$$\boxed{\psi(x,y,z) = c(xy + xz + zx)e^{-\alpha r^2}}$$

Spherically symmetric potential -> spherical coordinates

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}}$$

$$\boxed{\begin{aligned} &\psi(r,\theta,\varphi) = ce^{-\alpha r^2} r^2 (\sin^2\theta\cos\varphi\sin\varphi + \sin\theta\cos\theta\sin\varphi + \sin\theta\cos\theta\cos\varphi) = \\ &= ce^{-\alpha r^2} r^2 (\frac{1}{2}\sin^2\theta\sin 2\varphi + \sin\theta\cos\theta\sin\varphi + \sin\theta\cos\theta\cos\varphi) \end{aligned}}$$

Let's write the wavefunction in terms of spherical harmonics

$$\boxed{\psi(r,\theta,\varphi) = ce^{-\alpha r^2} r^2 \left(\boxed{\frac{1}{2}\sin^2\theta\sin 2\varphi} + \boxed{\sin\theta\cos\theta\sin\varphi} + \sin\theta\cos\theta\cos\varphi\right)}$$

l = 0[1] [edit]

$$Y_0^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

l = 1[1] [edit]

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta$$

$$Y_1^{0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \cos\theta$$

$$Y_1^{1}(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta$$

l = 2[1] [edit]

$$Y_2^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta$$

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$$Y_2^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot (3\cos^2\theta - 1)$$

$$Y_2^{1}(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot \cos\theta$$

$$Y_2^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta$$

$$\boxed{\frac{1}{2}\sin^2\theta\sin 2\varphi}$$

Goal: write this expression in function of spherical harmonics

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$$\boxed{\frac{1}{2}\sin^2\theta\sin 2\varphi} \longrightarrow Y_2^{-2} - Y_2^2 = -2i\sqrt{\frac{15}{32\pi}}\sin^2\theta\sin 2\varphi$$

Now we have to adjust the coefficients in front

$$\frac{1}{2} = -2i\sqrt{\frac{15}{32\pi}}a \longrightarrow a = i\sqrt{\frac{2\pi}{15}}$$

$$\boxed{\frac{1}{2}\sin^2\theta\sin 2\varphi = i\sqrt{\frac{2\pi}{15}}Y_2^{-2} - i\sqrt{\frac{2\pi}{15}}Y_2^2}$$

$$\psi(r,\theta,\varphi)=ce^{-\alpha r^2}r^2\left(\frac{1}{2}\sin^2\theta\sin 2\varphi+\boxed{\sin\theta\cos\theta\sin\varphi}+\sin\theta\cos\theta\cos\varphi\right)$$

$$\boxed{\sin\theta\cos\theta\sin\varphi}$$

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$$\boxed{1\sin\theta\cos\theta\sin\varphi} \longrightarrow Y_1^{-2} - Y_1^{2} = -2i\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\sin\varphi$$

Now we have to adjust the coefficients in front

$$1 = 2\sqrt{\frac{15}{8\pi}}b \longrightarrow b = i\sqrt{\frac{2\pi}{15}}$$

$$\boxed{\sin\theta\cos\theta\sin\varphi = i\sqrt{\frac{2\pi}{15}}Y_2^{-1} - i\sqrt{\frac{2\pi}{15}}Y_2^{1}}$$

$$\boxed{\psi(r,\theta,\varphi) = ce^{-\alpha r^2} r^2 \left(\frac{1}{2}\sin^2\theta \sin 2\varphi + \sin\theta\cos\theta\sin\varphi + \boxed{\sin\theta\cos\theta\cos\varphi}\right)}$$

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$$\boxed{1\sin\theta\cos\theta\cos\varphi} \Rightarrow Y_2^{-1}+Y_2^{1}=2\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\cos\varphi$$

