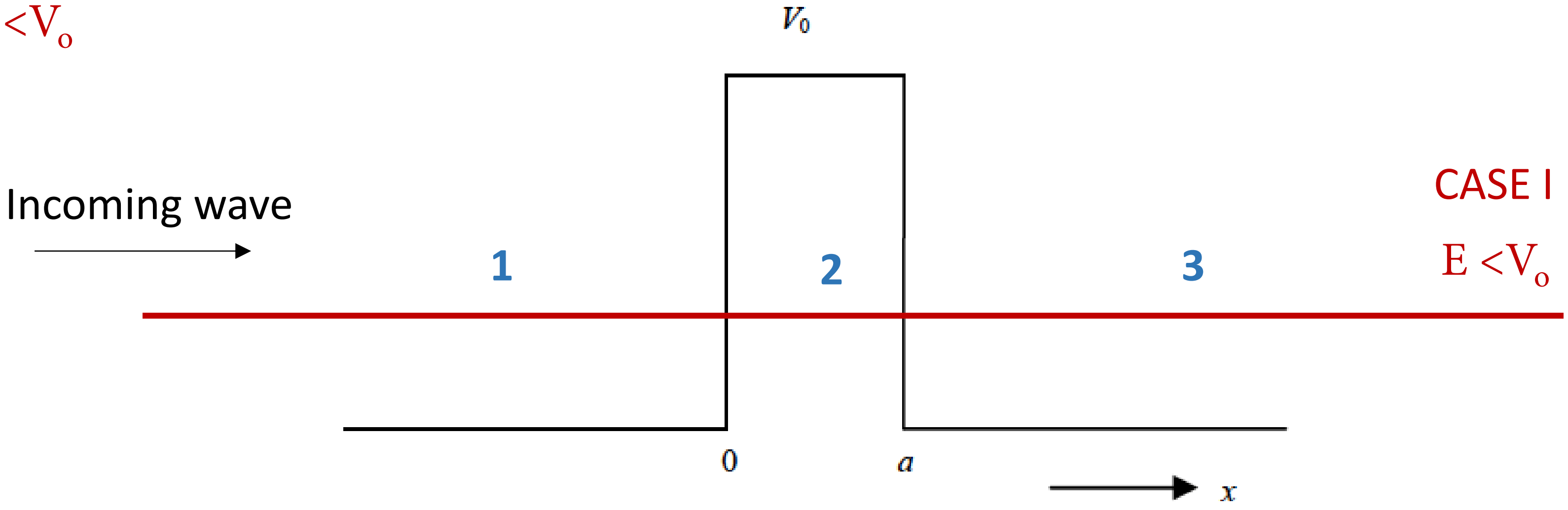
Tunnel effect

CASE I

$E < V_o$



$$H_{class} = \frac{p^2}{2m} \qquad H_{quant} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$H_{class} = \frac{p^2}{2m} + V_0 \qquad H_{quant} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0$$

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CASE I

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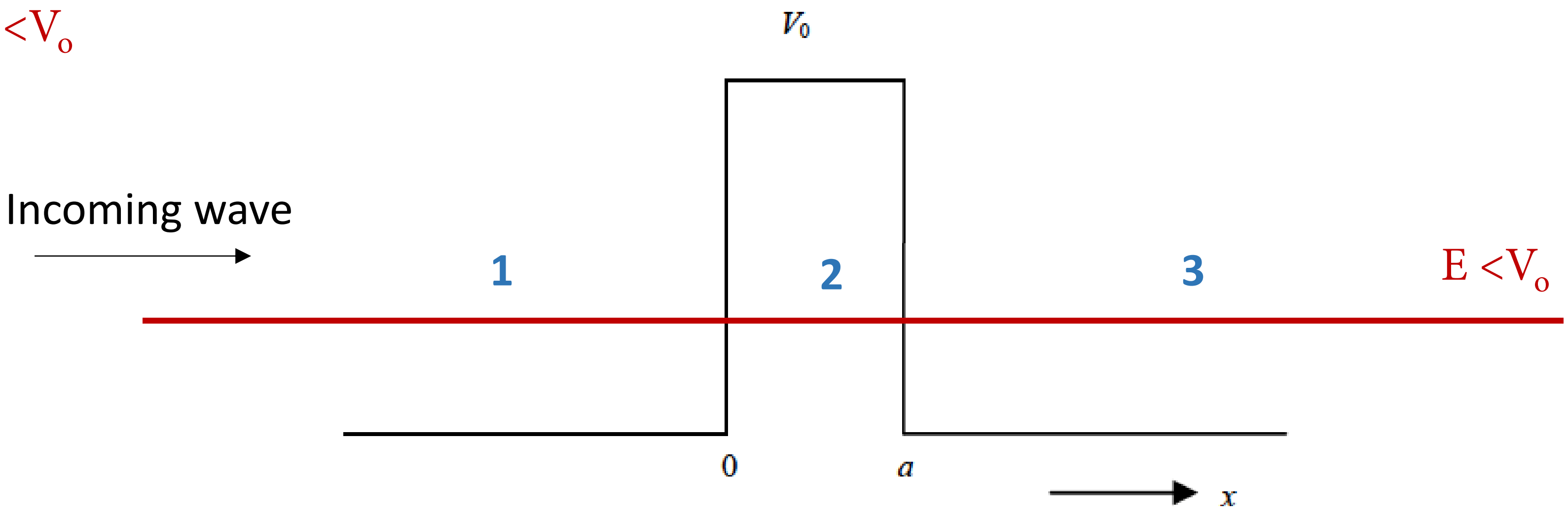


