Travel demand

Assignment

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Introduction to transportation systems

EPFL

Trip-based model: the 4-step approach

4-step approach

- ✓ Trip generation
- ✓ Trip distribution
- ✓ Modal split
- ▶ Assignment

Objective

Find the link flows

Context

Single mode

Origin-destination table

- f_{rs} for each pair of zones/centroids (r, s).

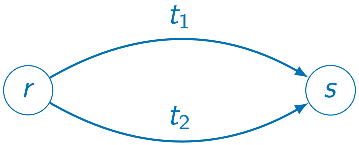
Transportation network

- Link performance functions: $t_\ell = t(x_\ell)$.
- Link-path incidence matrix P.

Assignment

Behavior

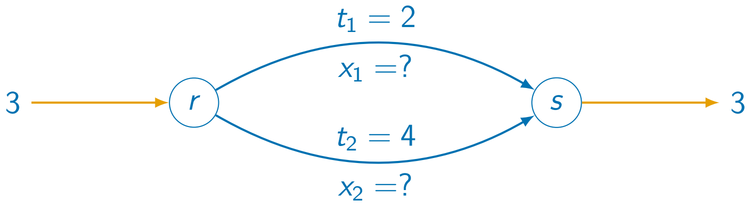
Route choice Assumption: utility maximizers, best path, "shortest" path



Example: $t_1 = 2$, $t_2 = 4$ Warning: all travelers have the same behavior The whole flow will take link 1: unrealistic But, we need to account for congestion

Assignment

All-or-nothing assignment



$x_1 = 3,\ x_2 = 0$

Congestion

Link performance functions

$$t_1(x) = 2 + x_1^2$$
$$t_2(x) = 4 + 2x_2^2$$

	Free flow		All on link 1		All on link 2	
	$\ell = 1$	$\ell = 2$	$\ell = 1$	$\ell = 2$	$\ell = 1$	$\ell = 2$
Flow	0	0	3	0	0	3
Cost	2	4	11	4	2	22

All-or-nothing does not make sense

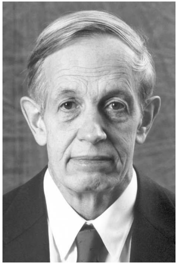
Congestion

Link performance functions

$$t_1(x) = 2 + x_1^2$$
$$t_2(x) = 4 + 2x_2^2$$

	x_1	t_1	x_2	t_2	Choice
Empty network	0	2	0	4	$\ell = 1$
First unit	1	3	0	4	$\ell = 1$
Second unit	2	6	0	4	$\ell = 2$
Third unit	2	6	1	6	Equilibrium

Equilibrium



nobelprize.org

Lisbon, 2010

Nash equilibrium

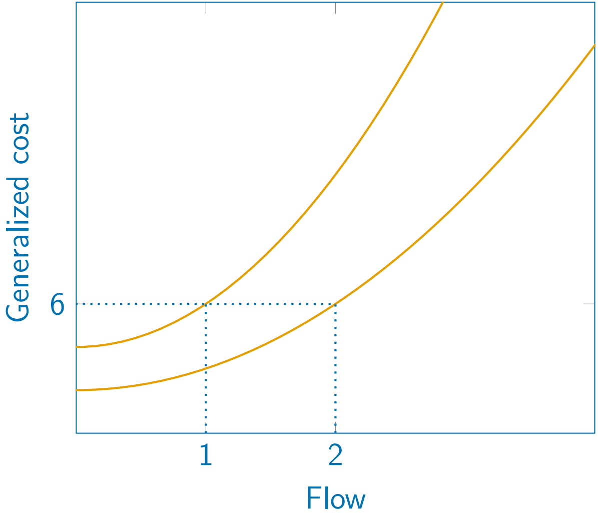
Situation where no traveler can improve her travel time by unilaterally changing routes.

John Forbes Nash Jr.

- 1928–2015
- Nobel laureate 1994
- PhD thesis on non cooperative games: 1950 (28 pages)

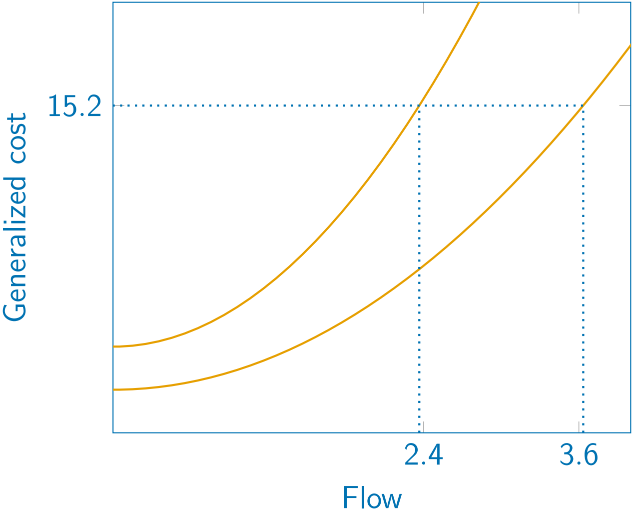
Equilibrium

$f_{rs} = 3$



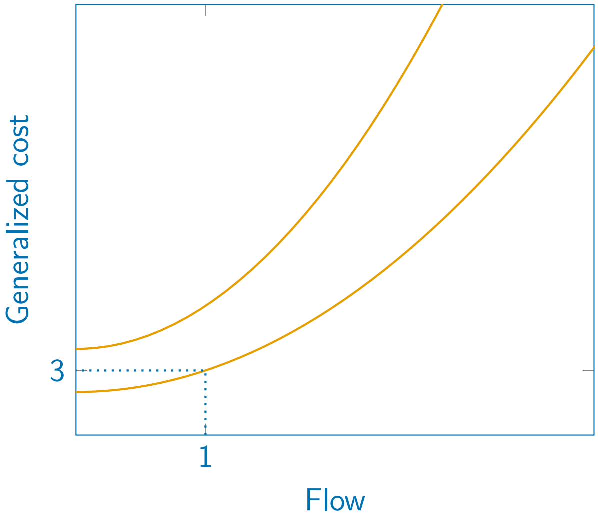
Equilibrium

$f_{rs} = 6$



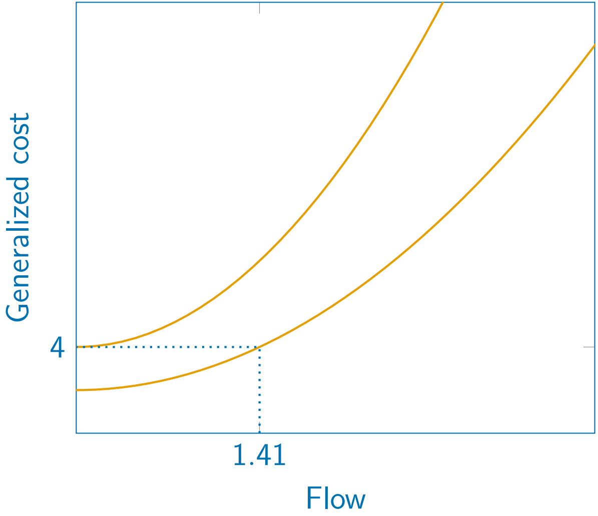
Equilibrium

$f_{rs} = 1$



Equilibrium

$f_{rs} = \sqrt{2}$



Nash equilibrium

Observations

- If $t_1(f_{rs}) \leq t_2(0)$, everybody uses link 1.
- If $t_1(f_{rs}) \geq t_2(0)$,
  - both links are used,
  - they have equal costs:

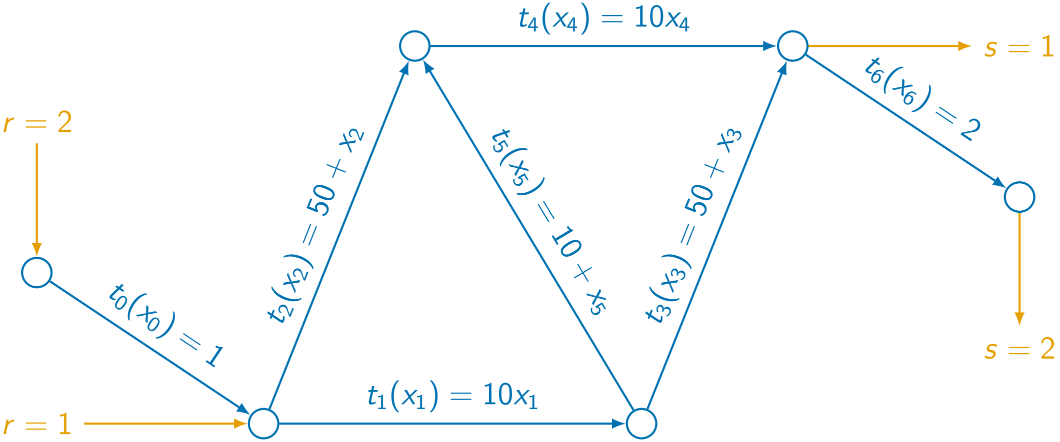
$$t_1(x_1) = t_2(x_2) \text{ and } x_1 + x_2 = f_{rs}$$

Nash equilibrium = user equilibrium

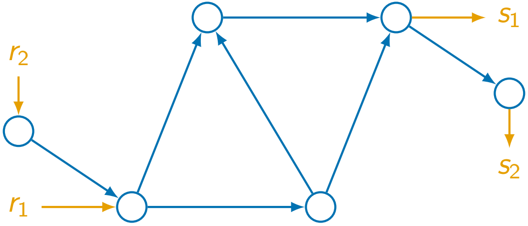
For each O-D pair, at user equilibrium,

- the generalized cost on all used paths is equal, and
- the generalized cost on all used paths is less or equal to the generalized cost on any unused path.

Network example



Network example



OD table: 3 entries

	s_1	s_2
r_1	3	1
r_2	2	0

Paths

- (r_1, s_1) : 
- (r_1, s_2) : 
- (r_2, s_1) : 

Empty network

p	flow	x_0	x_1	x_2	x_3	x_4	x_5	x_6	t_0	t_1	t_2	t_3	t_4	t_5	t_6	cost
r_1, s_1: $f_{rs} = 3$																
	0		0		0					0		50				50
	0			0		0					50		0			50
	0		0			0	0			0			0	10		10
r_1, s_2: $f_{rs} = 1$																
	0		0		0			0		0		50			2	52
	0			0		0		0			50		0		2	52
	0		0			0	0	0		0			0	10	2	12
r_2, s_1: $f_{rs} = 2$																
	0	0	0		0				1	0		50				51
	0	0		0		0			1		50		0			51
	0	0	0			0	0		1	0			0	10		11

Best free flow paths

p	flow	x_0	x_1	x_2	x_3	x_4	x_5	x_6	t_0	t_1	t_2	t_3	t_4	t_5	t_6	cost
r_1, s_1: $f_{rs} = 3$																
	0		6		0					60		50				110
	0			0		6					50		60			110
	3		6			6	6			60			60	16		136
r_1, s_2: $f_{rs} = 1$																
	0		6		0			1		60		50			2	112
	0			0		6		1			50		60		2	112
	1		6			6	6	1		60			60	16	2	138
r_2, s_1: $f_{rs} = 2$																
	0	2	6		0				1	60		50				111
	0	2		0		6			1		50		60			111
	2	2	6			6	6		1	60			60	16		137

Equilibrium

p	flow	x_0	x_1	x_2	x_3	x_4	x_5	x_6	t_0	t_1	t_2	t_3	t_4	t_5	t_6	cost
r_1, s_1: $f_{rs} = 3$																
	1		4		2					40		52				92
	1			2		4					52		40			92
	1		4			4	2			40			40	12		92
r_1, s_2: $f_{rs} = 1$																
	1		4		2			1		40		52			2	94
	0			2		4		1			52		40		2	94
	0		4			4	2	1		40			40	12	2	94
r_2, s_1: $f_{rs} = 2$																
	0	2	4		2				1	40		52				93
	1	2		2		4			1		52		40			93
	1	2	4			4	2		1	40			40	12		93

Another equilibrium

p	flow	x_0	x_1	x_2	x_3	x_4	x_5	x_6	t_0	t_1	t_2	t_3	t_4	t_5	t_6	cost
r_1, s_1: $f_{rs} = 3$																
	1		4		2					40		52				92
	0			2		4					52		40			92
	2		4			4	2			40			40	12		92
r_1, s_2: $f_{rs} = 1$																
	1		4		2			1		40		52			2	94
	0			2		4		1			52		40		2	94
	0		4			4	2	1		40			40	12	2	94
r_2, s_1: $f_{rs} = 2$																
	0	2	4		2				1	40		52				93
	2	2		2		4			1		52		40			93
	0	2	4			4	2		1	40			40	12		93

Modeling

Notations

- Number of links: K^{ℓ}
- Number of paths: K^{p}
- Number of ODs: K^{rs}
- Paths for OD q: $\mathcal{P}_q$

- Link flow: x
- Path flow: y
- Link cost: t
- Path cost: c

Modeling

Link-path incidence matrix

$$P \in \{0,1\}^{K^\ell \times K^p}$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Route choice matrix

$$R \in \mathbb{R}_+^{K^p \times K^{rs}}$$

$$R = \begin{pmatrix} 1/3 & 0 & 0 \\ 1/3 & 0 & 0 \\ 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}$$

Route choice matrix

Definition

$$R \in \mathbb{R}_{+}^{K^p \times K^{rs}}$$

R_{pq} is the proportion of travelers on OD pair q who choose route p.

