Series 9 - Solutions

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6.87

6.87 Oil (SAE 30) at 15.6 °C flows steadily between fixed, horizontal, parallel plates. The pressure drop per unit length along the channel is 30 kPa/m, and the distance between the plates is 4 mm. The flow is laminar. Determine: (a) the volume rate of flow (per meter of width), (b) the magnitude and direction of the shearing stress acting on the bottom plate, and (c) the velocity along the centerline of the channel.

(a) $$q = \frac{2h^3}{3\mu}\frac{\Delta p}{\ell} \qquad \text{(Eq. 6.136)}$$

For $h = \frac{4\,mm}{2} = 2\times10^{-3}\,m$, $\mu = 0.38\,\frac{N\cdot s}{m^2}$, and $\frac{\Delta p}{\ell} = 30\times10^3\,\frac{N}{m^3}$,

$$q = \frac{2\,(2\times10^{-3}\,m)^3\,(30\times10^3\,\frac{N}{m^3})}{3\,(0.38\,\frac{N\cdot s}{m^2})} = \underline{\underline{4.21\times10^{-4}\,\frac{m^2}{s}}}$$

(b) $$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \text{(Eq. 6.125d)}$$

Since $$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - h^2) \qquad \text{(Eq. 6.134)}$$

and $v = 0$

it follows that

$$\frac{\partial u}{\partial y} = \frac{1}{2\mu}\frac{\partial p}{\partial x}(2y) \qquad \frac{\partial v}{\partial x} = 0$$

and therefore

$$\tau_{yx} = \frac{\partial p}{\partial x}(y)$$

At the bottom plate, $y = -h$, and since $\frac{\partial p}{\partial x} = -\frac{\Delta p}{\ell}$,

$$\tau_{yx} = \frac{\Delta p}{\ell}(h) = (30\times10^3\,\frac{N}{m^3})(2\times10^{-3}\,m)$$

$= \underline{\underline{60\,\frac{N}{m^2}\ \text{acting in the direction of flow}}}$

(c) $$u_{max} = \frac{3}{2}V \qquad \text{(Eq. 6.138)}$$

$$= \frac{3}{2}\left(\frac{q}{2h}\right) = \frac{3}{2}\,\frac{(4.21\times10^{-4}\,\frac{m^2}{s})}{(2)(2\times10^{-3}\,m)} = \underline{\underline{0.158\,\frac{m}{s}}}$$

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6.89

6.89 A viscous, incompressible fluid flows between the two infinite, vertical, parallel plates of Fig. P6.89. Determine, by use of the Navier–Stokes equations, an expression for the pressure gradient in the direction of flow. Express your answer in terms of the mean velocity. Assume that the flow is laminar, steady, and uniform.

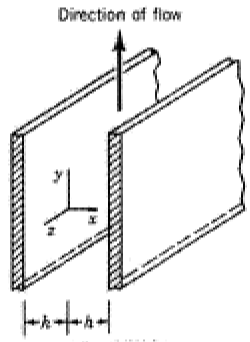

FIGURE P6.89

With the coordinate system shown $u=0$, $w=0$ and from the continuity equation $\frac{\partial v}{\partial y}=0$. Thus, from the y-component of the Navier-Stokes equations (Eq. 6.127b), with $g_y=-g$,

$$0=-\frac{\partial p}{\partial y}-\rho g+\mu\frac{d^2 v}{dx^2} \qquad (1)$$

Since the pressure is not a function of x, Eq. (1) can be written as

$$\frac{d^2 v}{dx^2}=\frac{\underline{P}}{\mu}$$

(where $\underline{P}=\frac{\partial p}{\partial y}+\rho g$) and integrated to obtain

$$\frac{dv}{dx}=\frac{\underline{P}}{\mu}x+C_1 \qquad (2)$$

From symmetry $\frac{dv}{dx}=0$ at $x=0$ so that $C_1=0$. Integration of Eq. (2) yields

$$v=\frac{\underline{P}}{\mu}\frac{x^2}{2}+C_2$$

Since at $x=\pm h$, $v=0$ it follows that $C_2=-\frac{\underline{P}}{2\mu}(h^2)$

and therefore

$$v=\frac{\underline{P}}{2\mu}(x^2-h^2)$$

The flowrate per unit width in the z-direction can be expressed as

$$q=\int_{-h}^{h}v\,dx=\int_{-h}^{h}\frac{\underline{P}}{2\mu}(x^2-h^2)\,dx=-\frac{2}{3}\frac{\underline{P}h^3}{\mu}$$

Thus, with V (mean velocity) given by the equation

$$V=\frac{q}{2h}=-\frac{1}{3}\frac{\underline{P}h^2}{\mu}$$

it follows that

$$\underline{\underline{\frac{\partial p}{\partial y}=-\frac{3\mu V}{h^2}-\rho g}}$$

6.95

6.95 Two immiscible, incompressible, viscous fluids having the same densities but different viscosities are contained between two infinite, horizontal, parallel plates (Fig. P6.95. The bottom plate is fixed and the upper plate moves with a constant velocity U. Determine the velocity at the interface. Express your answer in terms of U, μ_1, and μ_2. The motion of the fluid is caused entirely by the movement of the upper plate; that is, there is no pressure gradient in the x direction. The fluid velocity and shearing stress is continuous across the interface between the two fluids. Assume laminar flow.

