$\boxed{1}$

(8pt.) (a) $\mathcal{L}[\cos(2t)] = \mathcal{L}[g(2t)] \overset{\text{[scale]}}{=} \frac{1}{2} G(\frac{s}{2})$,

where $g(t) = \cos(t)$ and $G(s) \overset{\text{[formula]}}{=} \frac{s}{1+s^2}$.

Hence, $\mathcal{L}[\cos(2t)] = \frac{1}{2} \frac{s/2}{1+(s/2)^2} = \frac{\frac{s}{4}}{\frac{s^2+4}{4}} = \boxed{\frac{s}{s^2+4}}$

(8pt.) (b) Cosine F.s. of $\cos(2x)$ is $\boxed{\cos(2x)}$.

I.e., $a_2 = 1$, $a_n = 0$ for $n \in \mathbb{N}_0 \setminus \{2\}$.

(14pt.) (c) Sine F.s. of $\cos(2x)$ is $\boxed{\sum_{n=1}^{\infty} b_n \sin(nx)}$, where

$$b_n = \frac{2}{\pi}\int_0^\pi \cos(\underbrace{2x}_{b}) \sin(\underbrace{nx}_{a})\,dx \overset{\text{[formula]}}{=} \frac{2}{\pi}\int_0^\pi \frac{\sin(a+b)+\sin(a-b)}{2}\,dx$$

$$= \frac{1}{\pi}\int_0^\pi \{\sin[(n+2)x] + \sin[(n-2)x]\}\,dx$$

$$b_2 = -\frac{1}{\pi}\left.\frac{\cos(4x)}{4}\right|_0^\pi = 0.$$

$$b_{n\neq 2} = -\frac{1}{\pi}\left.\left\{\frac{\cos[(n+2)x]}{n+2} + \frac{\cos[(n-2)x]}{n-2}\right\}\right|_0^\pi$$

$$= -\frac{1}{\pi}\left\{\frac{\cos[(n+2)\pi]-1}{n+2} + \frac{\cos[(n-2)\pi]-1}{n-2}\right\}$$

$$= \frac{1}{\pi}\left[\frac{1-\cos(n\pi)}{n+2} + \frac{1-\cos(n\pi)}{n-2}\right]$$

$$= \frac{1}{\pi}[1-\cos(n\pi)]\left(\frac{1}{n+2}+\frac{1}{n-2}\right)$$

$$= \frac{1}{\pi}[1-(-1)^n]\frac{n-2+n+2}{(n+2)(n-2)} = \boxed{\frac{2}{\pi}[1-(-1)^n]\frac{n}{n^2-4}}$$

$\boxed{b_{2k} = 0}$ for $k \in \mathbb{N}$ and $\boxed{b_{2k-1}} = \boxed{\frac{4(2k-1)}{\pi[(2k-1)^2-4]}}$

$$\boxed{\cos 2x \sim \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{2k-1}{(2k-1)^2-4}\sin[(2k-1)x]}$$

$\boxed{1}$ (b) Alternative solution:

(8 pt)

Cosine F. series $= \boxed{\frac{a_0}{2} + \sum_{n=1} a_n \cos(nx)}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos(\underbrace{2x}_{b}) \cos(\underbrace{nx}_{a})\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \{\cos[(n-2)x] + \cos[(n+2)x]\}\,dx$$

$$= -\frac{1}{\pi}\left\{\frac{\sin[(n-2)x]}{n-2} + \frac{\sin[(n+2)x]}{n+2}\right\}\Bigg|_0^{\pi}$$

[for $n \neq 2$]

$\boxed{n \neq 2 \Rightarrow a_n = 0}$ because $\sin 0 = \sin(k\pi) = 0$, $k \in \mathbb{Z}$.

$$\boxed{a_2} = \frac{1}{\pi}\int_0^{\pi} \cos[(n-2)x]\,dx = \frac{1}{\pi}\int_0^{\pi} \cos 0\,dx = \frac{1}{\pi}\int_0^{\pi} dx = \frac{\pi}{\pi} = \boxed{1}$$