Notes

- $\sum_p R_{pq} = 1$, for each q.
- Each path is associated with exactly one OD pair.
- $R_{pq} = 0$ if $p \notin \mathcal{P}_q$.
- For all-or-nothing assignment, $R_{pq} \in \{0, 1\}$, $\forall p, q$.

Modeling

From OD table to path flows

$$y = Rf : \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 & 0 \\ 1/3 & 0 & 0 \\ 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$y_p = \sum_q R_{pq} f_q, \forall p.$$

Modeling

From path flows to link flows

$$x = Py : \begin{pmatrix} 2 \\ 4 \\ 2 \\ 2 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$x_\ell = \sum_p P_{\ell p} y_p, \forall \ell.$$

Modeling

From OD table to link flows

$$x = PRf : \begin{pmatrix} 2 \\ 4 \\ 2 \\ 2 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 2/3 & 1 & 1/2 \\ 1/3 & 0 & 1/2 \\ 1/3 & 1 & 0 \\ 2/3 & 0 & 1 \\ 1/3 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$x_\ell = \sum_p \sum_q P_{\ell p} R_{pq} f_q, \forall \ell.$$

Assignment matrix: $Q = PR, \; Q \in \mathbb{R}_+^{K^\ell \times K^{rs}}$.

Modeling

From link costs to path costs

$$c = P^T t : \begin{pmatrix} 92 \\ 92 \\ 92 \\ 94 \\ 94 \\ 94 \\ 93 \\ 93 \\ 93 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 40 \\ 52 \\ 52 \\ 40 \\ 12 \\ 2 \end{pmatrix}$$

$$c_p = \sum_{\ell} P_{\ell p} t_{\ell}.$$

Modeling

OD specific link-path incidence matrix

- P^q
- Columns of P corresponding to $\mathcal{P}_q$.

$$P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

OD specific route choice vector

- R^q
- Rows and column of R corresponding to q.

$$R_1 = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

Modeling

Path flows for OD q

$$y^q = R^q f_q : y_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} 3, \; y_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} 2.$$

Path costs for OD q

$$c^q = (P^q)^T t : c_1 = \begin{pmatrix} 92 \\ 92 \\ 92 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 40 \\ 52 \\ 52 \\ 40 \\ 12 \\ 2 \end{pmatrix}$$

Modeling

Lowest cost assumption

- Define the minimum cost for OD q:

$$c_q^* = \min_p c_p^q, \ \forall q.$$

Summary

	Links	Paths	OD pair
Flow	x_ℓ	y_p	f_q
Cost	t_ℓ	c_p	c_q^*

Equilibrium conditions

- Minimum cost for each OD pair q:

$$c_p^q \geq c_q^*, \ \forall q.$$

- For each OD pair q, the cost on all used paths is minimum:

$$y_p^q(c_p^q - c_q^*) = 0, \ \forall p, q.$$

- For each OD pair q, the whole demand is assigned:

$$\sum_p y_p^q = f_q, \ \forall q.$$

- Non negativity of path flows:

$$y_p^q \geq 0, \ \forall p, q.$$

Beckmann's model

Optimization problem

$$\min_{y} \sum_{\ell} \int_0^{x_\ell} t_\ell(z) dz$$

subject to

$$\sum_p y_p^q = f_q, \qquad \forall q,$$

$$y_p^q \geq 0, \quad \forall p, q,$$

where

$$x_\ell = \sum_p P_{\ell p} y_p^q, \forall \ell, q.$$

Assumptions

$$\frac{\partial t_\ell(x_\ell)}{\partial x_\ell} > 0, \qquad \forall \ell,$$

$$\frac{\partial t_\ell(x_\ell)}{\partial x_{\ell'}} = 0, \quad \forall \ell \neq \ell'.$$

Two link example

Link performance functions

$$t_1(x) = 2 + x_1^2$$
$$t_2(x) = 4 + 2x_2^2$$

Objective function

$$\int_0^{x_1} t_1(z)dz = 2z + \frac{1}{3}z^3\Big]_0^{x_1} = 2x_1 + \frac{1}{3}x_1^3.$$
$$\int_0^{x_2} t_2(z)dz = 4z + \frac{2}{3}z^3\Big]_0^{x_2} = 4x_2 + \frac{2}{3}x_2^3.$$

Two link example

Optimization problem

$$\min_{x_1, x_2} 2x_1 + \frac{1}{3}x_1^3 + 4x_2 + \frac{2}{3}x_2^3$$

subject to

$$x_1 + x_2 = 3$$
$$x_1, x_2 \geq 0.$$

$x_2 = 3 - x_1$

$f(x_1) = 2x_1 + \frac{1}{3}x_1^3 + 4(3 - x_1) + \frac{2}{3}(3 - x_1)^3$

$$f'(x_1) = 2 + x_1^2 - 4 - 2(3 - x_1)^2$$
$$= -x_1^2 + 12x_1 - 20$$
$$f''(x_1) = -2x_1 + 12$$

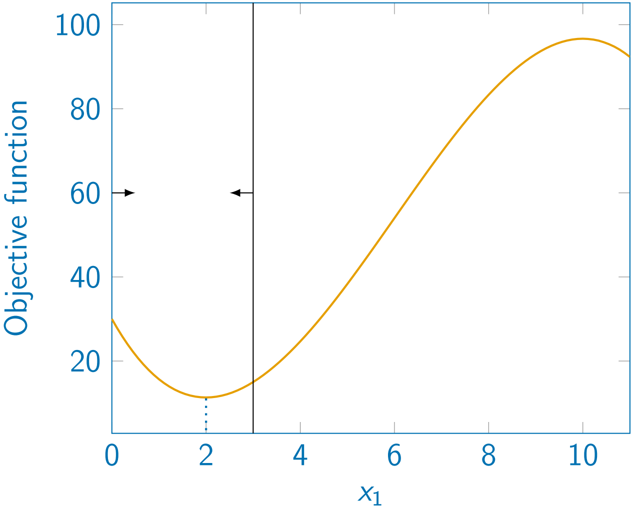
$f'(x_1) = 0$ if $x_1 = 2$ or $x_1 = 10$

$f''(2) = 8 > 0$, $f''(10) = -8 < 0$

Optimal solution: $x_1 = 2$,

$x_2 = 3 - x_2 = 1$.