$$H_{class} = \frac{p^2}{2m} \qquad H_{quant} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$\frac{\partial^2 \psi_1}{\partial x^2} - = \frac{2m}{\hbar^2} E\psi_1$$

$$\frac{\partial^2 \psi_1}{\partial x^2} = -\frac{2m}{\hbar^2} E\psi_1$$

$$k^2 = \frac{2m}{\hbar^2} E > 0$$

$$\frac{\partial^2 \psi_1}{\partial x^2} = -k^2 \psi_1$$

$$\boxed{\psi_1(x) = A_1 e^{ikx} + A_1' e^{-ikx}}$$

$$H_{class} = \frac{p^2}{2m} + V_0 \qquad H_{quant} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi_1}{\partial x^2} + V_0\psi_2 = E\psi_2$$

$$\frac{\partial^2 \psi_2}{\partial x^2} = -\frac{2m}{\hbar^2}(E - V_0)\psi_2$$

$$\frac{2m}{\hbar^2}(E - V_0) < 0 \qquad \rho^2 = -\frac{2m}{\hbar^2}(V_0 - E) > 0$$

$$\frac{\partial^2 \psi_2}{\partial x^2} = \rho^2 \psi_2$$

$$\boxed{\psi_2(x) = B_2 e^{\rho x} + B_2' e^{-\rho x}}$$

$$H_{class} = \frac{p^2}{2m} \qquad H_{quant} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$\frac{\partial^2 \psi_3}{\partial x^2} - = \frac{2m}{\hbar^2} E\psi_3$$

