Trigonometric identities

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$
$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$\sin(a)\sin(b) = \frac{\cos(a-b) - \cos(a+b)}{2}$$
$$\sin(a)\cos(b) = \frac{\sin(a+b) + \sin(a-b)}{2}$$
$$\cos(a)\cos(b) = \frac{\cos(a-b) + \cos(a+b)}{2}$$

Laplace transform

- Definition of the Laplace transform:

  $$F(s) = \mathcal{L}[f(t)] := \int_0^\infty f(t)e^{-st}dt,$$

- Linearity:

  $$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

- Laplace transform of $t^n f(t)$:

  $$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

- First shift theorem:

  $$\mathcal{L}\left[e^{-bt} f(t)\right] = F(s+b)$$

- Change of scale:

  $$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right), \quad a > 0$$

- Laplace transform of a derivative:

  $$\mathcal{L}\left[\frac{d^n}{dt^n} f(t)\right] = s^n F(s) - \sum_{j=1}^{n} s^{n-j} \frac{d^{j-1}}{dt^{j-1}} f(0)$$

- Laplace transform of an integral:

  $$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{F(s)}{s}$$

- Laplace transform of $\frac{f(t)}{t}$:

  $$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)du,$$

  if $\mathcal{L}\left[\frac{f(t)}{t}\right] \to 0$ as $s \to \infty$.

- Second shift theorem:

  $$\mathcal{L}[h(t-t_0)f(t-t_0)] = e^{-st_0} F(s),$$

  where $h(t)$ is the Heaviside function defined as

  $$h(t) := \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}.$$

- Several Laplace transforms:

  $$\mathcal{L}[1] = \frac{1}{s},$$
  $$\mathcal{L}[\cos t] = \frac{s}{1+s^2},$$
  $$\mathcal{L}[\sin t] = \frac{1}{1+s^2}.$$

Inverse Laplace transform

- Definition of the inverse Laplace transform:

  $$\mathcal{L}^{-1}[F(s)] = f(t)$$

- Linearity:

  $$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)]$$

- Initial value theorem:
  If $f(0)$ and the limit exist, then

  $$f(0) = \lim_{s\to\infty} sF(s).$$

- Final value theorem:
  If the limits exist, then

  $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

- Dirac delta function:

  $$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t)dt = (-1)^n f^{(n)}(0),$$
  $$\mathcal{L}\left[\delta^{(n)}(t)\right] = s^n.$$

- Several inverse Laplace transforms:

  $$\mathcal{L}^{-1}\left[\frac{1}{s} e^{-a\sqrt{s}}\right] = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right),$$
  $$\mathcal{L}^{-1}\left[e^{-a\sqrt{s}}\right] = \frac{a}{2\sqrt{\pi t^3}} e^{-a^2/(4t)}.$$

Definition of Fourier series

For a piecewise continuous function $f(x)$ which is periodic with a period of 2π, the Fourier series is defined as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)],$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

Convolution

$$f * g\ (t) = \int_0^t f(\tau) g(t-\tau) d\tau,$$
$$\mathcal{L}[f * g\ (t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

Complex numbers

$$e^{i\pi} = -1,$$
$$e^{i\phi} = \cos(\phi) + i\sin(\phi)$$

Complex Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Kronecker formula

If $p(x)$ is a polynomial of degree m and $f(x)$ is a continuous function, then:

$$\int p(x) f(x) dx = pF_1 - p'F_2 + p''F_3 - \cdots + (-1)^m p^{(m)} F_{m+1},$$

where $F_1 = \int f(x)dx$ and $F_{n+1} = \int F_n(x)dx$ for $n \in \mathbb{N}$.

Hyperbolic functions

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$
$$\sinh x = -i\sin(ix)$$
$$\cosh x = \cos(ix)$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$
$$1 = \cosh^2 x - \sinh^2 x$$

Sturm-Liouville problem for ODE $X''(x) + \lambda X(x) = 0$ with $0 < x < \pi$:

(1) If $X'(0) = X'(\pi)$, the solutions are:
$X_0(x) = A_0 = \text{const}$ for $\lambda = 0$ and
$X_n(\lambda) = A_n\cos(nx)$ for $\lambda = n^2$, $n \in \mathbb{N}$.
(2) If $X(0) = X(\pi)$, the solutions are:
$X_n(\lambda) = B_n \sin(nx)$ for $\lambda = n^2$, $n \in \mathbb{N}$.

Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$g(t) = f(t+a),\ a \in \mathbb{R} \Rightarrow G(\omega) = e^{i\omega a} F(\omega)$$
$$g(t) = f(at),\ a \in \mathbb{R}\setminus\{0\} \Rightarrow G(\omega) = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$
$$g(t) = \frac{d^n f(t)}{dt^n}, n \in \mathbb{N}_0 \Rightarrow G(\omega) = (i\omega)^n F(\omega)$$

Convolution and convolution theorem:

$$f * g(t) = \int_{-\infty}^{\infty} f(x) g\left(t - x\right) dx$$
$$\mathcal{F}\left[f * g\right](\omega) = F(\omega)G(\omega)$$

$$F_c(\omega) = \int_0^{\infty} f(t)\cos\left(\omega t\right) dt$$
$$f(t) = \frac{2}{\pi}\int_0^{\infty} F_c(\omega)\cos\left(\omega t\right) d\omega$$
$$F_s(\omega) = \int_0^{\infty} f(t)\sin\left(\omega t\right) dt$$
$$f(t) = \frac{2}{\pi}\int_0^{\infty} F_s(\omega)\sin\left(\omega t\right) d\omega.$$