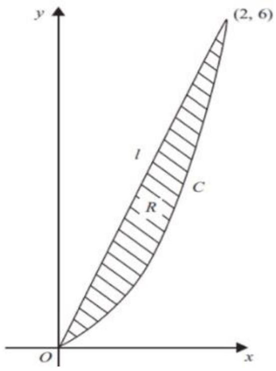
Solution Exercise 4 – 4th week 2025

A boat builder needs to compute the surface of the sail, R, to estimate the aerodynamic forces acting on the boat. The contour of the sail is defined by the linear segment l, which joins the origin (0,0) and the point (2,6), and the curve C, which is described by the equation



$$y = x\sqrt{x^3+1},\ \ 0 \le x \le 2$$

a) Complete the table below that provides values of y on the curve C.

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y	0				1.414				6

b) Compute the integral $\int_0^2 x\sqrt{x^3+1}\,dx$ using the most appropriate composite rule with the values from the table in a).

c) Use your answer calculated in b) to approximate the surface of the sail, R. Provide your solution with three significant digits.

d) Compute the surface of the sail, R, using the two points Gauss-Legendre formula and compare the result with c).

Solution:

a) To complete the table, we'll plug in each x value into the equation $y = x\sqrt{x^3+1}$ and calculate the corresponding y values:

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y	0	$0.25\sqrt{0.25^3+1} = 0.251$	$0.5\sqrt{0.5^3+1} = 0.530$	0.894	1.414	2.148	3.137	4.413	6

b) There are 9 points from the table in a), making this very conducive to the use of Simpson's 1/3rd rule for integration, because there is even number of subintervals (n=8). The composite formula of Simpson's 1/3 rule is as follows:

$$I_s = \frac{1}{3}h\left[y(x_0) + 4\sum_{i=1}^{n/2} y(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} y(x_{2i}) + y(x_n)\right]$$

$$I_s = \frac{1}{3}\frac{(2-0)}{8}\left[y(0) + 4\big(y(0.25)+y(0.75)+y(1.25)+y(1.75)\big) + 2\big(y(0.5)+y(1)+y(1.5)\big) + y(2)\right]$$

$$I_s = \frac{1}{3}0.25[0 + 4(0.251+0.894+2.148+4.413) + 2(0.530+1.414+3.137) + 6] = 3.91$$

c) To approximate the surface of sail R we need to subtract the integral surface I_s, calculated in b), form the triangular surface formed from the linear segment l.

$$R = \left(\frac{2*6}{2}\right) - 3.91 = 2.09$$

d) The formula for the two points Gauss-Legendre is as follows:

$$I_g = \frac{b-a}{2}[w_1 \cdot y(x_1) + w_2 \cdot y(x_2)]\,,$$

where $x_1 = \frac{b-a}{2}z_1 + \frac{b+a}{2}$, $x_2 = \frac{b-a}{2}z_2 + \frac{b+a}{2}$

$w_1 = w_2 = 1$, $z_1 = {-1}/{\sqrt{3}}$, $z_2 = {1}/{\sqrt{3}}$

Based on this formula:

$$x_1 = \frac{2-0}{2}\left(\frac{-1}{\sqrt{3}}\right) + \frac{2+0}{2} = \frac{-1+\sqrt{3}}{\sqrt{3}}$$

$$x_2 = \frac{2-0}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{2+0}{2} = \frac{1+\sqrt{3}}{\sqrt{3}}$$

$$I_g = \frac{2-0}{2}\left[1 \cdot y\left(\frac{-1+\sqrt{3}}{\sqrt{3}}\right) + 1 \cdot y\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right)\right]$$

$$I_g = \left[\left(\frac{-1+\sqrt{3}}{\sqrt{3}}\right)\sqrt{\left(\frac{-1+\sqrt{3}}{\sqrt{3}}\right)^3 + 1}\right] + \left[\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right)\sqrt{\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right)^3 + 1}\right] = 3.93$$

Therefore, the surface of sail R is:

$$R = \left(\frac{2*6}{2}\right) - 3.93 = 2.07$$

R (1/3 Simpson's rule) = 2.09
R (two points Gauss-Legendre) = 2.07

The results are quite close, with only a slight difference in the third significant digit between the two methods. In general, Simpson's 1/3 and Gauss-Legendre methods are of similar accuracy. However, the accuracy that we get when we use the composite Simpson's 1/3 rule is better than this of the 2-point Gauss-Legendre. This happens because with Gauss-Legendre you have only two function evaluations, whereas with Simpson's 1/3 you use 8 function evaluations.