$$\frac{\partial^2 \psi_3}{\partial x^2} = -\frac{2m}{\hbar^2} E\psi_3$$

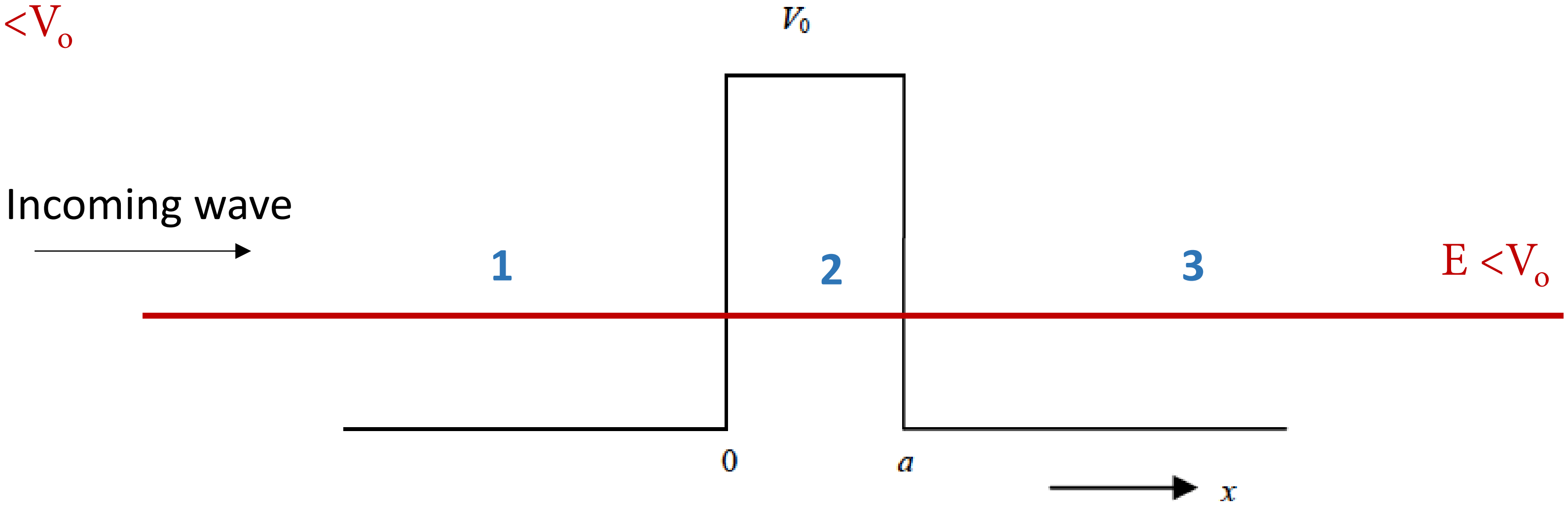
$$k^2 = \frac{2m}{\hbar^2} E > 0$$

$$\frac{\partial^2 \psi_3}{\partial x^2} = -k^2 \psi_3$$

$$\boxed{\psi_3(x) = A_3 e^{ikx} + A_3' e^{-ikx}}$$

CASE I

$E < V_o$



$$k^2 = \frac{2m}{\hbar^2}E > 0$$

$$\boxed{\psi_1(x) = A_1 e^{ikx} + A_1' e^{-ikx}}$$

$$\rho^2 = -\frac{2m}{\hbar^2}(V_0 - E) > 0$$

$$\boxed{\psi_2(x) = B_2 e^{\rho x} + B_2' e^{-\rho x}}$$

$$k^2 = \frac{2m}{\hbar^2}E > 0$$

$$\boxed{\psi_3(x) = A_3 e^{ikx} + A_3' e^{-ikx}}$$

We consider ONLY the particles that are originated at -∞

this implies: **$A_3'=0$**

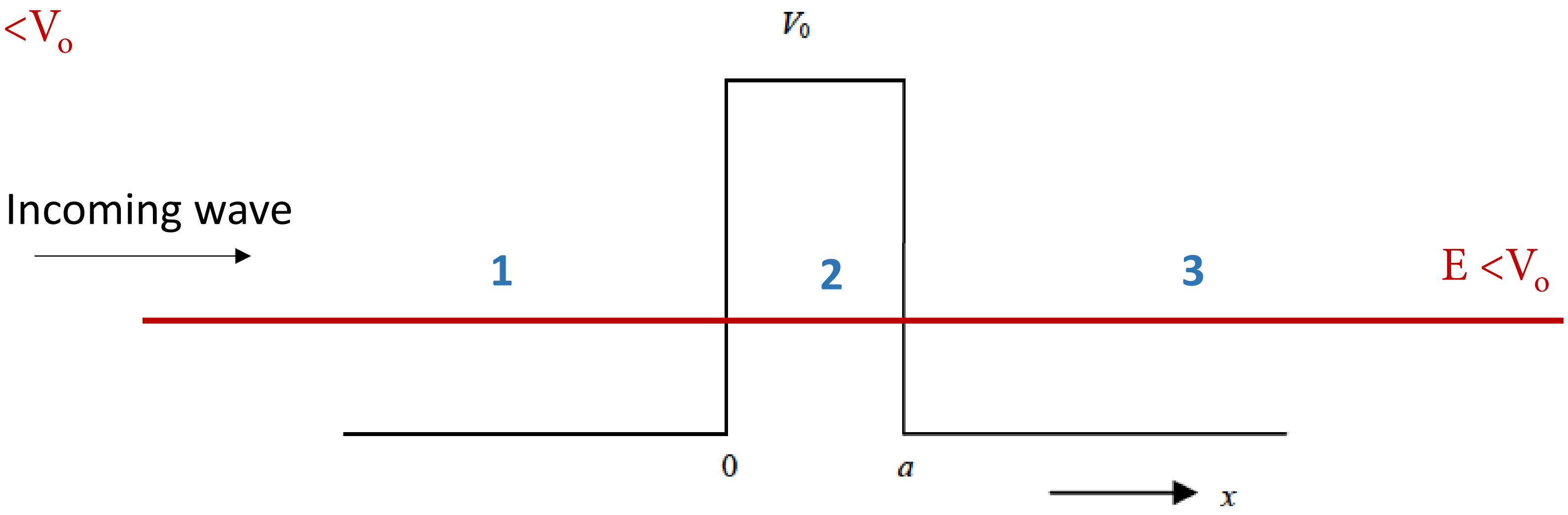
$$\boxed{\psi_1(x) = A_1 e^{ikx} + A_1' e^{-ikx}}$$