Two link example



Beckmann's model

Optimization problem

$$\min_{y} \sum_{\ell} \int_{0}^{x_\ell} t_\ell(z) dz$$

subject to

$$\sum_{p'} y_{p'}^{q'} = f_{q'}, \qquad \forall q', \ [\lambda_{q'}]$$

$$y_{p'}^{q'} \geq 0, \quad \forall p', q', \ [\mu_{p'q'}]$$

Lagrangian

$$L(y; \lambda, \mu) =$$

$$\sum_{\ell} \int_{0}^{x_\ell} t_\ell(z) dz$$

$$+ \sum_{q'} \lambda_{q'} (f_{q'} - \sum_{p'} y_{p'}^{q'})$$

$$- \sum_{p'} \sum_{q'} \mu_{p'q'} y_{p'}^{q'}$$

Beckmann's model

Lagrangian

$$L(y;\lambda,\mu) =$$
$$\sum_{\ell} \int_0^{x_\ell} t_\ell(z)dz$$
$$+ \sum_{q'} \lambda_{q'} (f_{q'} - \sum_{p'} y_{p'}^{q'})$$
$$- \sum_{p'} \sum_{q'} \mu_{p'q'} y_{p'}^{q'}$$

Necessary optimality conditions

$$\frac{\partial L}{\partial y_p^q} = 0, \ \forall p, q.$$

Inequality constraints

$$\mu_{pq} \geq 0, \ \forall p, q.$$

Complementarity slackness

$$\mu_{pq} y_p^q = 0, \forall p, q.$$

Beckmann's model

Objective function

$$f(y) = \sum_{\ell} \int_0^{x_\ell} t_\ell(z) dz$$

where

$$x_\ell = \sum_p P_{\ell p} y_p^q, \forall \ell, q.$$

$$\begin{aligned} \frac{\partial f}{\partial y_p^q} &= \sum_\ell \frac{\partial f}{\partial x_\ell} \frac{\partial x_\ell}{\partial y_{pq}} \\ &= \sum_\ell P_{\ell p} t_\ell(x_\ell) \\ &= c_p^q. \end{aligned}$$

Beckmann's model

Lagrangian

$$L(y;\lambda,\mu) =$$

$$f(y)$$

$$+\sum_{q'}\lambda_{q'}(f_{q'} - \sum_{p'} y_{p'}^{q'})$$

$$-\sum_{p'}\sum_{q'}\mu_{p'q'}y_{p'}^{q'}$$

Derivatives

$$\frac{\partial L}{\partial y_p^q} = c_p^q - \lambda_q - \mu_{pq}.$$

Necessary optimality conditions

$$\mu_{pq} = c_p^q - \lambda_q \geq 0,\ \forall p, q.$$

Complementarity slackness

$$y_p^q(c_p^q - \lambda_q) = 0.$$

Optimality conditions = equilibrium conditions

- Denote $\lambda_q = c^*_q$. It is the minimum cost:

$$c^q_p \geq c^*_q, \ \forall q. \text{[Optimality conditions]}$$

- For each OD pair q, the cost on all used paths is minimum:

$$y^q_p (c^q_p - c^*_q) = 0, \ \forall p, q. \text{[Compl. slackness]}$$

- For each OD pair q, the whole demand is assigned:

$$\sum_p y^q_p = f_q, \ \forall q. \text{[Primal constraints]}$$

- Non negativity of path flows:

$$y^q_p \geq 0, \ \forall p, q. \text{[Primal constraints]}$$

Beckmann's model

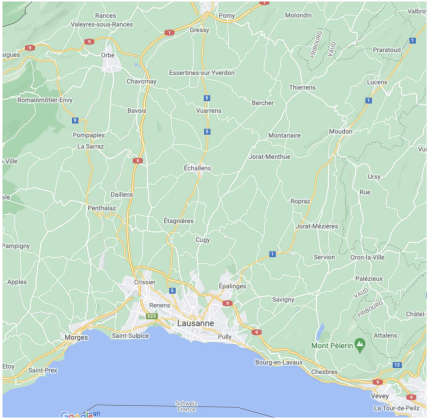
Equivalence

Solution of the optimization problem = equilibrium path flows

Uniqueness

- Optimization problem is strictly convex in the link flows.
- Link flow solution is unique.
- Path flow solution is not necessarily unique.

Complexity



- Path-based formulation is untractable for real networks.
- The number of paths grows exponentially with the number of centroids.
- We had a similar issue with the shortest path problem.
- For the shortest path problem: Dijkstra.
- We will also rely on Dijkstra here.

Solution algorithm

Initialization Empty network.

- Link costs: $t_\ell(0)$.
- Link flows from all-or-nothing assignment: x^0.
- $k = 0$.

Step 1 Calculate link costs: $t_\ell^k = t_\ell(x^k)$.

Step 2 Link flows from all-or-nothing assignment: $\tilde{x}^k$.

Step 3 Line search.

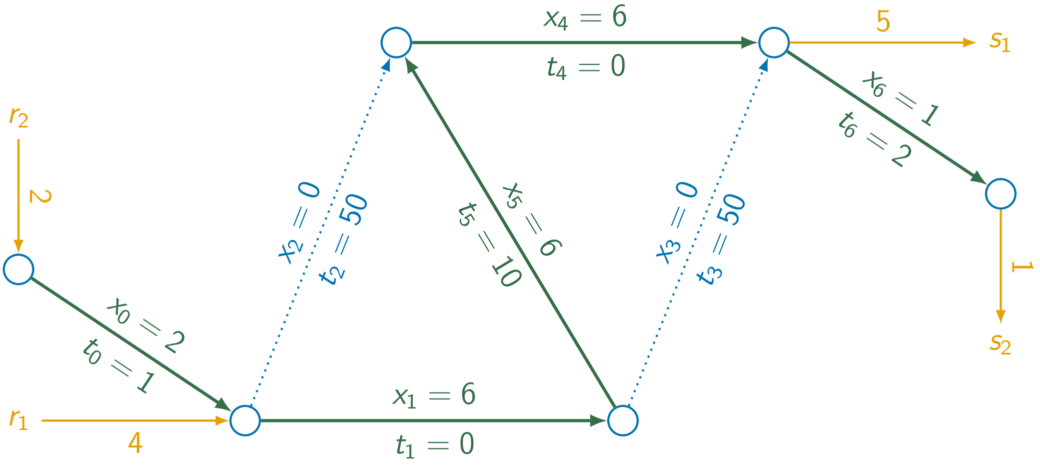
$$x^{k+1} = x^k + \alpha(\tilde{x}^k - x^k),\ 0 \leq \alpha \leq 1,$$

