Series 8 - Solutions

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6.21

6.21 A two-dimensional, incompressible flow is given by $u = -y$ and $v = x$. Show that the streamline passing through the point $x = 10$ and $y = 0$ is a circle centered at the origin.

For two-dimensional flow along a streamline

$$\frac{dy}{dx} = \frac{v}{u}$$

so that for the velocity components given

$$\frac{dy}{dx} = \frac{x}{-y}$$

and

$$-\int y\,dy = \int x\,dx$$

Thus,

$$-\frac{y^2}{2} = \frac{x^2}{2} + C \quad \text{(where } C \text{ is a constant)}$$

and

$$x^2 + y^2 = 2C = C' \qquad (1)$$

Equation (1) represents the equation for the family of streamlines. For a given value of C' the equation gives a circle centered at the origin with C' the square of the radius.

For $x = 10$ and $y = 0$

$$10^2 + 0 = C' = 100$$

and the equation of the streamline passing through this point is

$$x^2 + y^2 = 100$$

which is a circle of radius 10 centered at the origin.

6-21

6.22

6.22 In a certain steady, two-dimensional flow field the fluid density varies linearly with respect to the coordinate x; that is, $\rho = Ax$ where A is a constant. If the x component of velocity u is given by the equation $u = y$, determine an expression for v.

For a variable density flow,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} \qquad (\text{Eq. } 6.29)$$

With

$$\rho u = (Ax)(y) = Axy$$

it follows that

$$\frac{\partial(\rho u)}{\partial x} = Ay$$

Thus,

$$\frac{\partial(\rho v)}{\partial y} = -Ay \qquad (1)$$

Integrate Eq. (1) with respect to y to obtain

$$\int d(\rho v) = -\int Ay\,dy + f_1(x)$$

or

$$\rho v = -\frac{Ay^2}{2} + f_1(x)$$

With $\rho = Ax$

$$v = -\left(\frac{1}{Ax}\right)\left(\frac{Ay^2}{2}\right) + \frac{f_1(x)}{Ax}$$

or

$$\underline{\underline{v = -\frac{y^2}{2x} + f(x)}}$$

where $f(x)$ is an arbitrary function of x.

6-22

6.25

6.25 The stream function for an incompressible flow field is given by the equation

$$\psi = 3x^2y - y^3$$

where the stream function has the units of m^2/s with x and y in meters. **(a)** Sketch the streamline(s) passing through the origin. **(b)** Determine the rate of flow across the straight path AB shown in Fig. P6.25.

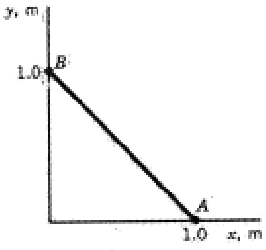


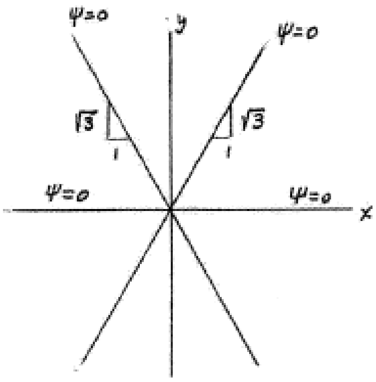
FIGURE P6.25

(a) Lines of constant ψ are streamlines. For $\psi = 3x^2y - y^3$ the streamline passing through the origin $(x=0, y=0)$ has a value $\psi = 0$. Thus, the equation for the streamlines through the origin is

$$0 = 3x^2y - y^3$$

or

$$y = \pm\sqrt{3}\,x \quad \text{or} \quad y = 0$$



A sketch of these streamlines is shown in the figure.

(b) $q = \psi_B - \psi_A$

At B $x = 0$, $y = 1\,m$ so that

$$\psi_B = 3(0)^2(1) - (1)^3 = -1\ m^3/s \text{ (per unit width)}$$

At A $x = 1\,m$, $y = 0$ so that

$$\psi_A = 3(1)^2(0) - (0)^3 = 0$$

Thus,

$$q = \psi_B = \underline{\underline{-1\ m^3/s \text{ (per unit width)}}}$$

The negative sign indicates that the flow is from <u>right</u> to <u>left</u> as we look from A to B.

6-25

6.33

6.33 A two-dimensional flow field for a nonviscous, incompressible fluid is described by the velocity components

$$u = U_0 + 2y$$
$$v = 0$$

where U_0 is a constant. If the pressure at the origin (Fig. P6.33) is p_0, determine an expression for the pressure at **(a)** point A, and **(b)** point B. Explain clearly how you obtained your answer. Assume the units are consistent and body forces may be neglected.