$$\boxed{\psi_2(x) = B_2 e^{\rho x} + B_2' e^{-\rho x}}$$

$$\boxed{\psi_3(x) = A_3 e^{ikx}}$$

CASE I

$E < V_o$



$$k^2 = \frac{2m}{\hbar^2}E > 0$$

$$\boxed{\psi_1(x) = A_1 e^{ikx} + A_1' e^{-ikx}}$$

$$\rho^2 = -\frac{2m}{\hbar^2}(V_0 - E) > 0$$

$$\boxed{\psi_2(x) = B_2 e^{\rho x} + B_2' e^{-\rho x}}$$

$$k^2 = \frac{2m}{\hbar^2}E > 0$$

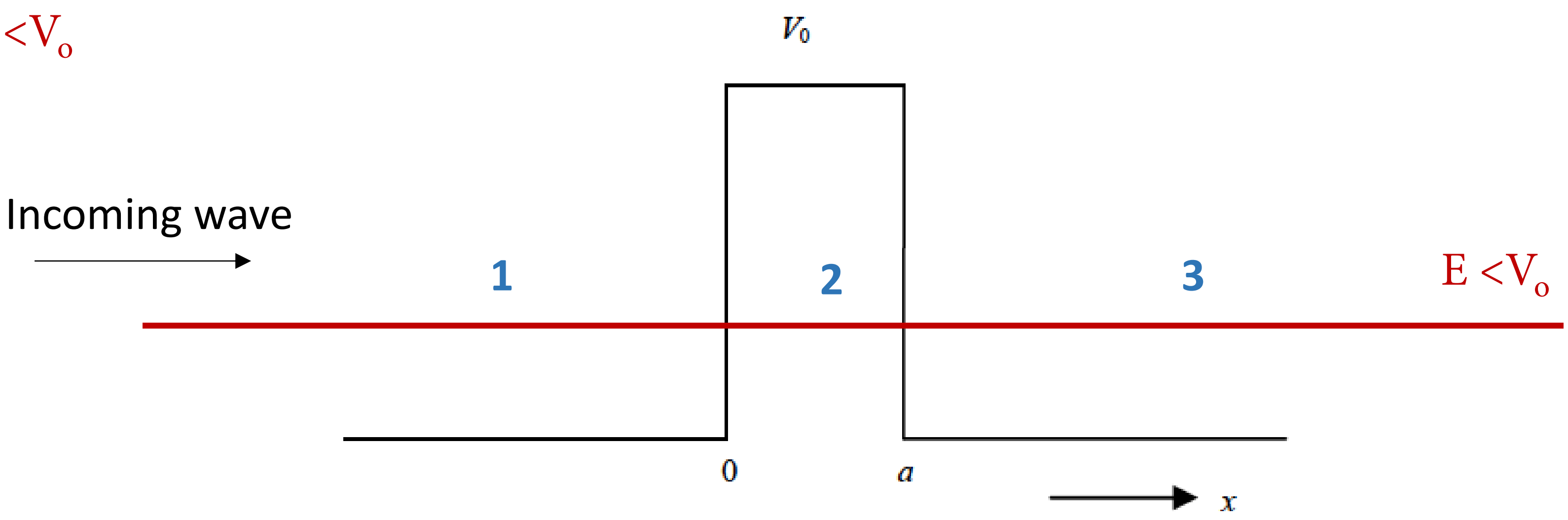
$$\boxed{\psi_3(x) = A_3 e^{ikx}}$$

<u>Boundary conditions</u>: equality of the wavefunctions in **0** and **a**

$$\begin{cases} \psi_1(0) = \psi_2(0) \\ \psi_1'(0) = \psi_2'(0) \\ \psi_2(a) = \psi_3(a) \\ \psi_2'(a) = \psi_3'(a) \end{cases} \qquad \begin{cases} A_1 + A_1' = B_2 + B_2' \\ ikA_1 - ikA_1' = \rho B_2 - \rho B_2' \\ B_2 e^{\rho a} + B_2' e^{-\rho a} = A_3 e^{ika} \\ B_2 \rho e^{\rho a} - B_2' \rho e^{-\rho a} = ikA_3 e^{ika} \end{cases}$$

CASE I

$E < V_o$



$$k^2 = \frac{2m}{\hbar^2}E > 0$$

$$\boxed{\psi_1(x) = A_1 e^{ikx} + A_1' e^{-ikx}}$$

Incident wave — *Reflected wave at 0*

$$\rho^2 = -\frac{2m}{\hbar^2}(V_0 - E) > 0$$

$$\boxed{\psi_2(x) = B_2 e^{\rho x} + B_2' e^{-\rho x}}$$

Transmitted wave in 0 — *Reflected wave at a*

$$k^2 = \frac{2m}{\hbar^2}E > 0$$

$$\boxed{\psi_3(x) = A_3 e^{ikx}}$$

Transmitted wave in a

$$R = \left|\frac{A_1'}{A_1}\right|^2$$

$$T = 1 - R = \left|\frac{A_3}{A_1}\right|^2$$

Probability of having a reflected wave is proportional to the coefficient in front of the reflected wave in region 1 (compared with the incident wave)