Now we have to adjust the coefficients in front

$$1=2\sqrt{\frac{15}{8\pi}}c \Rightarrow c=\sqrt{\frac{2\pi}{15}}$$

$$\boxed{\sin\theta\cos\theta\cos\varphi=\sqrt{\frac{2\pi}{15}}Y_2^{-1}+\sqrt{\frac{2\pi}{15}}Y_2^{1}}$$

$$\boxed{\psi(r,\theta,\varphi)=ce^{-\alpha r^2}r^2(\frac{1}{2}\sin^2\theta\sin 2\varphi+\sin\theta\cos\theta\sin\varphi+\sin\theta\cos\theta\cos\varphi)}$$

$$\psi(r,\theta,\varphi)=ce^{-\alpha r^2}r^2(i\sqrt{\frac{2\pi}{15}}Y_2^{-2}-i\sqrt{\frac{2\pi}{15}}Y_2^{2}+i\sqrt{\frac{2\pi}{15}}Y_2^{-1}-i\sqrt{\frac{2\pi}{15}}Y_2^{1}+\sqrt{\frac{2\pi}{15}}Y_2^{-1}+\sqrt{\frac{2\pi}{15}}Y_2^{1})$$

$$\psi(r,\theta,\varphi)=\sqrt{\frac{2\pi}{15}}ce^{-\alpha r^2}r^2(iY_2^{-2}-iY_2^{2}+iY_2^{-1}-iY_2^{1}+Y_2^{-1}+Y_2^{1})$$

$$\psi(r,\theta,\varphi)=\sqrt{\frac{2\pi}{15}}ce^{-\alpha r^2}r^2(iY_2^{-2}-iY_2^{2}+(i+1)Y_2^{-1}+(1-i)Y_2^{1})$$

We obtain a linear combination of spherical harmonics when the particle is in the states:

L = 2; m=-2

L = 2; m=2

L = 2; m=-1

L = 2; m=1

$$\hbar^2 l(l+1)=6\hbar^2$$

L=2 100% probability

$$P(m) = \frac{c_m * c_m}{\sum c_m * c_m}$$

$$\boxed{\psi(r,\theta,\varphi) = \sqrt{\frac{2\pi}{15}}\, c e^{-\alpha r^2} r^2 \left(iY_2^{-2} - iY_2^{2} + (i+1)Y_2^{-1} + (1-i)Y_2^{1}\right)}$$

$$P(m=-2, l=2) = \frac{c_m * c_m}{\sum c_m * c_m} = \frac{\left|\sqrt{\frac{2\pi}{15}}\sqrt{\frac{2\pi}{15}}(-i)i\right|}{\sqrt{\frac{2\pi}{15}}i\sqrt{\frac{2\pi}{15}}(-i) + \sqrt{\frac{2\pi}{15}}(-i)\sqrt{\frac{2\pi}{15}}i + \sqrt{\frac{2\pi}{15}}(i+1)\sqrt{\frac{2\pi}{15}}(-i+1) + \sqrt{\frac{2\pi}{15}}(1-i)\sqrt{\frac{2\pi}{15}}(1+i)}$$

$$P(m=-2, l=2) = \frac{c_m * c_m}{\sum c_m * c_m} = \frac{\frac{2\pi}{15}1}{\frac{2\pi}{15}6} = \frac{1}{6}$$

$$P(m=2, l=2) = \frac{c_m * c_m}{\sum c_m * c_m} = \frac{\frac{2\pi}{15}1}{\frac{2\pi}{15}6} = \frac{1}{6}$$

$$P(m=-1, l=2) = \frac{c_m * c_m}{\sum c_m * c_m} = \frac{\frac{2\pi}{15}(1-i)(1+i)}{\frac{2\pi}{15}6} = \frac{1}{3}$$

$$P(m=1, l=2) = \frac{c_m * c_m}{\sum c_m * c_m} = \frac{\frac{2\pi}{15}(1-i)(1+i)}{\frac{2\pi}{15}6} = \frac{1}{3}$$

$$P(m=0, l=2) = \frac{c_m * c_m}{\sum c_m * c_m} = \frac{0}{\frac{2\pi}{15}6} = 0$$