where α solves

$$\min_\alpha \sum_\ell \int_0^{x_\ell^{k+1}} t_\ell(z) dz$$

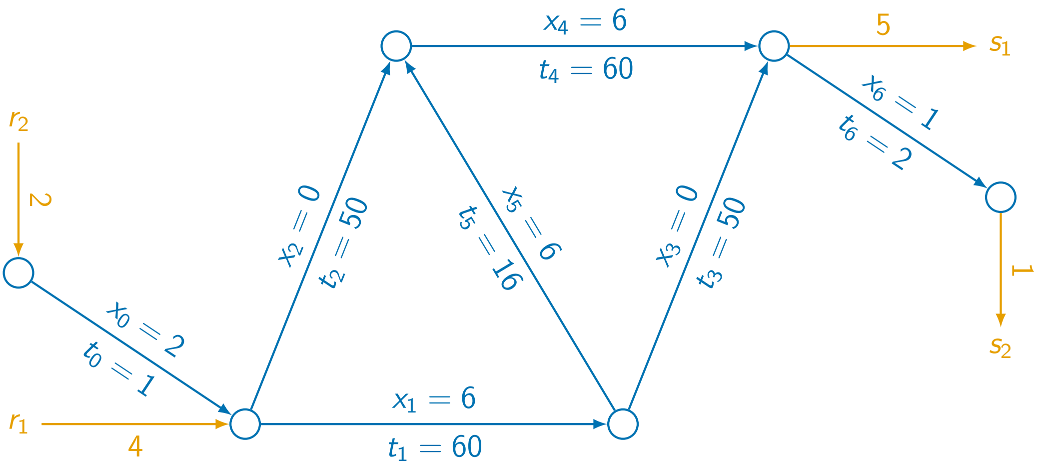
Step 4 Check convergence. If not, go to step 1.

All-or-nothing on empty network

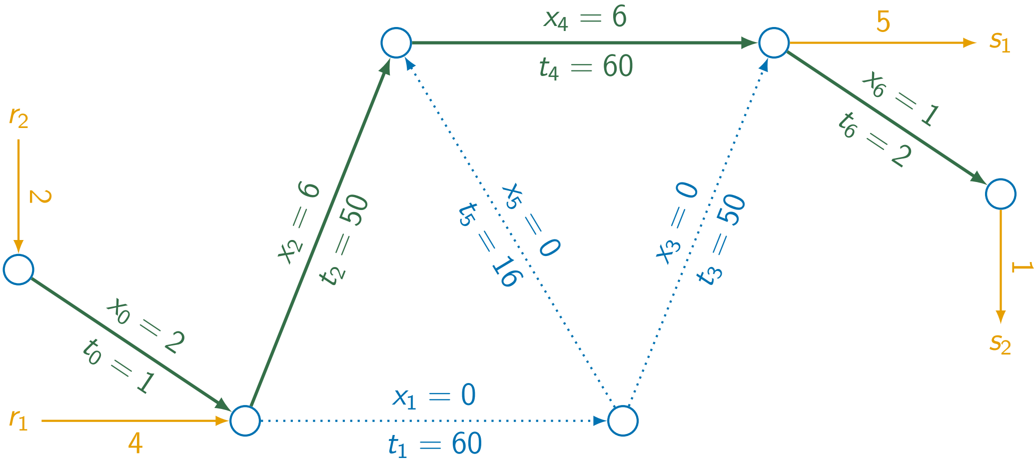


$c^*_{11} = 10,\ c^*_{12} = 12,\ c^*_{21} = 11$

Updated costs



All-or-nothing with updated costs

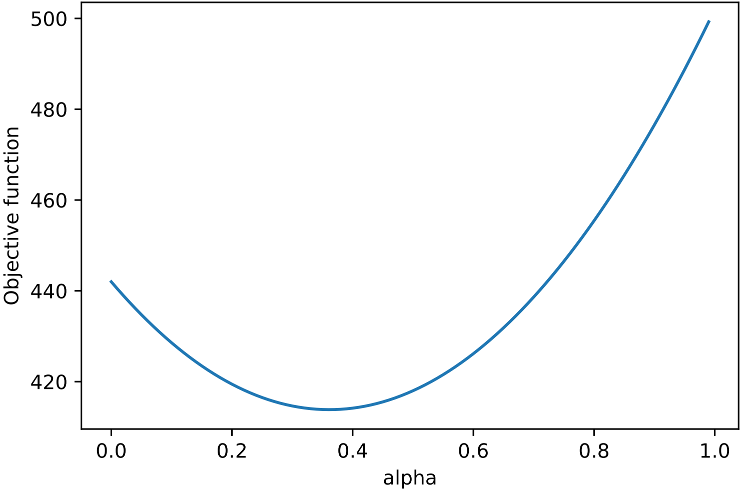


$$c_{11}^* = 110,\ c_{12}^* = 112,\ c_{21}^* = 111$$

Convex combination

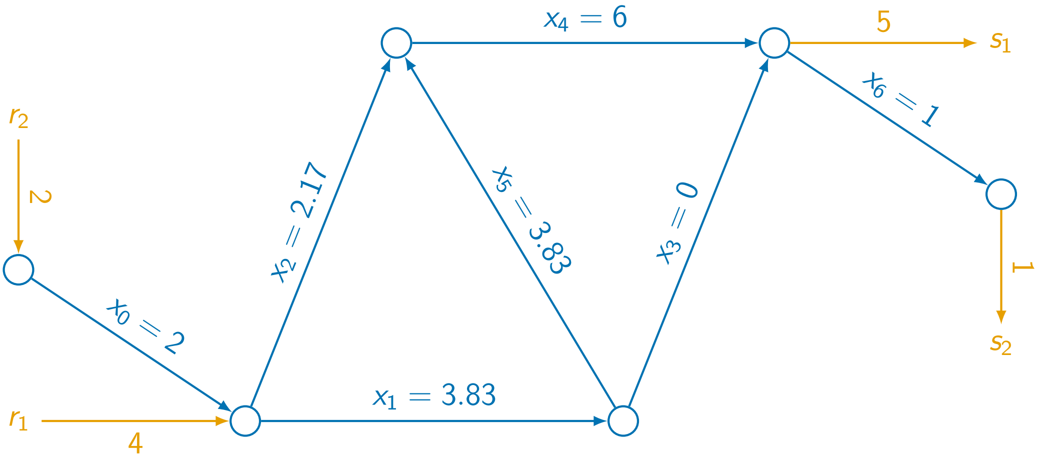
Arc	First flow	Second flow	Convex combination	
0	2	2	$2 + \alpha\ (2\text{-}2)$	$= 2$
1	6	0	$6 + \alpha\ (0\text{-}6)$	$= 6 - 6\ \alpha$
2	0	6	$0 + \alpha(6\text{-}0)$	$= 6\ \alpha$
3	0	0	$0 + \alpha(0\text{-}0)$	$= 0$
4	6	6	$6 + \alpha(6\text{-}6)$	$= 6$
5	6	0	$6 + \alpha\ (0\text{-}6)$	$= 6 - 6\ \alpha$
6	1	1	$1 + \alpha\ (1\text{-}1)$	$= 1$