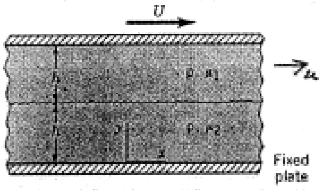

FIGURE P6.95

For the specified conditions, $v=0$, $w=0$, $\frac{\partial p}{\partial x}=0$, and $g_x=0$, so that the x-component of the Navier-Stokes equations (Eq. 6.127a) for either the upper or lower layer reduces to

$$\frac{d^2u}{dy^2}=0 \qquad (1)$$

Integration of Eq. (1) yields

$$u = Ay + B$$

which gives the velocity distribution in either layer.
In the upper layer at $y=2h$, $u=U$ so that

$$B_1 = U - A_1(2h)$$

where the subscript 1 refers to the upper layer.
For the lower layer at $y=0$, $u=0$ so that

$$B_2 = 0$$

where the subscript 2 refers to the lower layer. Thus,

$$u_1 = A_1(y-2h)+U$$

and

$$u_2 = A_2 y$$

At $y=h$, $u_1 = u_2$ so that

$$A_1(h-2h)+U = A_2 h$$

or

$$A_2 = -A_1 + \frac{U}{h} \qquad (2)$$

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Since the velocity distribution is linear in each layer the shearing stress

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu \frac{du}{dy}$$

is constant throughout each layer. For the upper layer

$$\tau_1 = \mu_1 A_1$$

and for the lower layer

$$\tau_2 = \mu_2 A_2$$

At the interface $\tau_1 = \tau_2$ so that

$$\mu_1 A_1 = \mu_2 A_2$$

or

$$\frac{A_1}{A_2} = \frac{\mu_2}{\mu_1} \qquad (3)$$

Substitution of Eq. (3) into Eq. (2) yields

$$A_2 = -\frac{\mu_2}{\mu_1} A_2 + \frac{U}{h}$$

or

$$A_2 = \frac{U/h}{1 + \mu_2/\mu_1}$$

Thus, velocity at the interface is

$$u_2 (y = h) = A_2 h = \underline{\underline{\frac{U}{1 + \frac{\mu_2}{\mu_1}}}}$$

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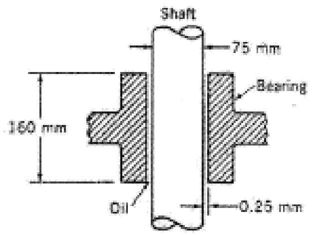
6.99

6.99 A vertical shaft passes through a bearing and is lubricated with an oil having a viscosity of 0.2 N-s/m² as shown in Fig. P6.99. Assume that the flow characteristics in the gap between the shaft and bearing are the same as those for laminar flow between infinite parallel plates with zero pressure gradient in the direction of flow. Estimate the torque required to overcome viscous resistance when the shaft is turning at 80 rev/min.



■ FIGURE P6.99

The torque due to force dF acting on a differential area, $dA = r_i \, \ell \, d\theta$, is (see figure at right)

$$dF = \tau \, dA = \tau r_i \ell \, d\theta$$

$\ell \sim$ shaft length

$$d\mathcal{T} = r_i \, dF = r_i^2 \tau \ell \, d\theta$$

where τ is the shearing stress. Thus,

$$\mathcal{T} = r_i^2 \tau \ell \int_0^{2\pi} d\theta = 2\pi r_i^2 \tau \ell \qquad (1)$$

In the gap,

$$u = U \frac{y}{b} \qquad (\text{Eq. } 6.142)$$

where $U = r_i \omega$ and b is the gap width. Also,

$$\tau = \mu \frac{du}{dy} = \frac{\mu U}{b}$$

Thus, from Eq. (1)

