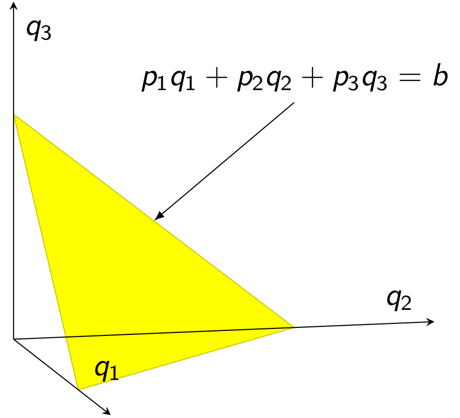
Context: consumption bundle

- Decision: quantities.
- Data: prices.

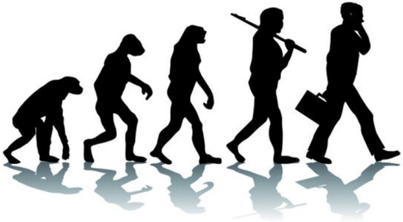
$$q = \begin{pmatrix} q_1 \\ \vdots \\ q_K \end{pmatrix}, \ p = \begin{pmatrix} p_1 \\ \vdots \\ p_K \end{pmatrix}$$

- Budget constraint

$$p^T q = \sum_{k=1}^{K} p_k q_k = b.$$



Behavioral assumptions



Concept

Homo economicus

Decision maker

- is consistently rational,
- is narrowly self-interested,
- optimizes her outcome.

Preferences

Operators $\succ$, $\sim$, and $\succsim$

- $q^k \succ q^\ell$: q^k is preferred to q^ℓ,
- $q^k \sim q^\ell$: indifference between q^k and q^ℓ,
- $q^k \succsim q^\ell$: q^k is at least as preferred as q^ℓ.

Preferences

Rationality

- Completeness: for all bundles k and ℓ,

$$q^k \succ q^\ell \text{ or } q^k \prec q^\ell \text{ or } q^k \sim q^\ell.$$

- Transitivity: for all bundles k, ℓ and m,

$$\text{if } q^k \succsim q^\ell \text{ and } q^\ell \succsim q^m \text{ then } q^k \succsim q^m.$$

- "Continuity": if q^k is preferred to q^ℓ and q^c is arbitrarily "close" to q^k, then q^c is preferred to q^ℓ.

Utility

Utility function

- Parameterized function:

$$\widetilde{u} = \widetilde{u}(q_1, \ldots, q_K; \theta) = \widetilde{u}(q; \theta)$$

- Consistent with the preference indicator:

$$q^k \succsim q^\ell \iff \widetilde{u}(q^k; \theta) \geq \widetilde{u}(q^\ell; \theta).$$

- Unique up to an order-preserving transformation.

$$q^k \succsim q^\ell \iff \widetilde{u}(q^k; \theta) \geq \widetilde{u}(q^\ell; \theta) \iff \exp \widetilde{u}(q^k; \theta) \geq \exp \widetilde{u}(q^\ell; \theta).$$

