5.2 The Radiative Transport Equation (RTE)

Dualism of light

- **Wave picture**: light is considered as an electromagnetic wave and modeled by the Maxwell equations

- **Particle picture**: light is considered as a stream of energetic particles - photons and modeled by energy conservation - the transport equation

Schematic picture of the radiative transport theory

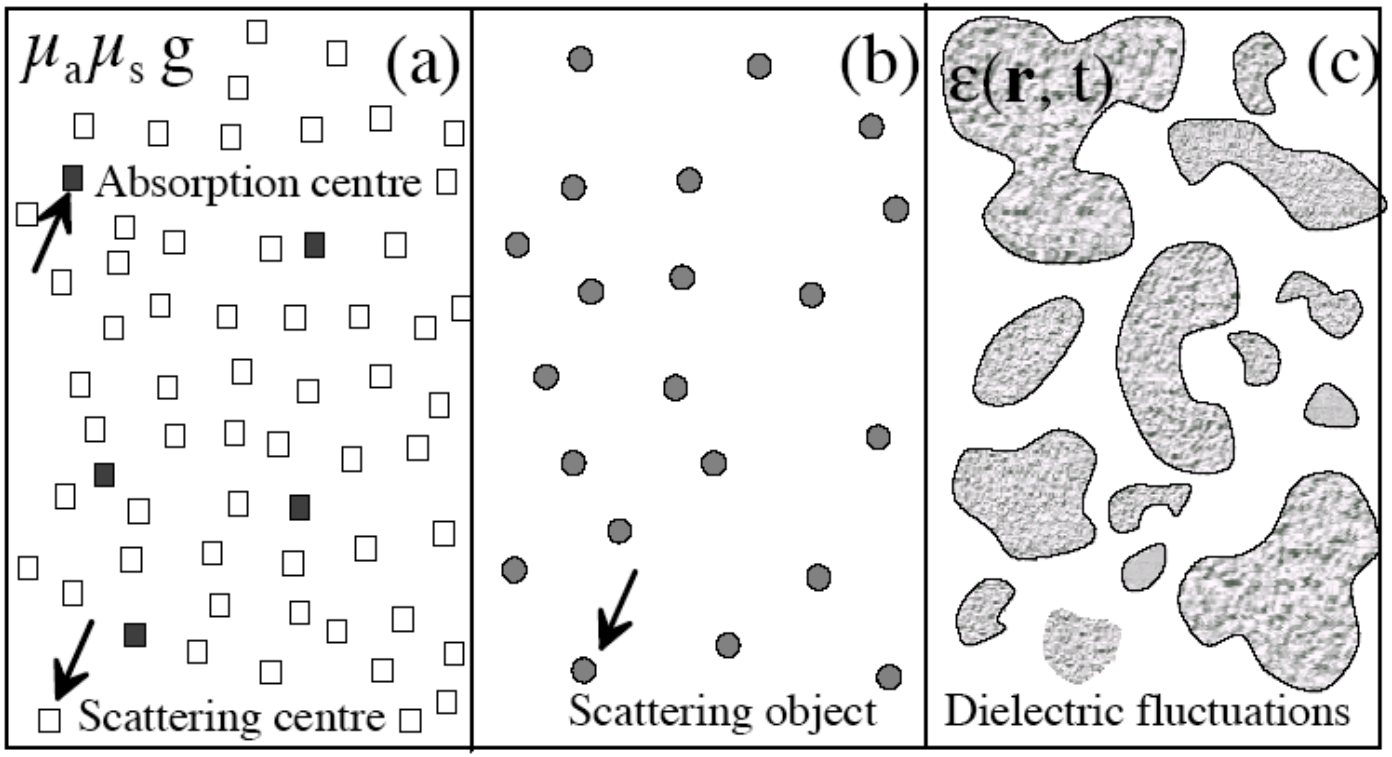


Figure 4.1 *Schematic picture of (a) the radiative transport theory and (b), (c) the electromagnetic theory. The former treats the tissue as it would if it contained randomly distributed scattering and absorption centres, while the later regards tissue as either an ensemble of single scattering particles (b) or as a medium with random dielectric fluctuations (c).*