Probability of having a transmitted wave is proportional to the coefficient in front of the transmitted wave in region 3 (compared with the incident wave)

CASE I

$E < V_o$

$$\begin{cases} A_1 + A_1' = B_2 + B_2' \\ ikA_1 - ikA_1' = \rho B_2 - \rho B_2' \\ B_2 e^{\rho a} + B_2' e^{-\rho a} = A_3 e^{ika} \\ B_2 \rho e^{\rho a} - B_2' \rho e^{-\rho a} = ikA_3 e^{ika} \end{cases}$$

$$\boxed{R = \left|\frac{A_1'}{A_1}\right|^2 \qquad T = 1 - R = \left|\frac{A_3}{A_1}\right|^2}$$

<u>We need A_1' and A_3 in terms of the same coefficient than A1 to cancel</u>

$$\begin{cases} A_1 + A_1' = \boxed{B_2 + B_2'} \\ ikA_1 - ikA_1' = \boxed{\rho B_2 - \rho B_2'} \\ \mathbf{a} \quad B_2 e^{\rho a} + B_2' e^{-\rho a} = A_3 e^{ika} \\ \mathbf{b} \quad B_2 \rho e^{\rho a} - B_2' \rho e^{-\rho a} = ikA_3 e^{ika} \end{cases}$$

Sum and subtraction of B_2 and B_2'

So then: **a+b** **a-b**

To express B_2 and B_2' in terms of A_3 ONLY

$$\begin{cases} \mathbf{a+b} \quad 2B_2 e^{\rho a} = A_3 e^{ika}\left(1 + \frac{ik}{\rho}\right) \\ \mathbf{a-b} \quad 2B_2' e^{-\rho a} = A_3 e^{ika}\left(1 - \frac{ik}{\rho}\right) \end{cases}$$

$$\begin{cases} B_2 = \frac{1}{2} A_3 e^{ika} e^{-\rho a}\left(1 + \frac{ik}{\rho}\right) \\ B_2' = \frac{1}{2} A_3 e^{ika} e^{\rho a}\left(1 - \frac{ik}{\rho}\right) \end{cases}$$

$$\begin{cases} \boxed{B_2 + B_2'} = \frac{1}{2} A_3 e^{ika}\left[e^{-\rho a}\left(1 + \frac{ik}{\rho}\right) + e^{\rho a}\left(1 - \frac{ik}{\rho}\right)\right] \\ \boxed{B_2 - B_2'} = \frac{1}{2} A_3 e^{ika}\left[e^{-\rho a}\left(1 + \frac{ik}{\rho}\right) - e^{\rho a}\left(1 - \frac{ik}{\rho}\right)\right] \end{cases}$$