$\boxed{2}$

(a) $\dot{a} = -3k\,a$

$\dot{b} = k\,a$

$\dot{c} = 2k\,a$

$\mathcal{L}$. tr. $s\,A(s) - a(0) = -3k\,A(s)$

$s\,B(s) - b(0) = k\,A(s)$

$s\,C(s) - c(0) = 2k\,A(s)$

Init. cond. $a(0) = a_0, \quad b(0) = c(0) = 0$:

$s\,A(s) - a_0 = -3k\,A(s)$

$s\,B(s) = k\,A(s)$

$s\,C(s) = 2k\,A(s)$

Solve: $A(s) = \dfrac{a_0}{s+3k}$

$B(s) = \dfrac{k\,A(s)}{s} = \dfrac{a_0 k}{s(s+3k)}$

(9 pts.) $C(s) = \dfrac{2k\,A(s)}{s} = \dfrac{2a_0 k}{s(s+3k)}$

$\mathcal{L}^{-1}$:

$$\boxed{a(t)} = \mathcal{L}^{-1}[A(s)] \overset{[\text{shift thm}]}{=\!=} \boxed{a_0\,e^{-3kt}}$$

Partial fractions $B(s) = a_0\left(\dfrac{\alpha}{s} + \dfrac{\beta}{s+3k}\right) = a_0\dfrac{(\alpha+\beta)s + 3k\alpha}{s(s+3k)}$

$\Rightarrow \quad \alpha+\beta = 0, \quad 3\alpha = 1 \quad \Rightarrow \quad \alpha = \frac{1}{3} = -\beta$

$B(s) = \dfrac{a_0}{3}\left(\dfrac{1}{s} - \dfrac{1}{s+3k}\right)$

$$\boxed{b(t)} = \mathcal{L}^{-1}[B(s)] \overset{[\text{shift}]}{=\!=} \boxed{\frac{a_0}{3}\left(1 - e^{-3kt}\right)}$$

$C(s) = 2B(s) \Rightarrow \boxed{c(t)} = 2\,b(t) = \boxed{\dfrac{2a_0}{3}\left(1 - e^{-3kt}\right)}$

or

$\Rightarrow \quad C(s) = \frac{2}{3}a_0\left(\frac{1}{s} - \frac{1}{s+3k}\right) \quad c(t) = \mathcal{L}^{-1}[C(s)]$

(10 pts)

$\boxed{2}$

(2pts) (b) $\boxed{\dfrac{c(t)}{b(t)} = 2}$

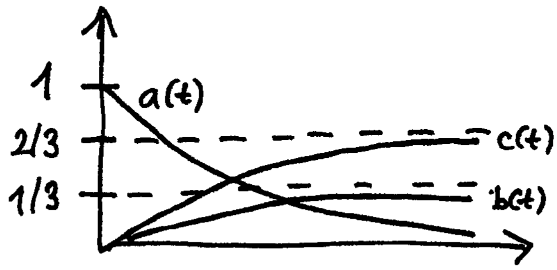
(3pts.) (c)

$$\boxed{\lim_{t\to\infty} a(t)} = \lim_{t\to\infty} a_0 e^{-3kt} = \boxed{0}$$

$$\boxed{\lim_{t\to\infty} b(t)} = \lim_{t\to\infty} \frac{a_0}{3}\left(1-e^{-3kt}\right) = \boxed{\frac{a_0}{3}}$$

$$\boxed{\lim_{t\to\infty} c(t)} = \lim_{t\to\infty} \frac{2a_0}{3}\left(1-e^{-3kt}\right) = \boxed{\frac{2a_0}{3}}$$

(6pts.) (d)



(a) Alternative solutions to find $c(t)$:

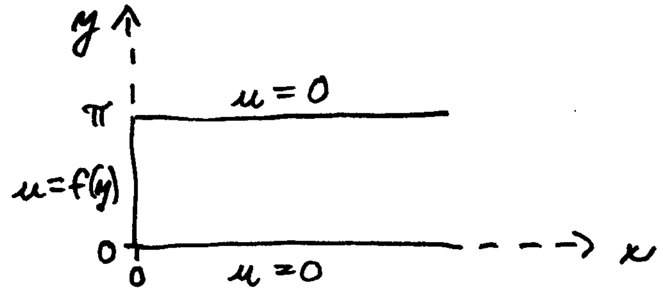
$$\dot{a}+\dot{b}+\dot{c} = (-3k+k+2k)a = 0 \implies$$

