Solution Exercise 2 - 1st week 2025

A production line manufactured a large batch of cameras with faulty lenses that produce blurry images (Figure below, left). To avoid the recall of cameras, the company decided to upgrade the camera software and perform numerical deblurring of the images as soon as they are captured (Figure below, right). The deblurring is performed by solving a system of linear equations $Ax = b$, where A is the deblurring matrix that captures the lens imperfections.

Blurred image (b) Deblurred image ($x = A^{-1}b$)

a) Assuming that the deblurring matrix is dense and non-symmetric, propose a method that would be the most efficient for this task. Explain your choice.
b) Apply the proposed method to compute x for

$$A = \begin{bmatrix} 3 & 4 & -5 \\ 6 & -3 & 4 \\ 8 & 9 & -2 \end{bmatrix} \quad b_1 = \begin{bmatrix} 1 \\ 9 \\ 9 \end{bmatrix} \text{ and } b_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

c) Compute the determinant of the matrix A.

Solution:

a) The system is dense (not sparse) suggesting to use one of the direct methods for solving it. Since the system matrix is non-symmetric and it is expected to solve the system many times for the same A (the lens imperfection does not change between different shots) and different b, we will use the LU decomposition.

b) We need first to decompose matrix A as:

$$A = \begin{bmatrix} 3 & 4 & -5 \\ 6 & -3 & 4 \\ 8 & 9 & -2 \end{bmatrix} = LU$$

where L and U have the following form:

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Following the procedure detailed in the course we obtain :

$$L = \begin{bmatrix} 3 & 0 & 0 \\ 6 & -11 & 0 \\ 8 & -\frac{5}{3} & \frac{304}{33} \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & \frac{4}{3} & -\frac{5}{3} \\ 0 & 1 & -\frac{14}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

Now we continue by expressing $LUx = b$, and assigning $y = Ux$, so that we have two systems that can be solved by back- or forward- substitution. We first look the problem for b_1 = [1 9 9]':

$$Ly = b \Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 6 & -11 & 0 \\ 8 & -\frac{5}{3} & \frac{304}{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 9 \end{bmatrix}$$

$$y_1 = \frac{1}{3}$$

$$2 - 11y_2 = 9\,; \quad y_2 = -\frac{7}{11}$$

$$\frac{8}{3} + \frac{35}{33} + \frac{304}{33} y_3 = 9; \quad y_3 = \frac{87}{152}$$

We now solve $Ux = y$

$$\begin{bmatrix} 1 & \frac{4}{3} & -\frac{5}{3} \\ 0 & 1 & -\frac{14}{11} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{7}{11} \\ \frac{87}{152} \end{bmatrix}$$

$$x_3 = \frac{87}{152}$$

$$x_2 - \frac{14}{11}\frac{87}{152} = -\frac{7}{11}\,; x_2 = \frac{14}{152}$$

$$x_1 + \frac{4}{3}\frac{14}{152} - \frac{5}{3}\frac{87}{152} = \frac{1}{3}\,; x_1 = \frac{177}{152}$$

For the second system, b_2 = [3 5 4]':

$$Ly = b \Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 6 & -11 & 0 \\ 8 & -\frac{5}{3} & \frac{304}{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

$$y_1 = 1$$

$$6 - 11y_2 = 5\,; y_2 = \frac{1}{11}$$

$$8 - \frac{5}{33} + \frac{304}{33} y_3 = 4\,; y_3 = -\frac{127}{304}$$

Then, from $L^T x = y$

$$\begin{bmatrix} 1 & \frac{4}{3} & -\frac{5}{3} \\ 0 & 1 & -\frac{14}{11} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{11} \\ -\frac{127}{304} \end{bmatrix}$$

$$x_3 = -\frac{127}{304}$$

$$x_2 + \frac{14}{11}\frac{127}{304} = \frac{1}{11}\,; x_2 = -\frac{134}{304}$$

$$x_1 - \frac{4}{3}\frac{134}{304} + \frac{5}{3}\frac{127}{304} = 1\,; x_1 = \frac{271}{304}$$

c) Here we will benefit from the specific structure of matrices L U and the fact that det det $LU = \det L \det U$. Therefore, $\det A = \det L = -3 \cdot 11 \cdot \frac{304}{33} = -304$