Line search



$\alpha^* = 0.361$

Updated flows



Iterations

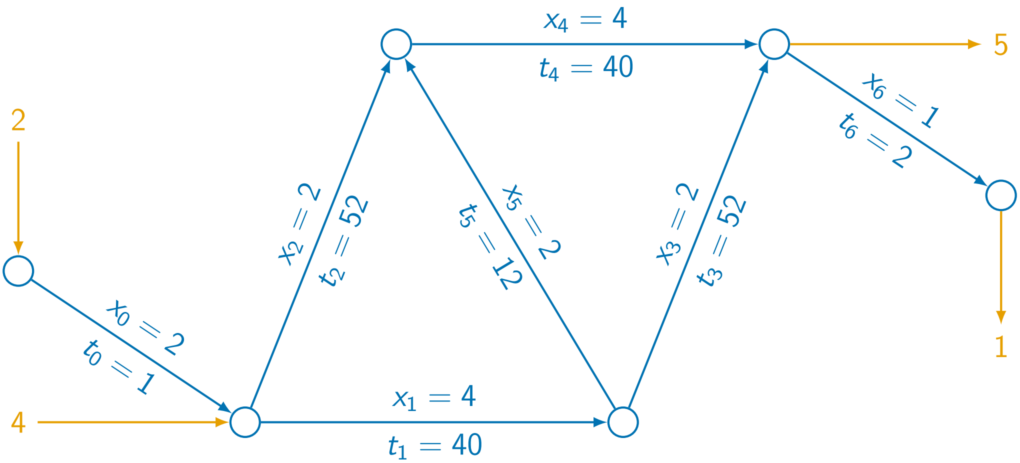
Iter	α	Objective function
0		442.00
1	0.361	413.83
2	0.309	391.72
3	0.0885	390.67
4	0.0538	390.31
5	0.0358	390.15
6	0.0249	390.08
7	0.0179	390.04
8	0.0131	390.02
9	0.00967	390.01
10	0.00722	390.01
11	0.00544	390.00

Comments

Complexity

- All-or-nothing: Dijkstra.
- Line search: link-based objective function
- No path enumeration is needed.
- Convergence may be slow.
- Convergence slower for highly congested networks.

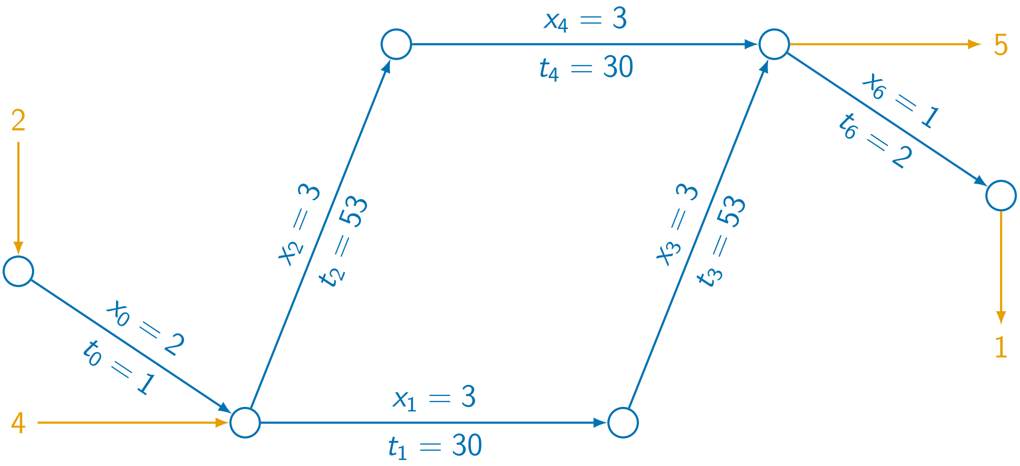
Equilibrium: level of service



$c^*_{11} = 92,\ c^*_{12} = 94,\ c^*_{21} = 93,$ Mean: $\frac{1}{6}(92 \cdot 3 + 94 \cdot 1 + 93 \cdot 2) = 92.7$

Level of service when a link is removed

Level of service when a link is removed



$c_{11}^* = 83,\ c_{12}^* = 85,\ c_{21}^* = 84,\ \text{Mean: } \frac{1}{6}(83 \cdot 3 + 85 \cdot 1 + 84 \cdot 2) = 83.7$

Braess paradox

Observation

- The capacity of the network is reduced.
- The performance of the network is improved.

Equivalently...

- The capacity of the network is increased.
- The performance of the network is deteriorated.

Is it a mathematical artifact? Or does it happen in reality?

Stuttgart, 1968

Events

- Schlossplatz
- Opening of a new traffic network.
- Consequences: big chaos.
- Solution: close Königstrasse

Source: [Knödel, 1969]

New-York, 1990

Events

- Earth Day (April 22)
- Closing of 42th street.
- Expectations: "earth day = doomsday"
- "You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem."
- Actually, the situation was better than expected.

Source: [Kolata, 1990]

Seoul, 2003

Before

After

Events

- Cheonggyecheon, Seoul
- Removal of a 6-lane highway.
- Expectations: catastrophe.
- In reality: "Many transportation professionals were surprised to learn that the city's traffic flow had actually improved, instead of worsening"

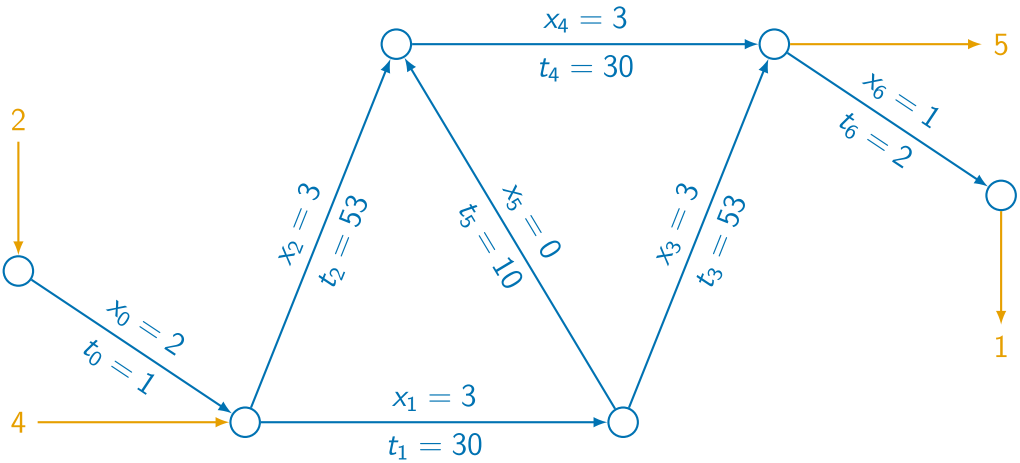
Source: [Baker, 2009]

Braess paradox

Why does it happen?

- People do not cooperate
- The new highway brings traffic in small roads.

What if we convince travelers to do the following?



$c^*_{11} = 83,\ c^*_{12} = 85,\ c^*_{21} = 84$, Mean: $\frac{1}{6}(83 \cdot 3 + 85 \cdot 1 + 84 \cdot 2) = 83.7$

Prisoner's dilemma

Context

- Joe and Averell have been arrested.
- They are separated and isolated.
- They are accused of a small robbery, with evidence.
- They are suspected of a major robbery, without evidence.