CASE I

$E < V_o$

$$\begin{cases} B_2 + B_2' = \frac{1}{2} A_3 e^{ika} \left[e^{-\rho a}\left(1+\frac{ik}{\rho}\right) + e^{\rho a}\left(1-\frac{ik}{\rho}\right) \right] \\ B_2 - B_2' = \frac{1}{2} A_3 e^{ika} \left[e^{-\rho a}\left(1+\frac{ik}{\rho}\right) - e^{\rho a}\left(1-\frac{ik}{\rho}\right) \right] \end{cases}$$

$$\boxed{\begin{cases} \sinh(x) = \frac{e^{x} - e^{-x}}{2} \\ \cosh(x) = \frac{e^{x} + e^{-x}}{2} \end{cases}}$$

$$\begin{cases} B_2 + B_2' = \frac{1}{2} A_3 e^{ika} \left[e^{-\rho a} + e^{\rho a} + \frac{ik}{\rho} e^{-\rho a} - \frac{ik}{\rho} e^{\rho a} \right] = A_3 e^{ika} \left[\cosh(\rho a) - \frac{ik}{\rho} \sinh(\rho a) \right] \\ B_2 - B_2' = \frac{1}{2} A_3 e^{ika} \left[e^{-\rho a} - e^{\rho a} + \frac{ik}{\rho} e^{-\rho a} + \frac{ik}{\rho} e^{\rho a} \right] = A_3 e^{ika} \left[-\sinh(\rho a) + \frac{ik}{\rho} \cosh(\rho a) \right] \end{cases}$$

$$\begin{cases} A_1 + A_1' = \boxed{B_2 + B_2'} \\ ikA_1 - ikA_1' = \boxed{\rho B_2 - \rho B_2'} \\ B_2 + B_2' = \boxed{A_3 e^{ika} \left[\cosh(\rho a) - \frac{ik}{\rho} \sinh(\rho a) \right]} \\ \rho(B_2 - B_2') = \boxed{\rho A_3 e^{ika} \left[-\sinh(\rho a) + \frac{ik}{\rho} \cosh(\rho a) \right]} \end{cases}$$

$$\begin{cases} A_1 + A_1' = A_3 e^{ika} \left[\cosh(\rho a) - \frac{ik}{\rho} \sinh(\rho a) \right] \\ ikA_1 - ikA_1' = \rho A_3 e^{ika} \left[-\sinh(\rho a) + \frac{ik}{\rho} \cosh(\rho a) \right] \end{cases}$$

CASE I

$E < V_o$

$$
\begin{cases}
A_1 + A_1' = A_3 e^{ika}\left[\cosh(\rho a) - \dfrac{ik}{\rho}\sinh(\rho a)\right] & \mathbf{a}\\[2ex]
ikA_1 - ikA_1' = \rho A_3 e^{ika}\left[-\sinh(\rho a) + \dfrac{ik}{\rho}\cosh(\rho a)\right] & \mathbf{b}
\end{cases}
$$

Now I have all the coefficients I was interested in

We can do **a+b** and **a-b** so to have A_1 and A_1' depending on A_3 ONLY

$$
\boxed{\begin{array}{l} R = \left|\dfrac{A_1'}{A_1}\right|^2 \\[2ex] T = 1 - R = \left|\dfrac{A_3}{A_1}\right|^2 \end{array}}
$$

$$
\begin{cases}
A_1 = \dfrac{1}{2} A_3 e^{ika}\left[2\cosh(\rho a) + \dfrac{(k^2 - \rho^2)}{ik\rho}\sinh(\rho a)\right]\\[2ex]
A_1' = A_3 e^{ika}\dfrac{(k^2 + \rho^2)}{ik\rho}\sinh(\rho a)
\end{cases}
$$