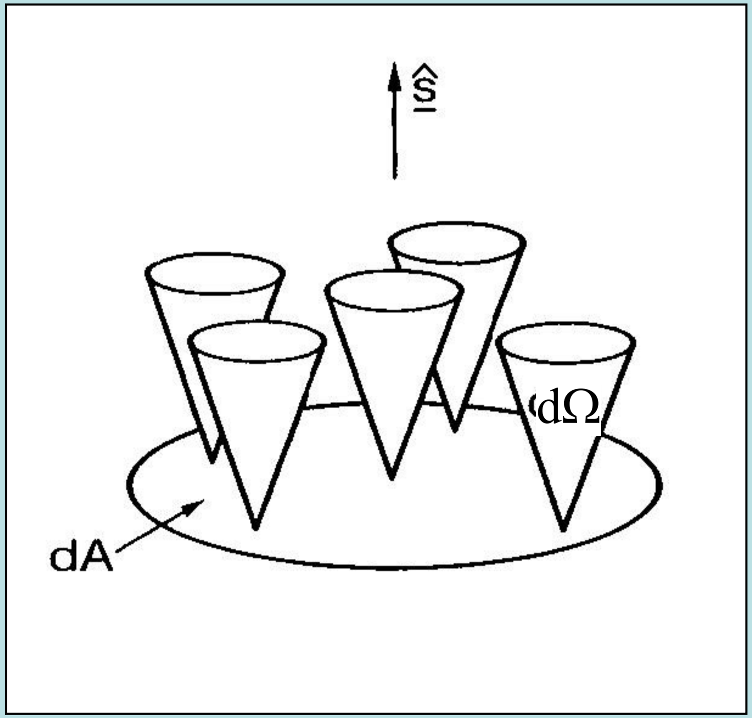
How do we reach an expression for light transport in tissue?

To get an expression that could be solved either analytically or numerically, we start to look into **conservation of energy** in a small volume of tissue

Radiance

Radiance, L(**r**, **s**), is the quantity used to describe the propagation of photon power.

Definition: Radiance is the power [W] that passes through or is emitted from a particular area *dA* [m^2] and falls within a given solid angle *d*Ω [steradian] in a specified direction **s.**



The photon distribution function N(r,s,t)

Definition: The photon distribution function $N(\mathbf{r},\mathbf{s},t)d^3r\, d\Omega$ is the number of photons in the volume d^3r within $d\Omega$ with the direction **s** at time *t*.

The unit of $N(\mathbf{r},\mathbf{s},t)$ is [photons $m^{-3}sr^{-1}$].

The radiance is obtained by multiplying *N* by the photon energy and the velocity of light in the medium:

$$L(\mathbf{r},\mathbf{s},t) = N(\mathbf{r},\mathbf{s},t)h\nu \cdot c$$

Radiant Energy Fluence Rate Φ (r,t)