Prisoner's dilemma

Bargain

- To Joe: you can stay silent, or betray Averell.
- To Averell: you can stay silent, or betray Joe.
- If both stay silent: 1 year in prison.
- If both betray each other: 2 years in prison.
- If Joe betrays Averell, and Averell stays silent, Joe is free and 3 years of prison for Averell.
- If Averell betrays Joe, and Joe stays silent, Averell is free and 3 years of prison for Joe.

Prisoner's dilemma: global point of view

Strategies

Decision		Penalty		
Joe	Averell	Joe	Averell	Total penalty
Silent	Silent	1	1	2
Silent	Betray	3	0	3
Betray	Silent	0	3	3
Betray	Betray	2	2	4

Best strategy

- Both stay silent.
- Optimal globally and individually.

Prisoner's dilemma: individual points of view

Joe's point of view

Assume that Averell stays silent

- If I stay silent: 1 year in prison.
- If I betray Averell: I am free.

Assume that Averell betrays me

- If I stay silent: 3 years in prison.
- If I betray Averell: 2 years in prison.

Whatever Averell does, I am better off betraying him.

Averell's point of view

Assume that Joe stays silent

- If I stay silent: 1 year in prison.
- If I betray Joe: I am free.

Assume that Joe betrays me

- If I stay silent: 3 years in prison.
- If I betray Joe: 2 years in prison.

Whatever Joe does, I am better off betraying him.

Prisoner's dilemma

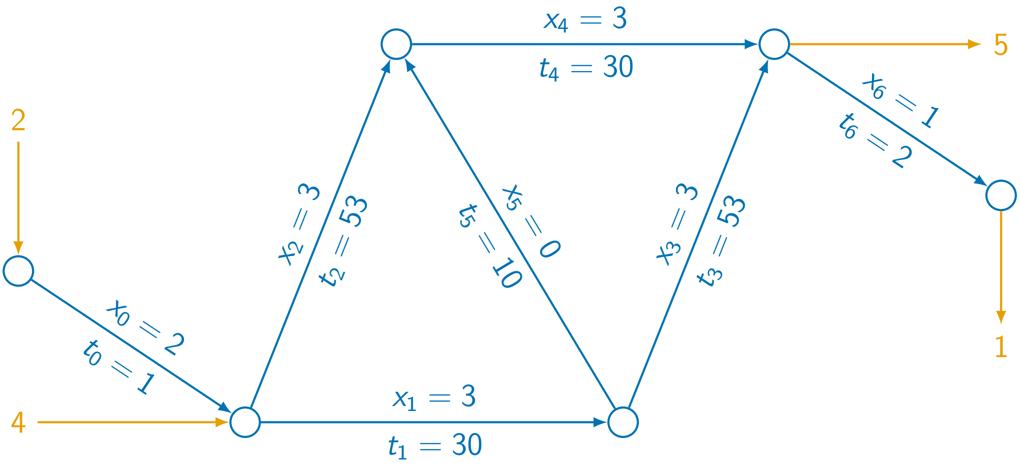
Nash equilibrium

- Equilibrium: betray the other.
- No player can improve the situation with a unilateral decision.

Cooperation

- Best joined decision: stay both silent.
- It requires cooperation and trust.

Can we convince travelers to do the following?



$c^*_{11} = 83,\ c^*_{12} = 85,\ c^*_{21} = 84$, Mean: $\frac{1}{6}(83 \cdot 3 + 85 \cdot 1 + 84 \cdot 2) = 83.7$

Traffic assignment

User equilibrium

$$y^* = \text{argmin}_y \sum_\ell \int_0^{x_\ell} t_\ell(z)dz$$

subject to

$$\sum_p y_p^q = f_q, \qquad \forall q,$$
$$y_p^q \geq 0, \quad \forall p, q.$$

System optimum

$$\tilde{y}^* = \text{argmin}_y \sum_\ell x_\ell t_\ell(x_\ell)$$

subject to

$$\sum_p y_p^q = f_q, \qquad \forall q,$$
$$y_p^q \geq 0, \quad \forall p, q.$$

$\sum_\ell x_\ell^* t_\ell(x_\ell^*) - \sum_\ell \tilde{x}_\ell^* t_\ell(\tilde{x}_\ell^*) \geq 0$: price of anarchy

Engineering point of view

Role

- Design
- Maintain
- Operate

Objective

- Minimize the price of anarchy.
- Benchmark: system optimum.

Actions

- Infrastructure.
- Influence the travelers.

Towards system optimum

Supply-based

- Traffic lights, speed limit, etc.
- Control strategies.
- Compliance guaranteed by law.
- See the course of Prof. Geroliminis.

Demand-based

- Information and incentives.
- Compliance not guaranteed.
- Pricing.

System optimum

Engineering and policy makers

- System optimum is about the average traveler.
- In the example, all travelers were better off when the link was removed.
- In practice, some travelers may pay a high price for the greater good.
- Concepts like equity, minimum level of service, etc. are important as well.

Summary

User equilibrium

- No traveler can improve her travel time by unilaterally changing routes.
- Minimum cost of all used paths.
- No flow on paths with higher costs.
- Equivalent optimization problem.

Braess paradox

- Decreasing capacity may improve the level of service.
- Increasing capacity may deteriorate the level of service.

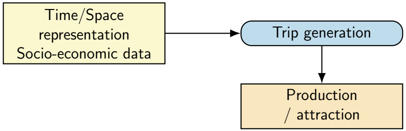
System optimum

- Requires cooperation among travelers.
- Prisoner's dilemma.
- Main objective for the engineer.

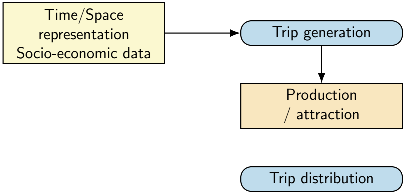
The 4-step approach: summary

Trip generation

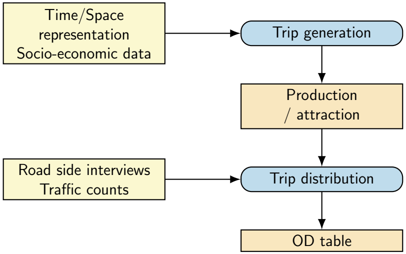
The 4-step approach: summary



The 4-step approach: summary



The 4-step approach: summary



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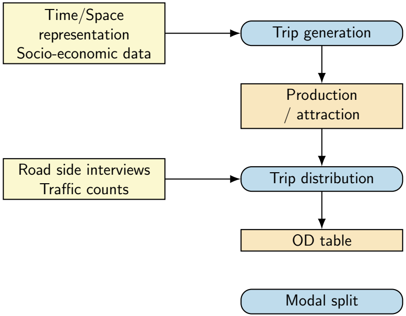