$\implies$ conservation of mass: $a(t)+b(t)+c(t) = a(0)+b(0)+c(0) = a_0$

[4pts] $\boxed{c(t)} = a_0 - a(t) - b(t) = a_0\left[1-e^{-3kt}\right] - \frac{a_0}{3}\left(1-e^{-3kt}\right)$

$$= \boxed{\frac{2}{3}a_0\left(1-e^{-3kt}\right)}$$

$\boxed{3}$



$f(y) = \sin y - \sin(2y)$.

Particular sol'n by separation of variables:

(a) (26 pts) $u(x,y) = X(x)Y(y)$

$$0 = u_{xx} + u_{yy} = X''(x)Y(y) + X(x)Y''(y)$$

[4 pts.] $$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

$$Y''(y) + \lambda Y(y) = 0 \quad , \quad Y(0) = Y(\pi) = 0$$

Sturm-Liouville problem $\Rightarrow$ $\lambda_n = n^2$, $n \in \mathbb{N}$,

[5pts.] $Y_n(y) \quad \propto \sin(ny)$

$$X''(x) - \lambda X(x) = 0 \quad \Rightarrow \quad X_n''(x) - n^2 X_n(x) = 0$$

$$X_n(x) = c_n e^{-nx} + d_n e^{+nx}$$

$|u(x,y)| \le M \Rightarrow |X_n(x)| \le M \Rightarrow d_n = 0 \Rightarrow X_n(x) = c_n e^{-nx}$ [5pts.]

General sol'n: $\boxed{u(x,y) = \sum_{n=1}^{\infty} c_n e^{-nx} \sin(ny)}$ [3 pts]

B. c. at $x=0$: $\sin(y) - \sin(2y) = f(y) = u(0,y) = \sum_{n=1}^{\infty} c_n \sin(ny)$

Sine F. series : $c_1 = 1$, $c_2 = -1$, $c_n = 0$ for $n \neq 1, 2$.

Final solution: $\boxed{u(x,y) = e^{-x} \sin y - e^{-2x} \sin(2y)}$ [9 pts]

(b) (4pts) $$\boxed{u\left(\ln 2, \frac{\pi}{2}\right)} = e^{-\ln 2} \underbrace{\sin\frac{\pi}{2}}_{=1} - e^{-2\ln 2} \underbrace{\sin(\pi)}_{=0} = \frac{1}{e^{\ln 2}} = \boxed{\frac{1}{2}}$$

$\boxed{3}$ Alternative way to find c_n:

[7pts]

$$c_n = \frac{2}{\pi}\int_0^{\pi}[\sin y - \sin(2y)]\sin(ny)\,dy$$

$$= \frac{2}{\pi}\int_0^{\pi}[\sin y\cdot\sin(ny) - \sin(2y)\cdot\sin(ny)]\,dy$$

$$\overset{(*)}{=} \frac{1}{\pi}\int_0^{\pi}\{\cos[(n-1)y] - \cos[(n+1)y] - \cos[(n-2)y] + \cos[(n+2)y]\}\,dy$$

$$c_n = -\frac{1}{\pi}\left\{\frac{\sin[(n-1)y]}{n-1} - \frac{\sin[(n+1)y]}{n+1} - \frac{\sin[(n-2)y]}{n-2} + \frac{\sin[(n+2)y]}{n+2}\right\}\Bigg|_0^{\pi}$$

$n \notin \{1,2\} \Rightarrow \boxed{c_n = 0}$

$$\boxed{c_1} = \frac{1}{\pi}\int_0^{\pi}\cos[(n-1)y]\,dy = \frac{1}{\pi}\int_0^{\pi}1\,dy = \boxed{1}$$

$$\boxed{c_2} = -\frac{1}{\pi}\int_0^{\pi}\cos[(n-2)y]\,dy = -\frac{1}{\pi}\int_0^{\pi}1\,dy = \boxed{-1}$$

← In both cases, the 3 other terms in (*) can be integrated as before and give 0.