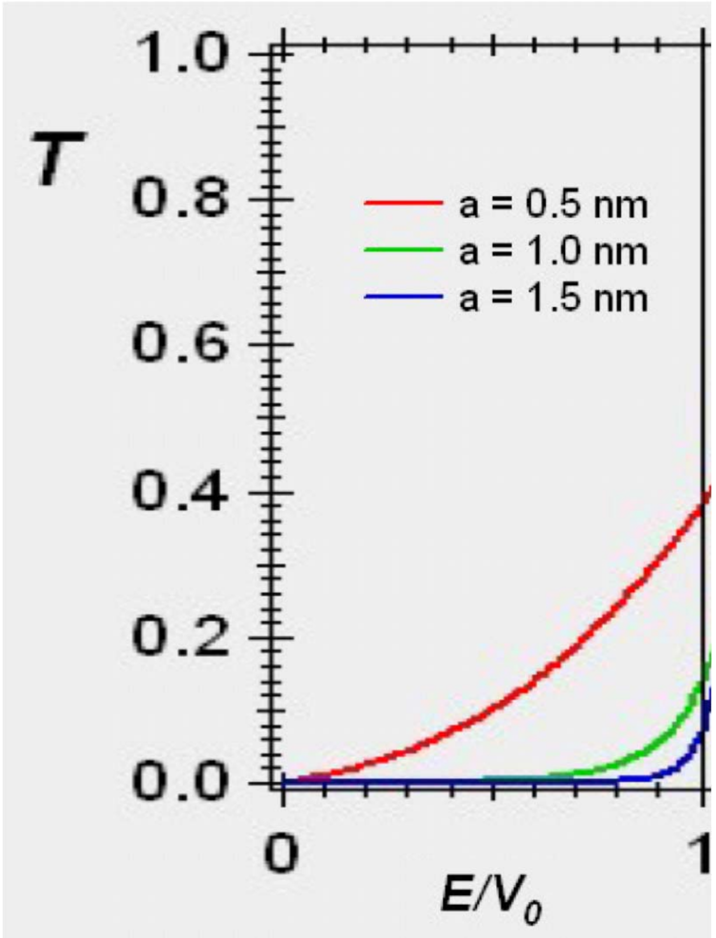
$$
\boxed{\begin{array}{l} R = \left|\dfrac{A_1'}{A_1}\right|^2 = \dfrac{(k^2 + \rho^2)^2 \sinh^2(\rho a)}{4k^2\rho^2 + (k^2 + \rho^2)^2 \sinh^2(\rho a)} \\ T = 1 - R \end{array}}
\longrightarrow
\boxed{T = \dfrac{1}{1 + \dfrac{V_0^{\,2}}{4E(E - V_0)}\sinh^2\left(\dfrac{a}{\hbar}\sqrt{2m(E - V_0)}\right)}}
$$

CASE I

E <V_o

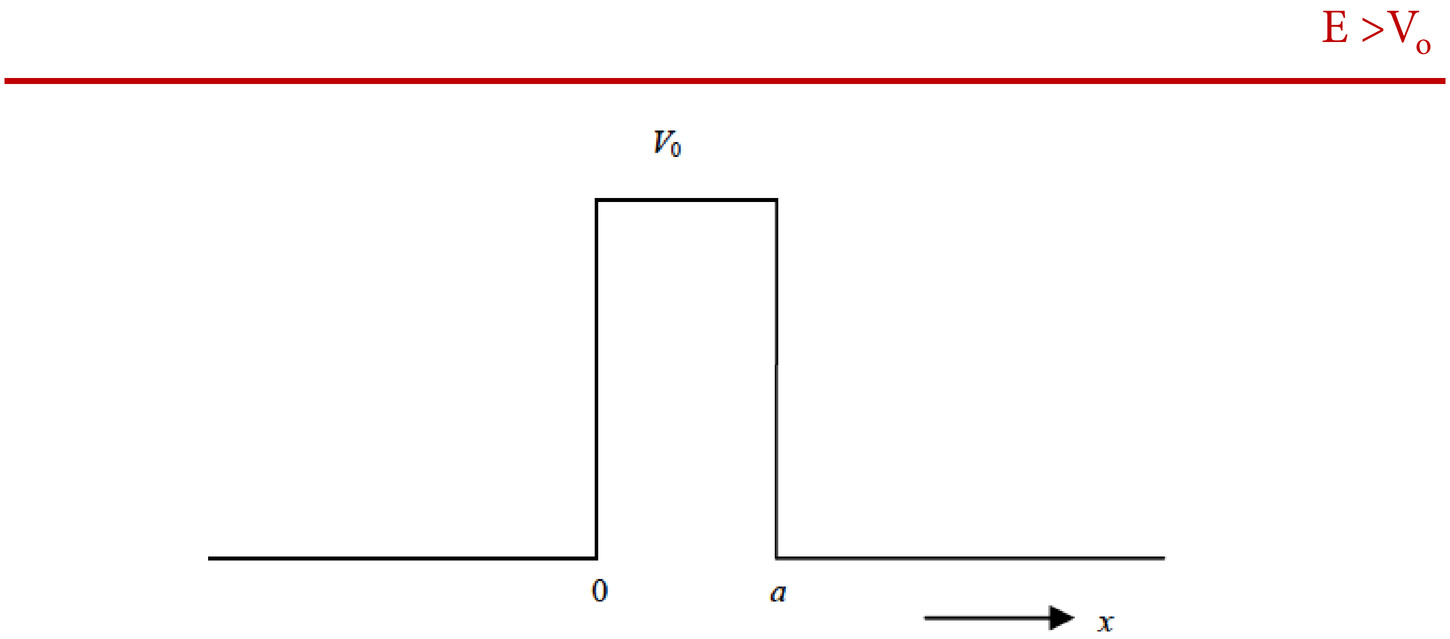
The lighter the molecules, the better the transmission (electrons, muons…)

