Let us consider

$$\langle \psi | \hat{H} | \psi \rangle = E[\psi] \langle \psi | \psi \rangle$$

In the coordinate representation this becomes

$$(*) \quad \int dx\, dx'\, \psi(x)\, h(x,x')\, \psi(x') = E[\psi] \int dx\, dx'\, \psi(x)\, \psi(x')\, \delta(x-x')$$

where $h(x,x') = \langle x | \hat{H} | x' \rangle$ and we have used the properties of the Dirac's delta to write the norm in terms of two integrals.
Note that we are assuming (for simplicity) that the wave function is real.

Then remember (see derivation of the 1d Sch. equation from abstract to coordinate representation) that

$$h(x,x') = \langle x | \hat{H} | x' \rangle = \langle x | \frac{\hat{p}^2}{2m} | x' \rangle + \langle x | V(\hat{x}) | x' \rangle$$

$$= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \delta(x-x') + V(x)\, \delta(x-x')$$

Substituting in (*)

$$\int dx\, dx'\, \psi(x) \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \delta(x-x')\, \psi(x')$$

$$= E[\psi] \int dx \int dx'\, \psi(x)\, \delta(x-x')\, \psi(x')$$

now consider

$$\frac{\delta}{\delta \psi(x'')} \int dx\, dx'\, \psi(x) \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \delta(x-x')\, \psi(x') \qquad \text{A}$$

$$= \int dx\, dx'\, \frac{\delta \psi(x)}{\delta \psi(x'')} \left[\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \delta(x-x')\, \psi(x')$$

$$+ \int dx\, dx'\, \psi(x) \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \delta(x-x')\, \frac{\delta \psi(x')}{\delta \psi(x'')}$$

$$= \int dx\, dx'\, \delta(x-x'') \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \delta(x-x')\, \psi(x')$$

$$+ \int dx\, dx'\, \psi(x) \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \delta(x-x')\, \delta(x'-x'')$$

$$= \int dx' \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx''^2} + V(x'') \right] \delta(x''-x')\, \psi(x')$$

$$+ \int dx\, \psi(x) \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \delta(x-x'')$$

$$= 2 \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx''^2} + V(x'') \right] \psi(x'')$$

where (in the second integral we have used the property of the δ function $\int dy\, g(y) \frac{\partial^2}{\partial y^2} \delta(x-y) = +\frac{d^2 g(x)}{dx^2}$ twice) now consider

$$E[\psi]\frac{\delta}{\delta\psi(x'')}\int dx\,dx'\;\psi(x)\,\delta(x-x')\,\psi(x') \qquad \Delta\Delta$$

$$= E[\psi]\int dx\,dx'\,\frac{\delta\psi(x)}{\delta\psi(x'')}\,\delta(x-x')\,\psi(x')$$

$$+ E[\psi]\int dx\,dx'\,\psi(x)\,\delta(x-x')\,\frac{\delta\psi(x')}{\delta\psi(x'')}$$

$$= E[\psi]\int dx\,dx'\,\delta(x-x'')\,\delta(x-x')\,\psi(x')$$

$$+ E[\psi]\int dx\,dx'\,\psi(x)\,\delta(x-x')\,\delta(x'-x'')$$

$$= E[\psi]\,\psi(x'') + E[\psi]\,\psi(x'')$$

$$= 2E[\psi]\,\psi(x'')$$

Combining Δ & $\Delta\Delta$ as $\Delta = \Delta\Delta$ we have

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx''^2} + V(x'')\right]\psi(x'') = E[\psi]\,\psi(x'')$$

which is the time independent Sch. equation in the coordinate representation