Part I exam

Solutions should be provided with **detailed calculus and expressions**. Whenever it is required to choose an appropriate method, you should **discuss your choice**.

1. **(15 points)** A production line manufactured a large batch of cameras with faulty lenses that produce blurry images (Figure below, left). To avoid the recall of cameras, the company decided to upgrade the camera software and perform numerical deblurring of the images as soon as they are captured (Figure below, right). The deblurring is performed by solving a system of linear equations $Ax = b$, where A is the deblurring matrix that captures the lens imperfections.

Blurred image (b)

Deblurred image ($x = A^{-1}b$)

   a) **(3 points)** Assuming that the deblurring matrix is dense and non-symmetric, propose a method that would be the most efficient for this task. Explain your choice.

   b) **(10 points)** Apply the proposed method to compute $\boldsymbol{x}$ for

$$A = \begin{bmatrix} 3 & 4 & -5 \\ 6 & -3 & 4 \\ 8 & 9 & -2 \end{bmatrix} \quad \boldsymbol{b_1} = \begin{bmatrix} 1 \\ 9 \\ 9 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b_2} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

   c) **(2 points)** Compute the determinant of the matrix A.

2. **(15 points)** A car laps a race track in 54 seconds. The speed of the car at each 6-second interval is determined using a radar gun and is given, from the beginning of the lap, in kilometer per hour, by the entries in the following table.

   a) **(8 points)** Estimate the length of the track by using the appropriate method from numerical integration (explain your method choice).

T (s)	0	6	12	18	24	30	36	42	48	54
X (km/h)	136.1	147	162.4	170.1	161.3	145.9	132.8	119.6	108.6	127.3

   b) **(5 points)** Determine the time points of the maximal and minimal car acceleration using the appropriate finite difference method.

   c) **(2 points)** Taylor series analysis indicates that the error to approximate the derivative of a smooth function $f(x) = e^x$ by the forward finite difference should follow:

$$\left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| = O(h)$$

however, after calculating the derivative of $f(x) = e^x$ at $x = 1$ in MATLAB with 16-decimal-digit accuracy for $h = 10^{-7}$ and $h = 10^{-14}$ the following was obtained:

$$\left| e^1 - \frac{e^{1+10^{-7}} - e^1}{10^{-7}} \right| \approx 1.3 \times 10^{-7}$$

and

$$\left| e^1 - \frac{e^{1+10^{-14}} - e^1}{10^{-14}} \right| \approx 0.053$$

Explain the above results.

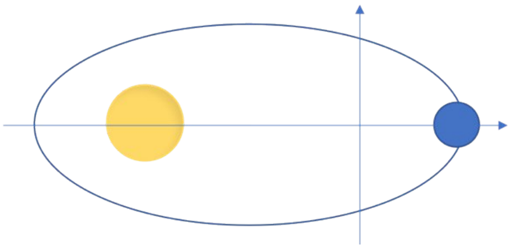
3. **(20 points)** Revolution of a planet around a star can be approximately described by the Kepler's law:

$$\omega t = \alpha - \varepsilon \cdot \sin\alpha$$

with ω being the angular frequency, t time, ε the orbit eccentricity, and α denotes the angular coordinate of the planet. We need to solve this equation for α to find the planet location at time t. Then, the planet coordinates, x and y can be calculated as:

$$x = \mathrm{A}\,(\cos(\alpha) - \varepsilon)$$
$$y = A\sqrt{1-\varepsilon^2}\cdot \sin(\alpha)$$

where A is the length of major semi-axis of the elliptic orbit. The figure shows the orbit and the planet position for $\alpha = 0$.



a) **(16 points)** If we assume that the eccentricity of the planetary orbit is $\varepsilon = 0.75$ and $\mathrm{A} = 1$ astronomical unit (AU), choose an appropriate method (and discuss your choice) to find the location of the planet for $\omega t = \frac{\pi}{6}, \frac{4\pi}{6}, \frac{10\pi}{6}$. with the equation error (in the equation of α) inferior to $\frac{\pi}{1800}$. For all calculations, take the initial guess as $\frac{\pi}{2}$. Present the results in the table format (Round your results to 3 decimal places):

ωt	α	x	y
0	0.000	0.250	0.000
$4\pi/6$			
$10\pi/6$			

b) **(4 points)** Which method would you choose to solve a system of nonlinear equations a) to guarantee convergence to the solution; b) to have the fastest convergence rate. Explain.

4. **(20 points)** A reaction of benzene hydrogenation on a supported Nickel-kieselguhr catalyst takes place in a tubular reactor. At temperatures below 200°C in the presence of a large excess of hydrogen a first-order kinetics can describe the reaction rate and the mass balance is

$$\frac{dy}{dx} = -\lambda y$$

with y being the dimensionless concentration of benzene and x is dimensionless axial coordinate of the reactor.

a) **(2 points)** Using the knowledge of the analytical solution of the 1st order linear ODE from above and knowing that $y = 1$ for $x = 0$ and $y = 0.41614 \cdot 10^{-9}$ for $x = 1$, calculate the constant λ.

b) **(5 points)** Use the Euler forward method with $h = 0.1$ to solve the system numerically at $x = [0.1, 0.2, 0.3, 0.4, 0.5]$. Compute the error between numerically computed values and analytical solution. Explain the obtained results.

c) **(5 points)** Repeat b) with $h = 0.02$ for $x = [0.1, 0.2, 0.3, 0.4, 0.5]$.

d) **(8 points)** Use the Adams-Bashforth-Moulton predictor-corrector algorithm to solve the system at $x = [0.02, 0.04, 0.06, 0.08, 0.1]$. Compute the error between numerically computed values and analytical solution. Compare the errors for $x = 0.1$ obtained in b), c) and d) and discuss.