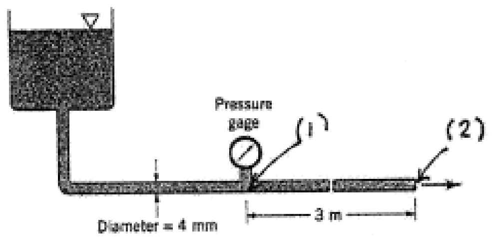
$$\mathcal{T} = 2\pi r_i^2 \left(\mu \frac{U}{b}\right) \ell = 2\pi r_i^3 \mu \omega \frac{\ell}{b}$$

$$= 2\pi \left(\frac{0.075}{2} \, m\right)^3 \left(0.2 \, \frac{N \cdot s}{m^2}\right) \left[\left(80 \, \frac{rev}{min}\right)\left(\frac{2\pi \, rad}{rev}\right)\left(\frac{min}{60 s}\right)\right] \frac{(0.160 m)}{(0.25 \times 10^{-3} m)}$$

$$= \underline{\underline{0.355 \, N \cdot m}}$$

6.103

6.103 A simple flow system to be used for steady flow tests consists of a constant head tank connected to a length of 4-mm-diameter tubing as shown in Fig. P6.103. The liquid has a viscosity of 0.015 N · s/m², a density of 1200 kg/m³, and discharges into the atmosphere with a mean velocity of 2 m/s. (a) Verify that the flow will be laminar. (b) The flow is fully developed in the last 3 m of the tube. What is the pressure at the pressure gage? (c) What is the magnitude of the wall shearing stress, τ_{rz}, in the fully developed region?



■ FIGURE P6.103

(a) Check Reynolds number to determine if flow is laminar:

$$Re = \frac{\rho V (2R)}{\mu} = \frac{(1200 \frac{kg}{m^3})(2 \frac{m}{s})(0.004 m)}{0.015 \frac{N \cdot s}{m^2}} = 640$$

Since the Reynolds number is well below 2100 the flow is laminar.

(b) For laminar flow,

$$V = \frac{R^2}{8\mu} \frac{\Delta p}{\ell} \qquad (Eq.\ 6.152)$$

Since $\Delta p = p_1 - p_2 = p_1 - 0$ (see figure)

$$p_1 = \frac{8 \mu V \ell}{R^2} = \frac{8 (0.015 \frac{N \cdot s}{m^2})(2 \frac{m}{s})(3 m)}{(\frac{0.004}{2} m)^2} = \underline{\underline{180\ kPa}}$$

(c)

$$\tau_{rz} = \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \qquad (Eq.\ 6.126f)$$

For fully developed pipe flow, $v_r = 0$, so that

$$\tau_{rz} = \mu \frac{\partial v_z}{\partial r}$$

Also,

$$v_z = V_{max} \left[1 - \left(\frac{r}{R} \right)^2 \right] \qquad (Eq.\ 6.154)$$

and with $V_{max} = 2V$, where V is the mean velocity

$$\tau_{rz} = 2V\mu \left(-\frac{2r}{R^2} \right)$$

Thus, at the wall, $r = R$,

$$\left| (\tau_{rz})_{wall} \right| = \left| -\frac{4V\mu}{R} \right| = \left| -\frac{4 (2 \frac{m}{s})(0.015 \frac{N \cdot s}{m^2})}{(\frac{0.004}{2} m)} \right| = \underline{\underline{60.0 \frac{N}{m^2}}}$$

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6.104

6.104 **(a)** Show that for Poiseuille flow in a tube of radius R the magnitude of the wall shearing stress, τ_{rz}, can be obtained from the relationship

$$|(\tau_{rz})_{wall}| = \frac{4\mu Q}{\pi R^3}$$

for a Newtonian fluid of viscosity μ. The volume rate of flow is Q. **(b)** Determine the magnitude of the wall shearing stress for a fluid having a viscosity of 0.004 N·s/m² flowing with an average velocity of 130 mm/s in a 2-mm-diameter tube.

(a) $$\tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \qquad \text{(Eq. 6.126f)}$$

For Poiseuille flow in a tube, $v_r = 0$, and therefore

$$\tau_{rz} = \mu \frac{\partial v_z}{\partial r}$$

Since,

$$v_z = v_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad \text{(Eq. 6.154)}$$

and $v_{max} = 2V$, where V is the mean velocity, it follows that

$$\frac{\partial v_z}{\partial r} = -\frac{4Vr}{R^2}$$

Thus, at the wall $(r=R)$,

$$(\tau_{rz})_{wall} = -\frac{4\mu V}{R}$$

and with $Q = \pi R^2 V$

$$|(\tau_{rz})_{wall}| = \underline{\underline{\frac{4\mu Q}{\pi R^3}}}$$

(b) $$|(\tau_{rz})_{wall}| = \frac{4\mu V}{R} = \frac{4\left(0.004 \frac{N \cdot s}{m^2}\right)\left(0.130 \frac{m}{s}\right)}{\left(\frac{0.002}{2}\, m\right)}$$

$$= \underline{\underline{2.08\ Pa}}$$

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