The narrower the barrier, the better the transmission



CASE I

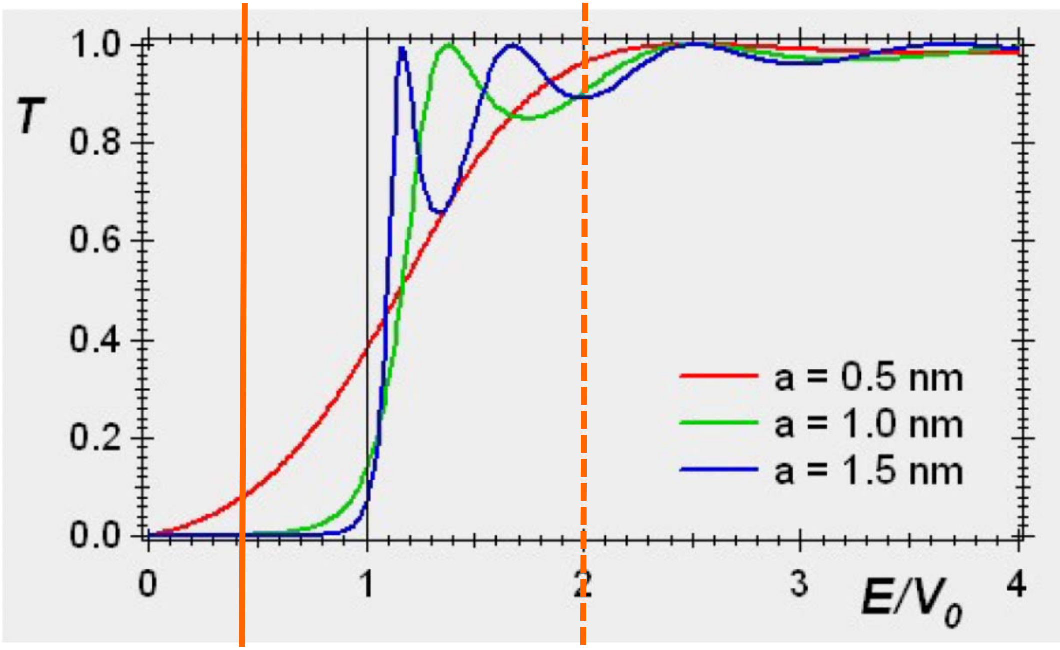
$E > V_o$



Same procedure…

$$R = \frac{1}{1 + \dfrac{4E(E - V_0)}{V^2 \sin^2\left(\dfrac{a}{\hbar}\sqrt{2m(E - V_0)}\right)}}$$

$$T = \frac{1}{1 + \dfrac{V_0^{\,2}}{4E(E - V_0)} \sin^2\left(\dfrac{a}{\hbar}\sqrt{2m(E - V_0)}\right)}$$



E/V_0<1 Tunnel effect
An incoming particle with energy $V_0/2$ is transmitted at ~10% with a barrier of 0.5nm, but it can't be transmitted throught a barrier of 1.5nm

E/V_0>1 Resonances
An incoming particle with energy $2V_0$ is reflected at ~3% with a barrier of 0.5nm, reflected at ~10% with a barrier of 1.5nm