Part I exam

All submitted solutions to exam problems have to be **handwritten** (either written on paper and scanned or handwritten on a tab/pad). Solutions without **detailed calculus and expressions** used to obtain them are not enough. Whenever it is required to choose an appropriate method, you should **discuss your choice**.

1. **(25 points)** Consider the following system of ODEs

$$\frac{dx_1}{dt} = 998x_1 + 1998x_2$$

$$\frac{dx_2}{dt} = -999x_1 - 1999x_2$$

The general solution of this system can be written in the form:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2c_1 + 2c_2 \\ -c_1 - c_2 \end{bmatrix} e^{-t} + \begin{bmatrix} -c_1 - 2c_2 \\ c_1 + 2c_2 \end{bmatrix} e^{-1000t}$$

a) Which method you would apply to solve numerically the above-mentioned system of ODEs? Discuss what would be the appropriate step size.

b) For $x_1(0) = 1$ and $x_2(0) = 1$, apply the proposed method with the proposed step size to solve the system in the interval $t \in (0,100)$. Provide details of calculating $x(t_1)$ using $x(t_0)$=$[x_1(0) \quad x_2(0)]^T$.

2. **(25 points)** A distillation column in a refinery transforms the incoming crude oil into the following fractions: kerosene (x), gasoline (y), and asphalt (z). Depending on the relative content of light (b_1), middle (b_2) and heavy (b_3) hydrocarbons, the column produces variable amounts of x, y and z according to the following system of linear equations:

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 8 & 1 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

a) Assuming that the refinery is processing thousands different types of crude oils (such as Brent Blend, Ekofisk Blend, Iran Heavy, West Texas Intermediate, Basrah Light, Dubai Crude, Oseberg Blend, etc.) that have varying relative content of hydrocarbons (i.e. varying b_1, b_2, and b_3) propose a method that should be used to compute the expected amounts of kerosene, gasoline and asphalt from all types of crude oil.

b) Use the proposed method to compute x, y and z for two types of crude oil: Iran Heavy $[2 \quad 4 \quad 8]^T$ and Tia Juana Light $[1 \quad 3 \quad 1]^T$.

3. **(18 points)** The level of liquid, x, in a reactor evolves according to the following equation:

$$\frac{dx}{dt} = F(t) - K\sqrt{2gx(t)}$$

where $K=0.00375$ and $g=9.81\ m/s^2$. The values of $x(m)$ measured at equidistant time points $t(s)$ are provided in the table:

t	0	10	20	30	40	50	60	70	80	90	100
x	1	1.19	1.215	1.043	0.912	0.875	0.836	0.815	0.808	0.813	0.819

a) Compute dx/dt using the appropriate finite difference method. Discuss your choice(s).

b) Using the results of a) compute the values of $F(t)$.

c) Knowing that $F(t)=V(t)/S$ with $V(t)$ being the volumetric flow rate, and $S = 1m^2$ being the surface of the reactor, compute the overall volume of the liquid that was pumped into the reactor over the observed period. Use the appropriate composite Simpson rule.

4. **(12 points)** Consider the following nonlinear equation:

$$g(x) = e^x - x - 2$$

a) Using the Newton-Raphson method find numerically the root of this equation with the precision better than 10^{-3}. Take as initial point $x_0=1$; the exact solution is $x^*=1.146193$.

b) Do three steps of Newton-Raphson iteration starting from $x_0=10$. Discuss shortly the convergence properties of the iterative scheme around this initial point.

c) How many iterations would be needed to attain the accuracy of 10^{-9} if we use the bisection method on the interval (0, 5)?