Alternative solution to exercises 6, problem 1

Using the separation of variables technique with

$$\phi(x,t) = \sum_{k=1}^{\infty} X_k(x) T_k(t) \tag{1}$$

gives the equation

$$\frac{T_k'}{T_k} = \kappa \frac{X_k''}{X_k} = \lambda_k. \tag{2}$$

We now have to solve the two ODEs

$$T_k' = \lambda_k T_k, \tag{3}$$

$$X_k'' = \frac{\lambda_k}{\kappa} X_k. \tag{4}$$

The solution to Eq. (3) is $T_k(t) = T_{k,0} e^{\lambda_k t}$.
We notice that Eq. (4) is a Sturm–Liouville problem with Dirichlet boundary condition $X_k(0) = X_k(\pi/4) = 0$ (note that our boundary here is at $\pi/4$ instead of the standard π). We have seen in the lecture that nontrivial solutions are only obtained when $\lambda_k/\kappa < 0$ and that the solution is $X_k(x) = b_k \sin(4n_k x)$. Yet, in the following, we derive the solution from scratch as if we did not know the discussion of the Sturm-Liouville problems from the lecture.
We apply the boundary conditions to the solution

$$X_k(x) = A_k e^{\sqrt{\lambda_k/\kappa}\, x} + B_k e^{-\sqrt{\lambda_k/\kappa}\, x}. \tag{5}$$

Applying $X_k(0) = 0$ gives $A_k = -B_k$. Therefore, we can rewrite Eq. (5) as

$$\begin{aligned} X_k(x) &= A_k\left(e^{\sqrt{\lambda_k/\kappa}\, x} - e^{-\sqrt{\lambda_k/\kappa}\, x}\right) \\ &= 2A_k \sinh\left(\sqrt{\lambda_k/\kappa}\, x\right). \end{aligned} \tag{6}$$

Applying $X_k(\pi/4) = 0$ gives

$$A_k \sinh\left(\frac{\pi\sqrt{\lambda_k}}{4\sqrt{\kappa}}\right) = 0. \tag{7}$$

This is satisfied if

$$\sqrt{\lambda_k/\kappa} = 4n_k i, \tag{8}$$

where we used that $\sinh(in\pi) = 0$ [note that $\sinh(ix) = i\sin(x)$].
Equation. (1) can now be written as

$$\phi(x,t) = \sum_{k=1}^{\infty} T_{k,0} e^{\lambda_k t} [2iA_k \sin(4n_k x)]. \tag{9}$$

Before we continue, let us define $b_k = 2iT_{k,0}A_k$ (without loss of generality) to simplify Eq. (9):

$$\phi(x,t) = \sum_{k=1}^{\infty} b_k e^{\lambda_k t} \sin(4n_k x). \tag{10}$$

We finally consider the initial condition $\phi(x,0) = x(\pi/4 - x)$:

$$\phi(x,0) = \sum_{k=1}^{\infty} b_k \sin(4n_k x) = x\left(\frac{\pi}{4} - x\right). \tag{11}$$

Now we can work out b_k and n_k by expanding the right hand side of Eq. (11) using the Fourier sine series [see the original solution to exercises 6 problem 1 for how to obtain the Fourier sine series of $x(\pi/4 - x)$]:

$$\sum_{k=1}^{\infty} b_k \sin(4n_k x) = \sum_{k=1}^{\infty} \frac{1}{2\pi(2k-1)^3} \sin[4(2k-1)x]. \tag{12}$$

From this we find

$$\begin{aligned} b_k &= \frac{1}{2\pi(2k-1)^3}, \\ n_k &= 2k - 1. \end{aligned} \tag{13}$$

We finally use that $\lambda_k = \kappa(4n_k i)^2 = -16\kappa(2k-1)^2$ [see Eq. (8) and Eq. (13)] to arrive at the final solution

$$\phi(x,t) = \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} e^{-16(2k-1)^2 \kappa t} \sin[4(2k-1)x]. \tag{14}$$