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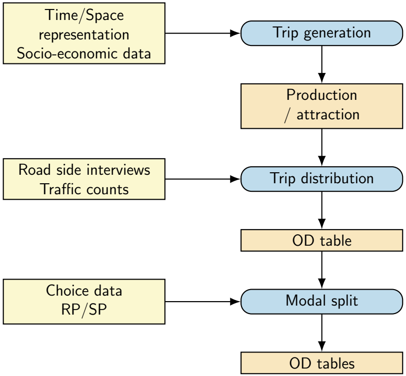

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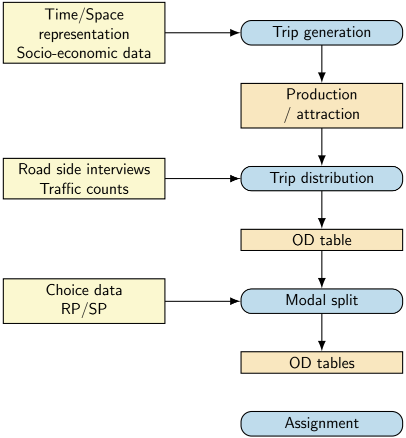

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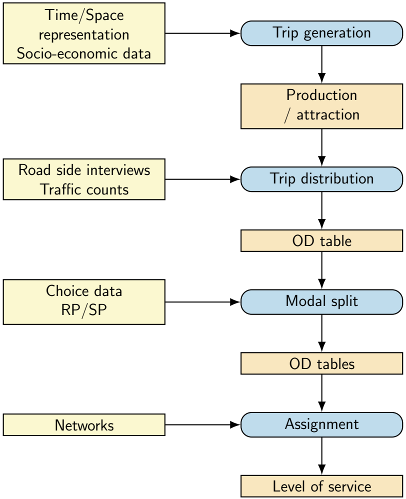

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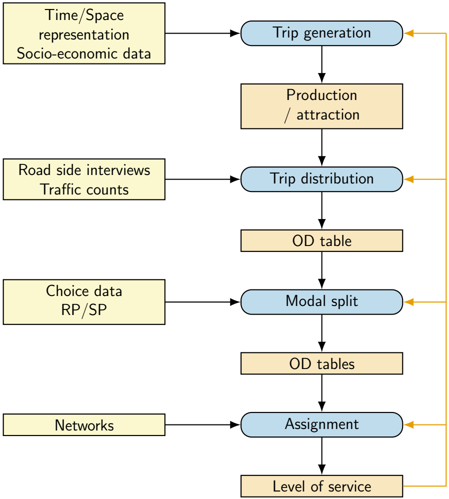

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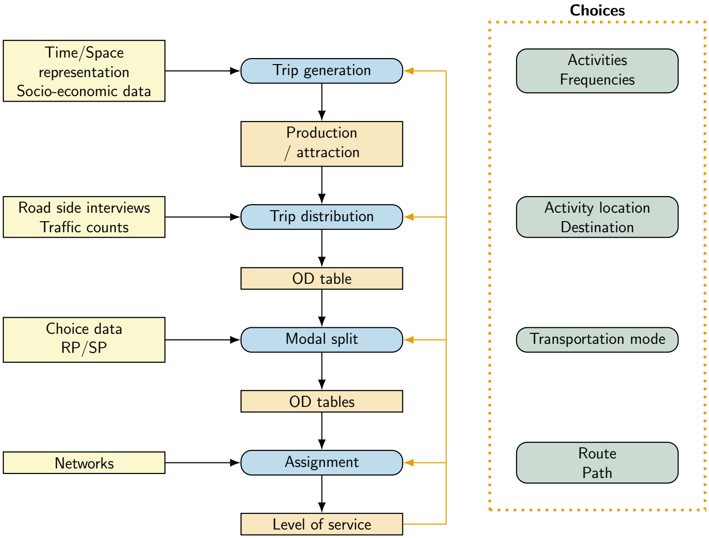

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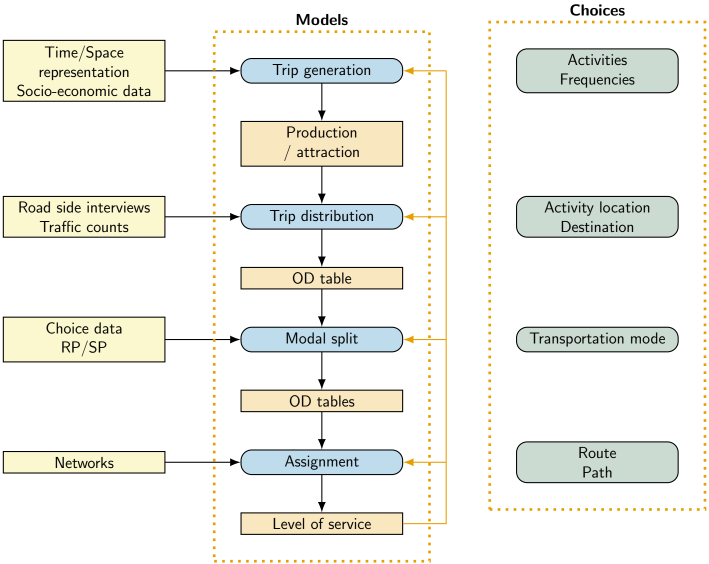

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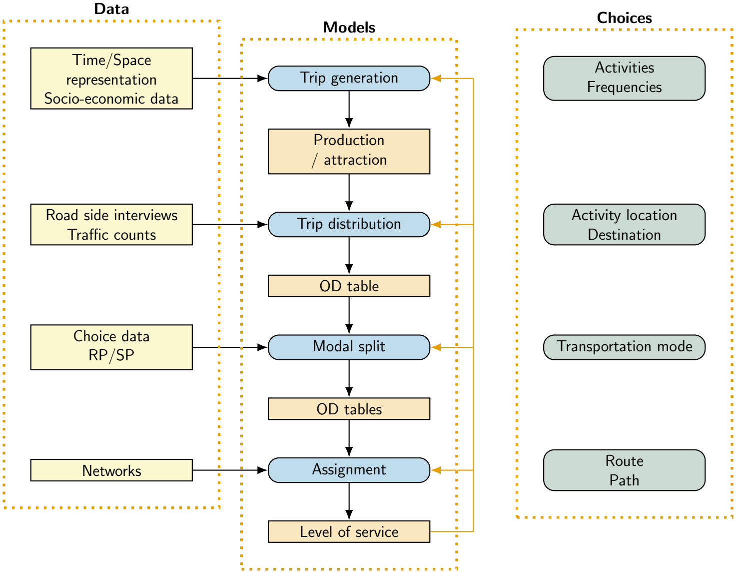

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