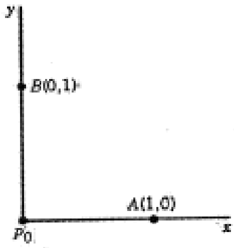


FIGURE P6.33

Check to see if flow is irrotational. Since

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \text{(Eq. 6.12)}$$

and for the given velocity distribution, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 2$, it follows that $\omega_z \neq 0$. Since flow is not irrotational cannot apply the Bernoulli equation between any two points in the flow field.

(a) Since $v = 0$, the origin and point A are on the same streamline. Thus,

$$\frac{p_0}{\gamma} + \frac{V_0^2}{2g} = \frac{p_A}{\gamma} + \frac{V_A^2}{2g} \qquad (1)$$

At the origin $V_0 = U_0$ and at A $V_A = U_0$ so that from Eq. (1)

$$\underline{\underline{p_A = p_0}}$$

(b) Point B is not on same streamline as origin so cannot apply Bernoulli equation between B and O. To find p_B use the y-component of Euler's equations:

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] \qquad \text{(Eq. 6.51b)}$$

Since $v = 0$ and $g_y = 0$,

$$\frac{\partial p}{\partial y} = 0$$

so that

$$\underline{\underline{p_B = p_0}}$$

6-34

6.41

6.41 The velocity potential for a certain inviscid, incompressible flow field is given by the equation

$$\phi = 2x^2y - (\tfrac{2}{3})y^3$$

where ϕ has the units of m^2/s when x and y are in meters. Determine the pressure at the point $x = 2$ m, $y = 2$ m if the pressure at $x = 1$ m, $y = 1$ m is 200 kPa. Elevation changes can be neglected and the fluid is water.

Since the flow is irrotational,

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} \qquad (1)$$

with $V^2 = u^2 + v^2$. For the velocity potential given,

$$u = \frac{\partial\phi}{\partial x} = 4xy \qquad v = \frac{\partial\phi}{\partial y} = 2x^2 - 2y^2$$

At point 1 let $x = 1\,m$ and $y = 1\,m$ so that

$$u_1 = 4(1)(1) = 4\,\frac{m}{s} \qquad v_1 = 2(1)^2 - 2(1)^2 = 0$$

and

$$V_1^2 = \left(4\,\frac{m}{s}\right)^2 = 16\,\frac{m^2}{s^2}$$

At point 2 $x = 2m$ and $y = 2m$ so that

$$u_2 = 4(2)(2) = 16\,\frac{m}{s} \qquad v_2 = 2(2)^2 - 2(2)^2 = 0$$

and

$$V_2^2 = \left(16\,\frac{m}{s}\right)^2 = 256\,\frac{m^2}{s^2}$$

Thus, from Eq. (1)

$$p_2 = p_1 + \frac{\gamma}{2g}\left(V_1^2 - V_2^2\right)$$

$$= 200 \times 10^3\,\frac{N}{m^2} + \frac{\left(9.80 \times 10^3\,\frac{N}{m^3}\right)}{2\left(9.81\,\frac{m}{s^2}\right)}\left(16\,\frac{m^2}{s^2} - 256\,\frac{m^2}{s^2}\right)$$

$$= \underline{\underline{80.1\ kPa}}$$

6-43

6.44

6.44 The velocity potential

$$\phi = -k(x^2 - y^2) \qquad (k = \text{constant})$$

may be used to represent the flow against an infinite plane boundary as illustrated in Fig. P6.44. For flow in the vicinity of a stagnation point it is frequently assumed that the pressure gradient along the surface is of the form

$$\frac{\partial p}{\partial x} = Ax$$

where A is a constant. Use the given velocity potential to show that this is true.

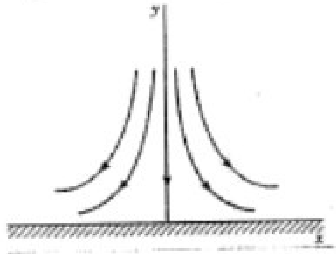


FIGURE P6.44

For the velocity potential given

$$u = \frac{\partial \phi}{\partial x} = -2kx \qquad (1)$$

$$v = \frac{\partial \phi}{\partial y} = 2ky \qquad (2)$$

and the stagnation point occurs at the origin.

For this steady, two-dimensional flow

$$-\frac{\partial p}{\partial x} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \qquad (\text{Eq. 6.51a})$$

and along the surface $(y=0)$ $v=0$ so that

$$-\frac{\partial p}{\partial x} = \rho u \frac{\partial u}{\partial x} \qquad (3)$$

From Eq. (1) $u = -2kx$ and therefore

$$\frac{\partial u}{\partial x} = -2k$$

and Eq. (3) becomes

$$-\frac{\partial p}{\partial x} = \rho(-2kx)(-2k) = 4k^2\rho x$$

or

$$\underline{\underline{\frac{\partial p}{\partial x} = Ax}}$$

where $A = -4k^2\rho$

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