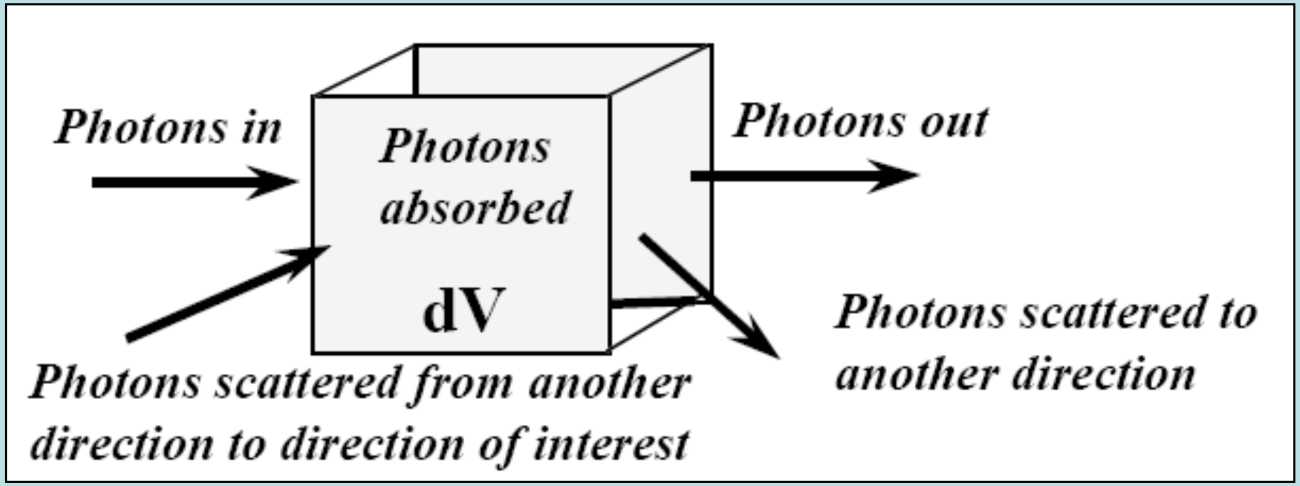
Definition: fluence rate $\Phi(\mathbf{r})$ [W/m2] is the integral of the radiance over all directions.
It corresponds to the radiant power incident on a small sphere, divided by the cross sectional area of that sphere.

$$\Phi(\mathbf{r},t) = \int_{4\pi} L(\mathbf{r},\mathbf{s},t)d\Omega = c \cdot h\nu \int_{4\pi} N(\mathbf{r},\mathbf{s},t)d\Omega$$

Because an absorption chromophore located at **r** can absorb photons irrespectively of their direction of propagation, the fluence rate has more practical significance than the radiance itself.

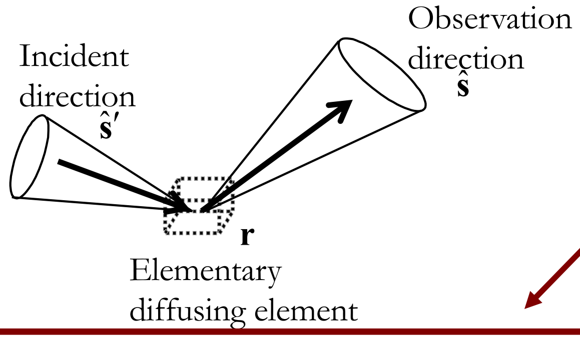
Radiative Transport Equation RTE (I)

- Consider a small volume dV and a direction **s**.
- Conservation of energy yields that photons can only be added to or subtracted from the photon distribution function in specific interactions.



Radiative Transport Equation

Radiance L ($W.m^{-2}.sr^{-1}$)



Change of radiance L

Loss: Absorption + scattering

$$\frac{1}{c}\frac{\partial L(\mathbf{r},\mathbf{s},t)}{\partial t} + \mathbf{s}\cdot\nabla L(\mathbf{r},\mathbf{s},t) = -(\mu_a(\mathbf{r})+\mu_s(\mathbf{r}))L(\mathbf{r},\mathbf{s},t)$$

$$+\mu_s(\mathbf{r})\int_{4\pi} p(\mathbf{s}',\mathbf{s})L(\mathbf{r},\mathbf{s}',t)d^2s' + q(\mathbf{r},\mathbf{s},t)$$

Gains/losses at boudaries

Gains: scattering + source

Integro-differential equation difficult to solve

➔Use the diffusion approximation equation.

First Term

Change of photon distribution

$$\int_V \frac{\partial N(\mathbf{r}, \mathbf{s}, t)}{\partial t} dV$$

Gauss Theorem

Gauss' theorem relates the flow of a vector field through a surface to the behavior of the vector field inside the surface.

More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence of the region inside the surface.