Behavioral assumption

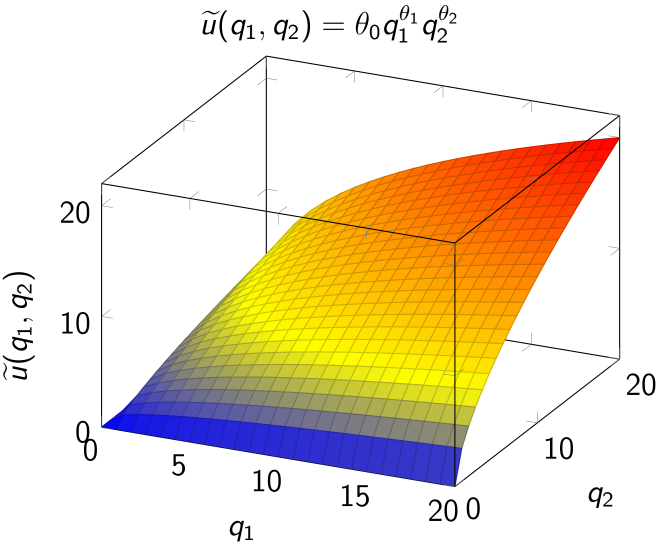
Decision = optimization problem

$$\max_q \widetilde{u}(q; \theta)$$

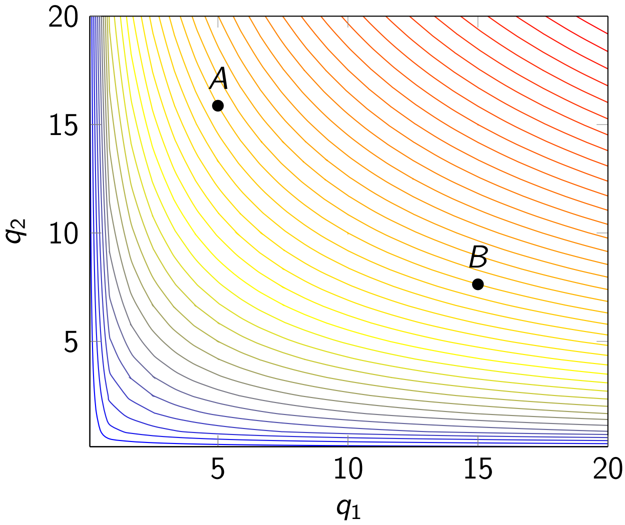
subject to

$$p^T q = b, \; q \geq 0.$$

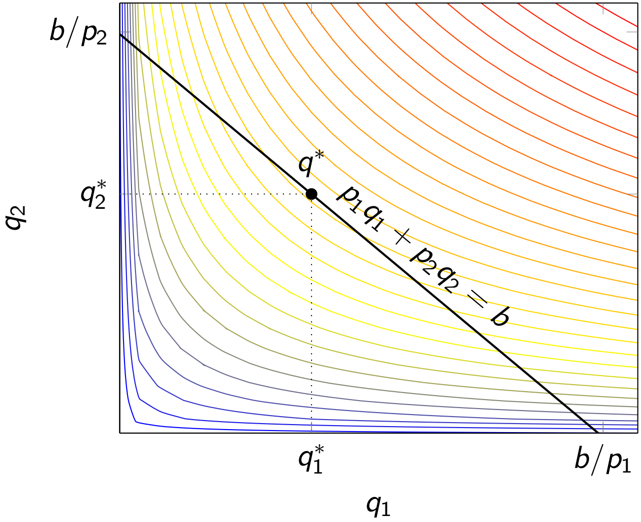
Example: Cobb-Douglas



Cobb-Douglas: indifference curves



Cobb-Douglas: utility maximization



Cobb-Douglas: utility maximization

Optimization problem

$$\max_{q_1,q_2} \widetilde{u}(q_1, q_2; \theta_0, \theta_1, \theta_2) = \theta_0 q_1^{\theta_1} q_2^{\theta_2}$$

subject to

$$p_1 q_1 + p_2 q_2 = b.$$

Equivalently

$$\max_{q_1,q_2} \ln \widetilde{u}(q_1, q_2; \theta_0, \theta_1, \theta_2) = \cancel{\ln \theta_0} + \theta_1 \ln q_1 + \theta_2 \ln q_2$$

subject to

$$p_1 q_1 + p_2 q_2 = b.$$

Cobb-Douglas: utility maximization

Lagrangian of the problem

$$L(q_1, q_2, \lambda) = \theta_1 \ln q_1 + \theta_2 \ln q_2 + \lambda(b - p_1 q_1 - p_2 q_2).$$

Necessary optimality condition

$$\nabla L(q_1, q_2, \lambda) = 0$$

Cobb-Douglas: utility maximization

$$L(q_1, q_2, \lambda) = \theta_1 \ln q_1 + \theta_2 \ln q_2 + \lambda(b - p_1 q_1 - p_2 q_2).$$

Gradient zero:
$\theta_1/q_1 - \lambda p_1 = 0 \iff \theta_1 = \lambda p_1 q_1$
$\theta_2/q_2 - \lambda p_2 = 0 \iff \theta_2 = \lambda p_2 q_2$
Add them and solve for λ (using the constraint):
$\lambda = (\theta_1 + \theta_2)/b$
Use $q_1 = \theta_1/\lambda p_1$ to obtain
$q_1 = \frac{b\theta_1}{p_1(\theta_1+\theta_2)}$
Similarly
$q_2 = \frac{b\theta_2}{p_2(\theta_1+\theta_2)}$
These are demand functions!!

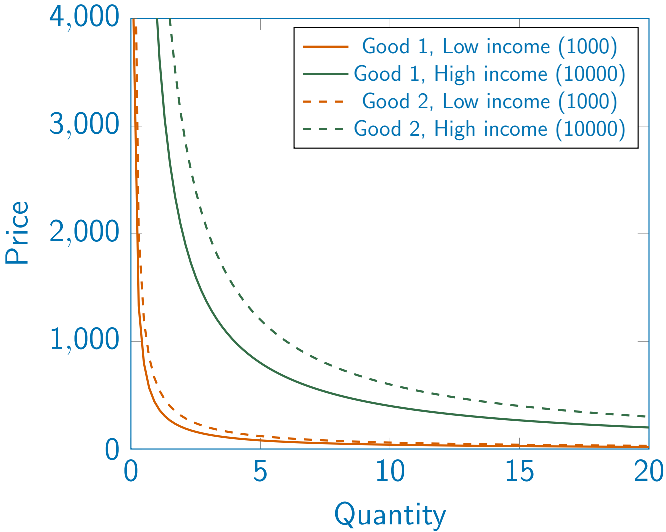
Optimal solution

Demand functions

$$q_1^* = \frac{b}{p_1} \frac{\theta_1}{\theta_1 + \theta_2}.$$

$$q_2^* = \frac{b}{p_2} \frac{\theta_2}{\theta_1 + \theta_2}.$$

Inverse demand functions



Summary

Supply and demand

- Supply: $p = f_s(q)$. Demand: $q = f_d(p)$.
- Fixed point: p^* solves $p^* = f_s(f_d(p^*))$.
- Modifications have impact on both.

Supply functions

- Characterize system performance.
- Details next semester with Prof. Geroliminis...

Summary

Demand elasticity

- Percentage change of q in response to a change in p.
- Point and arc elasticities.

Consumer surplus

- Difference between what consumers might be willing to pay for a service and what they actually pay.
- Measure of social welfare.

Demand functions

- Derived from behavioral assumptions.
- Utility maximization.
- Demand functions obtained from optimality conditions.

Summary

Change the supply function

- Build / improve infrastructure.
- Provide new services.

Change the demand function

- Modify perception/behavior.
- Provide incentives/penalties.

Bibliography