Solution Exercise 5 – 5th week 2025

Consider the following system of ODEs

$$\frac{dx_1}{dt} = 998x_1 + 1998x_2$$

$$\frac{dx_2}{dt} = -999x_1 - 1999x_2$$

The general solution of this system can be written in the form:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2c_1 + 2c_2 \\ -c_1 - c_2 \end{bmatrix} e^{-t} + \begin{bmatrix} -c_1 - 2c_2 \\ c_1 + 2c_2 \end{bmatrix} e^{-1000t}$$

a) Which method you would apply to solve numerically the above-mentioned system of ODEs? Discuss what would be the appropriate step size.

b) For $x_1(0) = 1$ and $x_2(0) = 1$, apply the proposed method with the proposed step size to solve the system in the interval $t \in (0,1)$ (provide details of calculating the first step).

Solution:

We see that the solution represents a stiff system, with the two components having vastly different timescales ($\lambda_1 = 1000$, $\lambda_2 = 1$). If we use an explicit scheme, we would need a timestep of around 1/500 to ensure we accurately capture the dynamics of the fast system, while we would need to integrate for quite a long while in order to ensure we capture the slow system until it reaches a steady state. In order to increase the timestep and hence decrease the computational cost, it is better to proceed with an implicit method such as the backward Euler method, with a timestep of h = 0.1. This of course limits the error to the same order of magnitude.

$$\frac{\boldsymbol{x}^{k+1} - \boldsymbol{x}^k}{h} = \boldsymbol{A}\boldsymbol{x}^{k+1} \Rightarrow [\boldsymbol{I} - \boldsymbol{A}h]\, \boldsymbol{x}^{k+1} = \boldsymbol{x}^k$$

$$[\boldsymbol{I} - \boldsymbol{A}h]\, \boldsymbol{x}^1 = \boldsymbol{x}^0,$$

$$\begin{pmatrix} \mathbf{-98.8} & \mathbf{-199.8} \\ \mathbf{99.9} & \mathbf{200.9} \end{pmatrix} \begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We can now use a linear system solver such as Gaussian elimination for example, to solve this system.

$$\begin{pmatrix} \mathbf{-98.8} & \mathbf{-199.8} \\ \mathbf{0} & \mathbf{-1.12449} \end{pmatrix} \begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 2.0111 \end{bmatrix}$$

$$\Rightarrow x_2^1 = \mathbf{-1.7884}$$

$$\Rightarrow x_1^1 = \frac{1 + 199.8 * (-1.7884)}{-98.8} = \mathbf{3.6066}$$