Intuitively, it states that *the sum of all sources minus the sum of all sinks gives the net flow out of a region.*

$$\int_V \nabla f \, dV = \oint_S f \, d\mathbf{S}$$

Second Term

Photons lost at boundaries with Gauss Theorem :

$$-c\int_V \mathbf{s}\cdot\nabla N(\mathbf{r},\mathbf{s},t)dV = -c\oint_S \mathbf{s}\cdot N(\mathbf{r},\mathbf{s},t)dS$$

Third term

Loss (II): Photons scattered from direction **s** to any other direction **s'**

$$-\int_V c\mu_s(\mathbf{r})N(\mathbf{r},\mathbf{s},t)dV$$

Forth term

Loss (III): Photons absorbed coming from direction **s**

$$-\int_V c\mu_a(\mathbf{r})N(\mathbf{r},\mathbf{s},t)dV$$

Fifth Term

Gain (I): Photons gained through scattering from any direction **s'** into direction **s.**

$$+\int_V c\mu_s(\mathbf{r})dV \int_{4\pi} p(\mathbf{s}', \mathbf{s})N(\mathbf{r}, t, \mathbf{s}')ds'$$

Sixth Term

Gain (II): Photons gained through a light source q

$$+\int_V q(\mathbf{r},\mathbf{s},t)dV$$

Final Transport Equation

$$\frac{1}{c}\frac{\partial L(\mathbf{r},\mathrm{s},t)}{\partial t} = -\mathrm{s}\cdot\nabla L(\mathbf{r},\mathrm{s},t) - (\mu_a + \mu_s)L(\mathbf{r},\mathrm{s},t)$$

$$+\mu_s \int_{4\pi} p(\mathrm{s}',\mathrm{s})L(\mathbf{r},\mathrm{s}')d\Omega' + q(\mathbf{r},\mathrm{s},t)$$

because:

$$L(\mathbf{r},\mathrm{s}) = N(\mathbf{r},\mathrm{s})h\nu\cdot c$$

Steady State Transport Equation

$$\mathbf{s} \cdot \nabla L(\mathbf{r}, \mathbf{s}) = \frac{\partial L(\mathbf{r}, \mathbf{s})}{\partial s} =$$

$$-(\mu_a + \mu_s) L(\mathbf{r}, \mathbf{s}) + \mu_s \int_{4\pi} p(\mathbf{s}', \mathbf{s}) L(\mathbf{r}, \mathbf{s}') d\Omega' + q(\mathbf{r}, \mathbf{s})$$

How to Solve RTE ?

Computer-based	Analytical
Numerical Techniques discretization of equation finite-difference and finite element method **difficult to implement, fast results,** **accuracy depends on discretization**	**Diffusion Approximation** Approximating RTE to find simple partial differential equations. **easy to implement** **fast results** **limited accuracy**
Monte Carlo Simulations stochastic solution methods. Many single photon are "randomly" propagated. indirect solution of integro-differential equations. **easy to implement, very time consuming** **very accurate**	**Kubelka Munk Approximation** 1-D Model based on the study of the propagation of diffuse flux. **easy to implement** **fast results** **limited accuracy** **1-D geometry and diffuse flux**

Approximation Criteria

- **Scattering dominant regime (red + near infrared)**

  $\mu_s' \gg \mu_a$

  Diffusion approximation can be used to describe light transport

- **Intermediate regime (visible)**

  $\mu_s' \approx \mu_a$

  Most difficult to handle rigorously

  → generally use Monte Carlo methods

Diffusion Approximation

- For a dense medium of primarily scatters, the transport equation can be simplified to the Diffusion Equation
- Can be solved analytically for special cases or more generally by numerical techniques
- Validity of diffusion equation is limited to tissue cases where the light has been highly scattered ($\mu_s' \gg \mu_a$); i.e., μ_s' should be at least 10 times greater than μ_a

Monte Carlo Approximation

- For situations where Diffusion theory breaks down, the most useful method is Monte Carlo modeling
- ✓ Computational technique which simulates multiple scattering trajectories of individual photons through a turbid medium
- ✓ Each interaction is governed by the random processes of absorption and scattering